PACIFIC JOURNAL OF MATHEMATICS
Vol. 169, No. 1, 1995

ON THE UNIQUENESS OF CAPILLARY SURFACES OVER AN INFINITE STRIP

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In 1987, Tam proved that the solution of the capillary surface equation without gravity over an infinite strip must be a rigid rotation of a cylinder. Here we give a simple proof for Tam's Theorem and generalize his result.

1. Introduction. Let Ω be a domain in $\mathbb{R}^n$. Consider the equation of prescribed mean curvature

$$\text{div } Tu = H \text{ in } \Omega \tag{1}$$

where $Tu = \frac{Du}{\sqrt{1+|Du|^2}}$ and Du is the gradient of u.

Finn [**4**] proved that if $H = n, \Omega$ contains the unit ball B_1 of $\mathbb{R}^n$, and (1) has a solution u, then Ω has to be exactly B_1 and u must be a lower hemisphere. We emphasize that no boundary condition is imposed.

In the case Ω is unbounded, Finn [**4**] conjectured that the only solution of (1) with $H = 2$ over an infinite strip of width 1 in $\mathbb{R}^2$ is a cylinder. Wang [**9**] and Collin [**2**] independently showed that other different solutions can appear, so Finn's conjecture is not true.

In [**7**], [**8**] Tam considered the problem related to Finn's conjecture as follows:

$$\begin{cases} \text{div } Tu = H & \text{in } \Omega, \\ Tu \cdot \nu = \cos\alpha & \text{on } \partial\Omega \end{cases} \tag{2}$$

where H and α are constants, Ω is the infinite strip $\left(-\frac{1}{2}, \frac{1}{2}\right) \times \mathbb{R}$, ν is the unit outward normal of $\partial\Omega$. The boundary value problem (2) determines the height of a capillary surface without gravity. Note that if (2) has a solution, then $H = 2\cos\alpha$ ([**7**]).

Tam [**7**] proved that any solution of (2) is of the form $\phi_\beta+$ constant for some $|\beta|<1$ if $\frac{\pi}{2}>\alpha>0$, where

$$\phi_\beta=-\frac{1}{\sqrt{1-\beta^2}}\sqrt{\left(\frac{1}{2\cos\alpha}\right)^2-x^2}+\frac{\beta}{\sqrt{1-\beta^2}}y,$$

that is, any solution of (2) must be a cylinder.

The statement above is still true for $\alpha=0$ and was proved by the author first, and then Tam modify his own method in [**7**] to give a unified proof for all cases $\frac{\pi}{2}>\alpha\geq 0$, (c.f. [**8**], where the author's name was spelled by Tam as "C. Wong").

In this paper, we do not give our original proof for the case $\alpha=0$, but give a simple proof for the cases $\frac{\pi}{2}>\alpha\geq 0$. We shall prove that if $\int_{\Gamma_{y_0}} Tu\cdot\nu_1\, d\sigma$ is given where y_0 is a constant, $\Gamma_{y_0}=\left\{(x,y_0)| -\frac{1}{2}\leq x\leq\frac{1}{2}\right\}$, $\nu_1=\langle 0,1\rangle$. Then the solution of (2) is unique up to an additive constant. Therefore we not only obtain Tam's Theorem but also generalize it (§2).

The author is grateful to Prof. Finn's kind suggestion on this paper.

2. Simple proof for Tam's Theorem. To simplify the proof for Tam's Theorem, the well-known inequality

$$(Du-Dv)\cdot(Tu-Tv)\geq\frac{|Du-Dv|^2}{\max((1+|Du|^2)^{\frac{3}{2}},(1+|Dv|^2)^{\frac{3}{2}})}$$

[**6**] will be sharpened, and actually we will use the following inequality (3) to deal with integral estimates:

Lemma 1. *Let $\Omega\subset\mathbb{R}^n$ and let $u,v\in C^1(\Omega)$. Then*

$$(3)\quad (Du-Dv)\cdot(Tu-Tv)\geq\frac{|Du-Dv|^2}{\sqrt{1+(|Du|+|Du-Dv|)^2}}\left(1-\frac{|Du|}{\sqrt{1+|Du|^2}}\right).$$

Proof. Let $|Du|=a$ and $|Du-Dv|=b$. After choosing a suitable coordinates, we may assume $Du-Dv=(b,0,\dots,0)$, $Du=(a\cos\theta,a_2,\dots,a_n)$ where $0\leq\theta\leq\pi$, $a^2\cos^2\theta+a_2^2+\cdots+a_n^2=a^2$.

