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CORRECTION TO "FREE BANACH-LIE ALGEBRAS, COUNIVERSAL BANACH-LIE GROUPS, AND MORE"

VLADIMIR PESTOV

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We correct a proof of the fact that the free Banach-Lie algebra on a normed space of dimension ≥ 2 is centreless, and observe that, as a corollary, every Banach-Lie algebra is a factor algebra of a Banach-Lie algebra faithfully representable in a Banach space.

1. All the major results of our paper [**2**] are based on the following statement, which appears as a part of Theorem 2.1.

Theorem A. *The free Banach-Lie algebra on a normed space E is either trivial (if $\dim E = 0$), or one-dimensional (if $\dim E = 1$), or centreless.*

Unfortunately, the proof of the above result presented in [**2**] is unsatisfactory, and it was Professor W.T. van Est who has kindly drawn the author's attention to this fact. Below we present a correct proof of Theorem A.

A 1973 investigation [**4**] of van Est and Świerczkowski was partly motivated by the question: is every Banach-Lie algebra a factor algebra of a Banach-Lie algebra faithfully representable in a Banach space? We can answer this in the positive.

Indeed, every Banach-Lie algebra $\mathfrak{g}$ is a factor Banach-Lie algebra of a free Banach-Lie algebra [**2**]. Since centreless Banach-Lie algebras are exactly those whose adjoint representation is faithful, the following direct corollary of Theorem A holds.

Theorem B. *Every Banach-Lie algebra is a factor algebra of a Banach-Lie algebra admitting a faithful representation in a Banach space.*

2. Denote by $\mathbb{K}$ the basic field (either $\mathbb{R}$ or $\mathbb{C}$), and let E be a normed space. For an $n > 0$, let $\mathcal{A}_n(E) = E^{\otimes_\pi n} \equiv E \otimes_\pi E \otimes_\pi \cdots \otimes_\pi E$ be an n-fold (non-completed) projective tensor product. ([**3**, III.6.3.]) Endow the space $\mathcal{B}^n(E)$ of all n-linear continuous functionals on E^n with a norm:

$$\| f \| \stackrel{def}{=} \sup\{|f(x_1, \ldots, x_n)| : \| x_i \| \leq 1,\ i = 1, 2, \ldots, n\}.$$

The spaces $\mathcal{A}_n(E)$ and $\mathcal{B}^n(E)$ admit a canonical pairing, which determines an isometric embedding of $\mathcal{A}_n(E)$ into the strong dual $\mathcal{B}^n(E)'$ ([**3**, exer. III.21, (a)]).

Let $\mathcal{A}_+(E)$ stand for the free associative (non-unital) algebra on E, $\mathcal{A}_+(E) = \oplus_{n=1}^{\infty}\mathcal{A}_n(E)$, endowed with an l_1-type norm, $\left\|\sum_{n=1}^k x_n\right\| = \sum_{n=1}^k \| x_n \|$. Denote by $\hat{\mathcal{A}}_+(E)$ the Banach associative algebra completion of $\mathcal{A}_+(E)$. It is easy to verify that $\hat{\mathcal{A}}_+(E)$ contains an isometric copy of E in such a way that an arbitrary linear contraction f from E to an associative algebra A endowed with a complete submultiplicative norm extends to a unique algebra homomorphism $\bar{f}\colon \hat{\mathcal{A}}_+(E) \to A$ of norm ≤ 1. We call $\hat{\mathcal{A}}_+(E)$ the *free Banach algebra* on E. Denote the Banach space completion of $\mathcal{A}_n(E)$ by $\hat{\mathcal{A}}_n(E)$; then $\hat{\mathcal{A}}_+(E)$ is the l_1-type sum of $\hat{\mathcal{A}}_n(E)$, $n = 1, 2, \ldots$.

