

PERMUTATION MODEL FOR SEMI-CIRCULAR SYSTEMS AND QUANTUM RANDOM WALKS

PHILIPPE BIANE

We give an approximation model for semi-circular families of non-commutative random variables, which is based on the permutation group. This model is compared to the random matrix model of Voiculescu, and a common refinement of the two approximations by explicit matrices with 0 – 1 entries is exhibited.

Introduction.

The recent theory of semi-circular systems, introduced by D.Voiculescu [16] has proved to be a very efficient tool in the theory of free group factors (see e.g. [13] and the references therein), mainly due to the fact that semi-circular systems can be approximated by gaussian random matrices [17]. It is thus an interesting problem to try to find other simple approximations of semi-circular systems. In this article we introduce such an approximation, which is based on permutation groups.

Let us describe briefly the key fact which is at the basis of this approximation. Let \mathfrak{S}_{n+1} be the permutation group of $\{0, 1, \dots, n\}$, and for $1 \leq k \leq n$, let $\tau_k \in \mathfrak{S}_{n+1}$ be the transposition which exchanges 0 and k , and denote again by τ_k the corresponding left translation operator on $L^2(\mathfrak{S}_{n+1})$, then the trace of the spectral measure of the operator $\frac{1}{\sqrt{n}}(\tau_1 + \dots + \tau_n)$ converges, as n goes to infinity, towards the semi-circle distribution. This fact is a consequence of Theorem 1 below.

In the second part of the paper, we show how this permutation model is related to the gaussian random matrix model of Voiculescu. We first describe a convenient way of obtaining a gaussian hermitian matrix, which is to let a brownian motion evolve on the set of hermitian matrices. This brownian motion was first studied by Dyson, after Wigner's work on the semi-circle law. We show that the evolution of the spectrum of a brownian hermitian matrix is very much linked to the theory of tensor product representations of the unitary groups. This enables us to approximate in law a hermitian random matrix by explicit finite dimensionnal matrices. This gives us a family of finite dimensionnal matrices depending on two integer parameters n and d . The random matrix model of Voiculescu is obtained when we let

n go to infinity first and then d go to infinity. The permutation model is obtained when d goes first to infinity, and then n does.

1. A permutation model for semi-circular systems.

1.1. Preliminaries. We first recall some definitions from Voiculescu [16].

1.1.1. Definition. A \star -non-commutative probability space (A, φ) is composed of A a unital \star -algebra and φ a state on A .

Elements of A are called (non-commutative) random variables.

In what follows, traces on matrix algebras will be normalized to be equal to 1 at the identity.

1.1.2. Let I be an index set, and $\mathbb{C}(y_\iota, y_\iota^*)_{\iota \in I}$ be the free \star -algebra with generators $(y_\iota)_{\iota \in I}$. For every family $(X_\iota)_{\iota \in I}$ of elements of A there is a unique \star -homomorphism $h : \mathbb{C}(y_\iota, y_\iota^*)_{\iota \in I} \rightarrow A$ which sends y_ι and y_ι^* to X_ι and X_ι^* respectively.

Definition. The law of the family $(X_\iota)_{\iota \in I}$ is the map $\varphi \circ h$ on $\mathbb{C}(y_\iota, y_\iota^*)_{\iota \in I}$.

Convergence of a sequence of laws is defined as simple convergence on $\mathbb{C}(y_\iota, y_\iota^*)_{\iota \in I}$.

1.1.3. Definition. A family of \star -subalgebras $(A_\iota)_{\iota \in I}$ of A is called free if, for all $n \geq 2$, $\varphi(a_1 a_2 \dots a_n) = 0$ whenever $a_j \in A_{\iota(j)}$ with $\iota(j) \neq \iota(j+1)$ for $1 \leq j \leq n-1$ and $\varphi(a_j) = 0$ for $1 \leq j \leq n$. A family of random variables $(X_\iota)_{\iota \in I}$ is free if the family of \star -algebras $(A_\iota)_{\iota \in I}$ is free, where A_ι is the \star -algebra generated by X_ι for $\iota \in I$.

1.1.4. Definition. A family of random variables $(X_\iota)_{\iota \in I}$ is a semi-circular system if it is a free family, the $(X_\iota)_{\iota \in I}$ are self adjoint ($X_\iota = X_\iota^*$) and $\varphi(X_\iota^{2n}) = \frac{1}{n+1} \mathbf{C}_{2n}^n$, $\varphi(X_\iota^{2n+1}) = 0$ for $n \in \mathbb{N}$.