Then $Dv = (\alpha\cos\theta - b, \alpha_2, \dots, \alpha_n)$. Hence

$$\begin{aligned}
&\left(\frac{Du}{\sqrt{1+|Du|^2}} - \frac{Dv}{\sqrt{1+|Dv|^2}}\right)\cdot(Du - Dv) \\
&= \left(\frac{\alpha\cos\theta}{\sqrt{1+\alpha^2}} - \frac{\alpha\cos\theta - b}{\sqrt{1+(\alpha\cos\theta-b)^2+\alpha_2^2+\cdots+\alpha_n^2)}}\right) b \\
&= b\left(\alpha\cos\theta\left(\frac{1}{\sqrt{1+\alpha^2}} - \frac{1}{\gamma}\right) + \frac{b}{\gamma}\right) \\
&= b\left(\alpha\cos\theta\left(\frac{-2\alpha b\cos\theta + b^2}{\sqrt{1+\alpha^2}\gamma(\sqrt{1+\alpha^2}+\gamma)}\right) + \frac{b}{\gamma}\right) \\
&= \frac{b^2}{\gamma}\left(\frac{\alpha\cos\theta(-2\alpha\cos\theta + b)}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}+\gamma)} + 1\right),
\end{aligned}$$

where $\gamma = \sqrt{1+\alpha^2 - 2\alpha b\cos\theta + b^2}$.

Since

$$\begin{aligned}
\sqrt{1+\alpha^2} + \sqrt{1+\alpha^2-2\alpha b\cos\theta+b^2} &\geq |\alpha| + |\alpha\cos\theta - b| \\
&\geq |-2\alpha\cos\theta + b|
\end{aligned}$$

and since $\sqrt{1+\alpha^2-2\alpha b\cos\theta+b^2} \leq \sqrt{1+(\alpha+b)^2}$, we have

$$(Tu - Tv)\cdot(Du - Dv) \geq \frac{b^2}{\sqrt{1+(\alpha+b)^2}}\left(1 - \frac{\alpha}{\sqrt{1+\alpha^2}}\right).$$

This completes the proof. □

Now we generalize Tam's Theorem.

THEOREM 2. *Let Ω be the infinite strip $\left(-\frac{1}{2}, \frac{1}{2}\right) \times \mathbb{R}$ in $\mathbb{R}^2$. Let u, v be $C^2(\Omega)$ functions satisfying $Tu, Tv \in C^0(\bar{\Omega}) \cap C^1(\Omega)$. Suppose that for every ε with $0 < \varepsilon < \frac{1}{2}$, $|Du|$ is uniformly bounded in $\left[-\frac{1}{2}+\varepsilon, \frac{1}{2}-\varepsilon\right] \times \mathbb{R}$. And if*

$$(4)\quad \begin{cases} \operatorname{div} Tv = \operatorname{div} Tu & \text{in } \Omega \\ Tv\cdot\nu = Tu\cdot\nu & \text{on } \partial\Omega \\ \int_{\Gamma_{y_0}} Tv\cdot\nu_1\, d\sigma = \int_{\Gamma_{y_0}} Tu\cdot\nu_1\, d\sigma & \text{for some constant } y_0, \end{cases}$$

where ν is the unit outward normal of $\partial\Omega, \nu_1 = \langle 0, 1\rangle$, $\Gamma_{y_0} = \bar{\Omega} \cap \{(x,y)|y = y_0\}$, then we have $v \equiv u+$ constant.