It is clear that the free Banach-Lie algebra $\mathcal{FL}(E)$ is naturally isometric to the l_1-type direct sum of a family of complete normed spaces $\mathcal{FL}_n(E)$, $n \in \mathbb{N}$, where $\mathcal{FL}_n(E)$ is the completion of $\mathcal{L}_n(E)$. (Here $\mathcal{L}_1(E) = E$, and the linear subspaces $\mathcal{L}_n(E)$ of the free Lie algebra, $\mathcal{L}(E) = \oplus_{k=1}^{\infty}\mathcal{L}_k(E)$, on the vector space E [**1**], are defined in a usual recursive fashion.) The symbol $E_{1/2}$ will stand for the normed space $(E, (1/2)\|\cdot\|)$. We will denote by $\hat{\mathcal{A}}_+^{\lrcorner}(E)$ an algebra obtained from the free Banach algebra $\hat{\mathcal{A}}_+(E_{1/2})$ by doubling its norm. The doubled norm $\|\cdot\|^{\lrcorner}$ is Lie-submultiplicative, and the identity map Id_E extends to a contracting Lie algebra morphism $i\colon \mathcal{FL}(E) \to \hat{\mathcal{A}}_+^{\lrcorner}(E)$. The restriction of i to $\mathcal{L}(E)$ is well known to be mono [**2**]. Since the identity map $\mathcal{A}_+^{\lrcorner}(E) \to \hat{\mathcal{A}}_+(E_{1/2})$ has norm $1/2$, its composition with i is a contracting Lie algebra homomorphism $\mathcal{FL}(E) \to \hat{\mathcal{A}}_+(E_{1/2})$, which we denote by i as well.

Assertion 1. *Let $n = 1, 2, \ldots$. The restriction i_n of $i\colon \mathcal{FL}(E) \to \hat{\mathcal{A}}_+(E_{1/2})$ to $\mathcal{FL}_n(E)$ is an isomorphic embedding of normed spaces; namely, for each $x \in \mathcal{FL}_n(E)$ one has*

$$\| i(x) \|_{\hat{\mathcal{A}}_n(E_{1/2})} \leq \| x \| \leq \frac{2^n}{n}\| i(x) \|_{\hat{\mathcal{A}}_n(E_{1/2})}. \tag{1}$$

Proof. Define recursively the n-fold commutator, $[x_1, \ldots, x_n]$, by $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$. The map $(x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n]$ from $E \times E \times \cdots \times E \subset \mathcal{A}_n(E)$ to $\mathcal{L}_n(E)$ is n-linear and

$$\| [x_1, \ldots, x_n] \| \leq \| x_1 \|_E \cdots \| x_n \|_E = 2^n \frac{1}{2}\| x_1 \|_E \cdots \frac{1}{2}\| x_n \|_E.$$

Therefore, it extends to a unique bounded linear operator (of norm $\leq 2^n$), $\nu\colon \hat{\mathcal{A}}_n(E_{1/2}) \to \mathcal{FL}_n(E)$, having the property that if $x_1, x_2, \ldots, x_n \in E$, then

$\nu(x_1x_2\ldots x_n) = [x_1,\ldots,x_n]$. The restriction of ν to $\mathcal{A}_n(E)$ is the familiar *Specht-Wever map* [**1**].

While the left hand side inequality in (1) follows from a definition of i, suppose that $\| i(x) \| \leq 1$. One can assume without loss in generality that $x \in \mathcal{L}_n(E)$. Let an $\varepsilon > 0$ be arbitrary. For a finite collection of elements $x_{i,j} \in E$ one has ([**3**, III.6.3]) $i(x) = \sum_i x_{i,1}x_{i,2}\ldots x_{i,n}$ and

$$\| i(x) \| \geq \sum_i \frac{1}{2}\| x_{i,1} \|_E \frac{1}{2}\| x_{i,2} \|_E \cdots \frac{1}{2}\| x_{i,n} \|_E - \varepsilon$$

that is,

$$\sum_i \| x_{i,1} \|_E \| x_{i,2} \|_E \cdots \| x_{i,n} \|_E \leq 2^n(1+\varepsilon),$$

and therefore

$$\left\| \sum_i [x_{i,1}, x_{i,2}, \ldots, x_{i,n}] \right\| \leq 2^n(1+\varepsilon).$$

According to the Specht-Wever theorem [**1**], $\nu(x) \equiv \sum_i [x_{i,1}, x_{i,2}, \ldots, x_{i,n}] = nx$, whence $x = (1/n)\sum_i [x_{i,1}, x_{i,2}, \ldots, x_{i,n}]$ and $\| x \| \leq (2^n/n)(1+\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, one has $\| x \| \leq 2^n/n$, as desired. □