REMARK. In (4), we assume neither div Tu = constant nor $Tu\cdot\nu$ = constant, hence we generalize Tam's Theorem.

As for more general domains Ω, the results are stated in Theorem 4.

Proof of Theorem 2. For any two numbers y_1, y_2 with $y_1 < y_2$, we set $\Omega_{y_1,y_2} = \Omega \cap \{(x,y) | -y_1 < y < y_2\}$. By divergence theorem, we have

$$\int_{\partial\Omega_{y_1,y_2}} (Tu \cdot \nu - Tv \cdot \nu)\, d\sigma = \int\int_{\Omega_{y_1,y_2}} \text{div } Tu - \text{div } Tv = 0.$$

Since $(Tu - Tv) \cdot \nu = 0$ on $\partial\Omega$, $\nu = \nu_1$ on Γ_{y_2} and $\nu = -\nu_1$ on Γ_{y_1}, we obtain

$$\int_{\Gamma_{y_2}} (Tu - Tv) \cdot \nu_1\, d\sigma - \int_{\Gamma_{y_1}} (Tu - Tv) \cdot \nu_1\, d\sigma = 0.$$

Hence

(5)

$$\int_{\Gamma_{y_1}} (Tu - Tv) \cdot \nu_1\, d\sigma = \text{ constant } = \int_{\Gamma_{y_0}} (Tu - Tv) \cdot \nu_1\, d\sigma = 0$$

for every $y_1 \in \mathbb{R}$.

Similarly, applying divergence theorem again, we have

$$\begin{aligned}(6) \qquad & \int_{\Gamma_{y_2}} \tan^{-1}(u-v)(Tu - Tv) \cdot \nu_1\, d\sigma \\ & - \int_{\Gamma_{y_1}} \tan^{-1}(u-v)(Tu - Tv) \cdot \nu_1\, d\sigma \\ & = \int\int_{\Omega_{y_1,y_2}} \frac{(Du - Dv)}{1 + (u-v)^2} \cdot (Tu - Tv) \\ & + \int\int_{\Omega_{y_1,y_2}} \tan^{-1}(u-v)(\text{div } Tu - \text{div } Tv) \\ & = \int\int_{\Omega_{y_1,y_2}} \frac{(Du - Dv)}{1 + (u-v)^2} \cdot (Tu - Tv) \geq 0, \quad (y_1 < y_2).\end{aligned}$$

For each $y \in \mathbb{R}$, let us write

$$(7) \qquad f(y) = \int_{\Gamma_y} \tan^{-1}(u-v)(Tu - Tv) \cdot \nu_1.$$

By (6), $f(y)$ is increasing in y, hence $\lim_{y\to\pm\infty} f(y)$ exist. We claim that $\lim_{y\to+\infty} f(y) = 0$. Otherwise, there exist two constants y_0' and C_1 such that $0 < C_1 < 1$ and

$$(8) \qquad |f(y)| \geq C_1 \text{ for every } y \geq y_0'.$$

Define $m(y) = \frac{1}{|\Gamma'_y|}\int_{\Gamma'_y} \tan^{-1}(u-v)\, d\sigma$ for every $y \geq y'_0$ where

$$\Gamma'_y = \left[-\frac{1}{2}+\frac{C_1}{8\pi}, \frac{1}{2}-\frac{C_1}{8\pi}\right] \times \{y\},$$

$|\Gamma'_y|$ is the length of Γ'_y. It follows that $|m(y)| \leq \frac{\pi}{2}$ for every $y \geq y'_0$. From (5), we see that

$$\text{(9)} \qquad \int_{\Gamma_y} m(y)(Tu - Tv)\cdot \nu_1 = m(y)\int_{\Gamma_y}(Tu-Tv)\cdot\nu_1 = 0 \text{ for every } y \geq y'_0.$$

By hypothesis, $|Du|$ is uniformly bounded in

$$\left[-\frac{1}{2}+\frac{C_1}{8\pi}, \frac{1}{2}-\frac{C_1}{8\pi}\right] \times \mathbb{R},$$

so there exists a positive constant C_2 (independent of y) such that

$$\text{(10)} \qquad |Du| \leq C_2 \text{ in } \left[-\frac{1}{2}+\frac{C_1}{8\pi}, \frac{1}{2}-\frac{C_1}{8\pi}\right] \times \mathbb{R}.$$

Hence for each $y \geq y'_0$,

$$\text{(11)} \qquad \left|\int_{\Gamma'_y}(\tan^{-1}(u-v)-m(y))(Tu-Tv)\cdot\nu_1\right| \geq \left|\left|\int_{\Gamma_y}(\tan^{-1}(u-v)-m(y))(Tu-Tv)\cdot\nu_1\right| - \left|\int_{\Gamma_y\setminus\Gamma'_y}(\tan^{-1}(u-v)-m(y))(Tu-Tv)\cdot\nu_1\right|\right|.$$

By direct computation, we have

$$\left|\int_{\Gamma_y\setminus\Gamma'_y}(\tan^{-1}(u-v)-m(y))(Tu-Tv)\cdot\nu_1\right| \leq 2\cdot\frac{C_1}{8\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\cdot 2 = \frac{C_1}{2}.$$

Combining this result with (7)-(9) and (11), we have

$$\text{(12)} \qquad \left|\int_{\Gamma'_y}\tan^{-1}(u-v)-m(y))(Tu-Tv)\cdot\nu_1\right| \geq \frac{C_1}{2} \text{ for every } y \geq y'_0.$$

Applying Poincaré inequality to the left hand side of (12), we obtain

$$
\begin{aligned}
\frac{C_1}{2} &\le 2\int_{\Gamma'_y} |\tan^{-1}(u-v) - m(y)| \\
&\le 2|\Gamma'_y| \int_{\Gamma'_y} \frac{|Du - Dv|}{1+(u-v)^2} \\
&\le 2|\Gamma'_y| \left(\int_{\Gamma'_y \cap \{|Du-Dv| < 8^{-1}|\Gamma'_y|^{-2}C_1\}} \frac{|Du - Dv|}{1+(u-v)^2} \right. \\
&\qquad \left. + \int_{\Gamma'_y \cap \{|Du-Dv| \ge 8^{-1}|\Gamma'_y|^{-2}C_1\}} \frac{|Du - Dv|}{1+(u-v)^2} \right) \\
&\le \frac{C_1}{4} + 2|\Gamma'_y| \int_{\Gamma'_y \cap \{|Du-Dv| \ge 8^{-1}|\Gamma'_y|^{-2}C_1\}} \frac{|Du - Dv|}{1+(u-v)^2}.
\end{aligned}
$$

Hence for every $y \ge y'_0$, we have

$$
\frac{C_1}{8|\Gamma'_y|} \le \int_{\Gamma'_y \cap \{|Du-Dv| \ge 8^{-1}|\Gamma'_y|^{-2}C_1\}} \frac{|Du - Dv|}{1+(u-v)^2}. \tag{13}
$$

From (10), we have

(14)

$$
\begin{aligned}
&\frac{|Du - Dv|^2}{\sqrt{1 + (|Du| + |Du - Dv|)^2}} \left(1 - \frac{|Du|}{\sqrt{1+|Du|^2}}\right) \\
&\ge \frac{|Du - Dv|^2}{\sqrt{1 + (C_2 + |Du - Dv|)^2}} \left(1 - \frac{C_2}{\sqrt{1+C_2^2}}\right) \\
&\ge |Du - Dv| \frac{8^{-1}|\Gamma'_y|^{-2}C_1}{\sqrt{1 + (C_2 + 8^{-1}|\Gamma'_y|^{-2}C_1)^2}} \left(1 - \frac{C_2}{\sqrt{1+C_2^2}}\right)
\end{aligned}
$$

in $\Gamma'_y \cap \{|Du - Dv| \ge 8^{-1}|\Gamma'_y|^{-2}C_1\}$. Combining (13) with the inequalities (3) and (14), we obtain the estimate