3. Proof of Theorem A. We can assume that $\dim E \geq 2$. Let an $x \in \mathcal{FL}(E)$, $x \neq 0$ be arbitrary, $x = \sum_{k=1}^{\infty} x_k$, where $x_k \in \mathcal{FL}_n(E)$. For at least one $n = 1, 2, \ldots$, one has $x_n \neq 0$. It remains to find a $z \in E$ such that $[z, x_k] \neq 0$, for clearly then $[z, x] \neq 0$ as well. If E is of finite dimension, then such is $\mathcal{L}_n(E)$; if an element $x \in \mathcal{FL}_n(E) \equiv \mathcal{L}_n(E)$ commutes with every element of E, it must belong to the centre of $\mathcal{L}(E)$, which is trivial if $\dim E > 1$. In infinite dimensions, however, this argument fails (which was essentially author's blunder in [**2**]).

Denote by ad the adjoint representation of $\mathcal{FL}(E)$ in the underlying Banach space, $\mathcal{FL}(E)_+$.

Assertion 2. *Assume that* $\dim E = \infty$. *Let* $n \in \mathbb{N}$ *and let* $x \in \mathcal{FL}_n(E)$, $\| x \| = 1$. *Then* $\| \operatorname{ad} x \| \geq n2^{-n}$.

Proof. Since $\| \operatorname{ad} \| \leq 1$, it is enough to check the desired property for $x \in \mathcal{L}_n(E)$: indeed, the unit sphere of $\mathcal{L}_n(E)$ is dense in the unit sphere of $\mathcal{FL}_n(E)$, and if $x_n \to x$ as $n \to \infty$, then $\operatorname{ad} x_n \to \operatorname{ad} x$ in End $(\mathcal{FL}(E)_+)$ and $\| \operatorname{ad} x_n \| \to \| \operatorname{ad} x \|$.

The norm of $i(x)$ in $\hat{\mathcal{A}}_+(E_{1/2})$ is $\geq n2^{-n}$, according to Assertion 1. Assume that $i(x) = \sum_i x_{1,i} \otimes x_{2,i} \otimes \cdots \otimes x_{n,i}$, where $x_{j,i} \in E$. Let an $\varepsilon > 0$ be arbitrary.

Choose an $f \in \mathcal{B}^n(E_{1/2})$ with $\| f \| \leq 1$ and $|f(i(x))| \geq n2^{-n} - \varepsilon$. Due to infinite-dimensionality of E, there exists a linear functional $g\colon E \to \mathbb{K}$ of norm 1 with $g(x_{n,i}) = 0$ for all (finitely many) values of i. Let $y \in E$ be such that $\| y \| = 1$ and $g(y) = 1$. (The kernel of g, being one-dimensional, admits a projection from E of norm 1.) The mapping $f \otimes g$ of the form $a \otimes b \mapsto f(a) \cdot g(b)$, $a \in E^n_{1/2}$, $b \in E_{1/2}$, is an $(n+1)$-linear functional of norm ≤ 1 on $E^{n+1}_{1/2}$. Since

$$(f \otimes g)(y \otimes i(x)) = \sum_i f(y \otimes x_{1,i} \otimes \cdots \otimes x_{n-1,i}) \cdot g(x_{n,i}) = 0,$$

one has

$$\begin{aligned} |(f \otimes g)(i(x) \otimes y - y \otimes i(x))| &= |(f \otimes g)(i(x) \otimes y)| \\ &= |f(i(x))| \cdot 1 \geq \frac{n}{2^n} - \varepsilon, \end{aligned}$$

and in view of arbitrariness of $\varepsilon > 0$, the norm of the element $i([y,x]) = i(x)y - yi(x)$ in $\hat{\mathcal{A}}_+(E_{1/2})$ is $\geq n2^{-n}$. In view of Assertion 1, the norm of $[y,x]$ in $\mathcal{FL}_n(E)$ is $\geq n2^{-n}$ as well. $\square$

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Victoria University of Wellington
P.O. Box 600, Wellington, New Zealand
E-mail address: vladimir.pestov@vuw.ac.nz