$$
\begin{aligned}
\frac{C_1}{8|\Gamma'_y|} \le 8|\Gamma'_y|^2 C_1^{-1} \sqrt{1 + (C_2 + 8^{-1}|\Gamma'_y|^{-2}C_1)^2} \left(1 - \frac{C_2}{\sqrt{1+C_2^2}}\right)^{-1} \\
\cdot \int_{\Gamma'_y \cap \{|Du-Dv| \ge 8^{-1}|\Gamma'_y|^{-2}C_1\}} \frac{(Du - Dv)\cdot(Tu - Tv)}{1+(u-v)^2}.
\end{aligned}
$$

Since $\Gamma'_y \subset \Gamma_y$, it is easy to see that

$$C_3 \le \int_{\Gamma_y} \frac{(Du - Dv)\cdot(Tu - Tv)}{1 + (u - v)^2} \tag{15}$$

for every $y \ge y'_0$ where

$$C_3 = \frac{C_1^2}{64|\Gamma_y|^3\sqrt{1 + (C_2 + 8^{-1}|\Gamma_y|^{-2}C_1)^2}}\left(1 - \frac{C_2}{\sqrt{1 + C_2^2}}\right).$$

By (6), (7), (15) and Fubini's Theorem, for every $y \ge y'_0$, we have

$$\begin{aligned}
f(y) - f(y_0)
&= \int_{\Gamma_y} \tan^{-1}(u - v)(Tu - Tv)\cdot \nu_1 \\
&\quad - \int_{\Gamma_{y'_0}} \tan^{-1}(u - v)(Tu - Tv)\cdot \nu_1 \\
&= \int\int_{\Omega_{y'_0, y}} \frac{(Du - Dv)}{1 + (u - v)^2}\cdot(Tu - Tv) \\
&= \int_{y'_0}^{y}\left(\int_{\Gamma_t} \frac{(Du - Dv)}{1 + (u - v)^2}\cdot(Tu - Tv)\, d\sigma\right) dt \\
&\ge \int_{y'_0}^{y} C_3|\Gamma_t|\, dt \\
&= C_3(y - y'_0).
\end{aligned} \tag{16}$$

Note that $|\int_{\Gamma_y} \tan^{-1}(u-v)(Tu-Tv)\cdot\nu_1| \le \pi\cdot|\Gamma_y| = \pi$. Let $y \to +\infty$ in (16), we get a contradiction. Thus we obtain $\lim_{y\to+\infty} f(y) = 0$. Similarly, we have $\lim_{y\to-\infty} f(y) = 0$. Using (6) again and letting $y_2 \to +\infty, y_1 \to -\infty$, we have $\int\int_\Omega \frac{(Du-Dv)}{1+(u-v)^2}\cdot(Tu - Tv) = 0$, hence $(Du-Dv)\cdot(Tu-Tv) \equiv 0$ in Ω, and $Du \equiv Dv$ in Ω. This completes the proof. □

Tam's Theorem is an immediate consequence of Theorem 2.

COROLLARY 3. *Every solution u of (2) is of the form $\phi_\beta +$ constant for some $|\beta| < 1$, where Ω is the infinite strip $\left(-\frac{1}{2}, \frac{1}{2}\right) \times \mathbb{R}$,*

$$\phi_\beta = -\frac{1}{\sqrt{1-\beta^2}}\sqrt{\left(\frac{1}{2\cos\alpha}\right)^2 - x^2} + \frac{\beta}{\sqrt{1-\beta^2}}y,$$

β satisfies $\int_{\Gamma_{y_0}} T\phi_\beta \cdot \nu_1 = \int_{\Gamma_{y_0}} Tu \cdot \nu_1$, for some constant y_0.

Tam's Theorem for more general domains Ω is stated in the following Theorem :

THEOREM 4. *Let $g_1, g_2 \in C^1(\mathbb{R})$ satisfy $g_2 < g_1$ in $\mathbb{R}$. Set $\Omega = \{(x,y)|g_2(y) < x < g_1(y)\}$. Let $u, v \in C^2(\Omega)$ satisfy $Tu, Tv \in C^0(\bar{\Omega}) \cap C^1(\Omega)$. Suppose that for every $\varepsilon > 0$, $|Du|$ is uniformly bounded in $\{(x,y)|g_2(y) + \varepsilon \le x$ and $x \le g_1(y) - \varepsilon\}$. And if*

$$(4')\quad \begin{cases} \text{div } Tv = \text{div } Tu & \text{in } \Omega, \\ Tv \cdot \nu = Tu \cdot \nu & \text{on } \partial\Omega, \\ \int_{\Gamma_{y_0}} Tv \cdot \nu_1 \, d\sigma = \int_{\Gamma_{y_0}} Tu \cdot \nu_1 \, d\sigma & \text{for some constant } y_0, \\ |\Gamma_y| = O(|y|^{\frac{1}{3}-\alpha}) & \text{as } y \to \pm\infty \\ & \text{for some constant } \alpha \\ & \text{with } 0 < \alpha < \frac{1}{3}, \end{cases}$$

where $\Gamma_y = [g_2(y), g_1(y)] \times \{y\}$, then we have $u \equiv v+$ constant in Ω.

Proof. The proof is similar to that of Theorem 2. Here we only sketch the proof.

For each $y \in \mathbb{R}$, let us write

$$(7)\qquad f(y) = \int_{\Gamma_y} \tan^{-1}(u-v)(Tu - Tv) \cdot \nu_1.$$

By (6), $f(y)$ is increasing in y, hence $\lim_{y\to\pm\infty} f(y)$ exist. We claim that $\lim_{y\to+\infty} f(y) = 0$. Otherwise, there exist two positive constants y_0' and C_1 such that

$$(8')\qquad |f(y)| \ge C_1 \text{ for every } y \ge y_0' > 0.$$

Hence $C_1 \le |\int_{\Gamma_y} \tan^{-1}(u-v)(Tu-Tv)\cdot\nu_1| \le \pi|\Gamma_y|$, and we have $|\Gamma_y| \ge \frac{C_1}{\pi}$ for every $y \ge y_0'$. Define $m(y) = \frac{1}{|\Gamma_y'|}\int_{\Gamma_y'} \tan^{-1}(u-v)\, d\sigma$ for every $y \ge y_0'$, where $\Gamma_y' = \left[g_2(y) + \frac{C_1}{8\pi}, g_1(y) - \frac{C_1}{8\pi}\right] \times \{y\}$.

By hypothesis, $|Du|$ is uniformly bounded in $\bigcup_{y\ge y_0'} \Gamma_y'$, so there exists a positive constant C_2 (independent of y) such that

$$(10')\qquad |Du| \le C_2 \text{ in } \bigcup_{y \ge y_0'} \Gamma_y'.$$

Hence for each $y \geq y_0'$, we have

$$(12') \quad \left|\int_{\Gamma_y'} (\tan^{-1}(u-v) - m(y))(Tu - Tv)\cdot \nu_1\right| \geq |f(y)| - \frac{C_1}{2} \geq \frac{|f(y)|}{2}.$$

Applying Poincaré inequality to the left hand side of (12′), we obtain

$$\begin{aligned}\frac{|f(y)|}{2} &\leq 2\int_{\Gamma_y'} |\tan^{-1}(u-v) - m(y)| \\ &\leq 2|\Gamma_y'| \int_{\Gamma_y'} \frac{|Du - Dv|}{1+(u-v)^2} \\ &\leq \frac{C_1}{4} + 2|\Gamma_y'| \int_{\Gamma_y' \cap \{|Du-Dv| \geq 8^{-1}|\Gamma_y'|^{-2}C_1\}} \frac{|Du-Dv|}{1+(u-v)^2}.\end{aligned}$$

Hence for every $y \geq y_0'$, we have

$$(13') \quad \frac{|f(y)|}{8|\Gamma_y'|} \leq \int_{\Gamma_y' \cap \{|Du-Dv| \geq 8^{-1}|\Gamma_y'|^{-2}C_1\}} \frac{|Du-Dv|}{1+(u-v)^2}.$$

Combining (13′) with the inequalities (3) and (14), we obtain

$$\begin{aligned}\frac{|f(y)|}{8|\Gamma_y'|} &\leq 8|\Gamma_y'|^2 C_1^{-1}\sqrt{1+(C_2+8^{-1}|\Gamma_y'|^{-2}C_1)^2}\left(1 - \frac{C_2}{\sqrt{1+C_2^2}}\right)^{-1} \\ &\quad \cdot \int_{\Gamma_y' \cap \{|Du-Dv| \geq 8^{-1}|\Gamma_y'|^{-2}C_1\}} \frac{(Du-Dv)\cdot(Tu-Tv)}{1+(u-v)^2}.\end{aligned}$$

Since $\Gamma_y' \subset \Gamma_y$,

$$|\Gamma_y'|^2\sqrt{1+(C_2+8^{-1}|\Gamma_y'|^{-2}C_1)^2} \leq |\Gamma_y|^2\sqrt{1+(C_2+8^{-1}|\Gamma_y|^{-2}C_1)^2},$$

it is easy to see that

$$\begin{aligned}|f(y)| &\leq 64|\Gamma_y|^3 C_1^{-1}\sqrt{1+(C_2+8^{-1}|\Gamma_y|^{-2}C_1)^2} \\ &\quad \cdot \left(1 - \frac{C_2}{\sqrt{1+C_2^2}}\right)^{-1} \int_{\Gamma_y} \frac{(Du-Dv)(Tu-Tv)}{1+(u-v)^2},\end{aligned}$$

hence we have

$$C_4|\Gamma_y|^{-3}|f(y)| \le \int_{\Gamma_y} \frac{(Du - Dv)\cdot(Tu - Tv)}{1+(u-v)^2} \tag{15'}$$

where

$$C_4 = 64^{-1}C_1\left(1+\left(C_2+8^{-1}\left(\frac{C_1}{\pi}\right)^{-2}C_1\right)^2\right)^{-\frac{1}{2}}\left(1-\frac{C_2}{\sqrt{1+C_2^2}}\right).$$

By Fubini's Theorem, we have

$$\int_{\Gamma_y} \frac{(Du - Dv)\cdot(Tu - Tv)}{1+(u-v)^2} = f'(y).$$

Hence

$$\frac{f'(y)}{|f(y)|} \ge C_4|\Gamma_y|^{-3} \ge C_5 y^{-1+3\alpha} \text{ for } y \ge y_0' > 0 \tag{17}$$

where C_5 is a positive constant. Integrate (17) from y_0' to y, $y > y_0'$, we obtain

$$|\log|f(y)| - \log|f(y_0')|| \ge \frac{C_5}{1-3\alpha}(y^{3\alpha} - y_0'^{3\alpha}). \tag{18}$$

Note that $|f(y)| \le \pi|\Gamma_y| = O\left(y^{\frac{1}{3}-\alpha}\right)$. Let $y \to +\infty$ in (18), we get a contradiction. The remainder of the proof is similar to that of Theorem 2. □

REMARK. Let Ω be a domain (bounded or unbounded) in $\mathbb{R}^n$ and let k be a positive constant, where $n \ge 2$ is an integer. Suppose that α is defined on $\partial\Omega$ with $0 \le \alpha \le 2\pi$. Then the boundary value problem

$$\begin{cases} \text{div } Tu = ku & \text{in } \Omega \\ Tu \cdot \nu = \cos\alpha & \text{on } \partial\Omega \end{cases} \tag{19}$$

determines the height u of a capillary surface in a uniform gravitational field.

It was proved by Finn and the author [**5**] that if (19) has a solution then it is unique. We point out that neither growth condition of u nor condition on the form of Ω at infinity is imposed.

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Received October 29, 1992. Partially supported by grant NSC 82-0208-M-001-078.

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