1.2. Permutations and semi-circular systems.

1.2.1. Let E be a countable set, and let $\mathfrak{S}(E)$ be the group of permutations of E which fix all elements of E except a finite number. The group $\mathfrak{S}(E)$ is the inductive limit of permutation groups of finite subsets of E .

Let $L(E)$ be the von Neumann algebra generated by the left regular representation of $\mathfrak{S}(E)$. It is a hyperfinite II_1 factor, and its canonical trace is characterized by $tr(e) = 1$, $tr(\sigma) = 0$ for $e \neq \sigma \in \mathfrak{S}(E)$ (e is the unit element of $\mathfrak{S}(E)$).

We consider elements of $\mathfrak{S}(E)$ as random variables in the non-commutative probability space $(L(E), tr)$.

1.2.2. Let $E = \{\delta\} \cup (\bigcup_{\iota \in I} E_\iota)$ be a partition of E such that $\text{card} E_\iota = \infty$ for all $\iota \in I$. We put $E' = E \setminus \{\delta\}$.

For $\iota \in I$, let $e_\iota^1, e_\iota^2, \dots, e_\iota^n, \dots$ be an enumeration of the elements of E_ι , and let τ_ι^n , for $n \in \mathbb{N}$ be the element in $L(E)$ which corresponds to the transposition (δ, e_ι^n) .

Let $a_\iota^n = \frac{1}{\sqrt{n}} \sum_{k=1}^n \tau_\iota^k \in L(E)$, and let $(a_\iota)_{\iota \in I}$ be a semi-circular system in some non-commutative probability space (A, φ) .

1.2.3.

Theorem 1. *The law of the family of random variables $(a_\iota^n)_{\iota \in I}$ in $(L(E), tr)$ converges towards the law of $(a_\iota)_{\iota \in I}$ as n goes to infinity.*

1.2.4. Before we proceed to the proof of the theorem in Section 1.3 we shall make some comments on this result.

For each $\varepsilon \in E'$ let G_ε be a copy of the group $\mathbb{Z}/2\mathbb{Z}$, let ν_ε be the non trivial element of G_ε and let $G = \star_{\varepsilon \in E'} G_\varepsilon$ be the free product group. There is a surjective morphism $G \rightarrow \mathfrak{S}(E)$ which sends ν_ε to the transposition (δ, ε) . The elements ν_ε in the factor generated by the left regular representation of G form a free family and if we replace in the definition of a_ι^n the transpositions (δ, ε) by ν_ε the Theorem 1.2.3 remains true, as follows from Voiculescu's central limit theorem ([15]).

More generally, let H be a group with generators $(\theta_\varepsilon)_{\varepsilon \in E'}$ satisfying $\theta_\varepsilon^2 = e$ and such that there are surjective morphisms $\xi : G \rightarrow H$ and $\zeta : H \rightarrow \mathfrak{S}(E)$ with $\xi(\nu_\varepsilon) = \theta_\varepsilon$ and $\zeta(\theta_\varepsilon) = \tau_\varepsilon$, then the theorem remains true if we replace τ_ε in the definition of a_ι^n by θ_ε considered as a left translation operator on $L^2(H)$. This means that, in some sense the relations between the transpositions (δ, ε) for $\varepsilon \in E'$ "do not commute too much". This sentence may be clarified by the following remark. The transpositions $(0, k)$ for $1 \leq k \leq n$ form a system of generators of the permutation group of $\{0, 1, \dots, n\}$. Observe that, if we take as generators of this permutation group the transpositions $t_k = (k, k + 1)$, for $0 \leq k \leq n - 1$, then the limit as n goes to infinity of the trace of the spectral measure of the operator $\frac{1}{\sqrt{n}}(t_1 + \dots + t_n)$ is the gaussian law.

1.3. Proof of the Theorem 1.

1.3.1. Let us first recall how to evaluate moments of semi-circular systems.

Definition. A partition $p = (V_1, \dots, V_l)$ of the set $\{1, \dots, m\}$ is called *crossing* if there exists $1 \leq k_1 \neq k_2 \leq l$ and $u_1, v_1 \in V_{k_1}, u_2, v_2 \in V_{k_2}$ such that $u_1 < u_2 < v_1 < v_2$, and *non-crossing* in the opposite case.

This definition occurs in Kreweras [8], see also Speicher [14] where non-crossing partitions are called *admissible*.

For $m \in \mathbb{N}$, let $\mathcal{P}_{nc}(m)$ be the set of non-crossing partitions of $\{1, \dots, m\}$ which consist of $\frac{m}{2}$ pairs, and for any $p \in \mathcal{P}_{nc}(m)$, choose a sequence k_1, \dots, k_m in $\{1, \dots, m\}$ such that $p = (\{k_1, k_2\}, \dots, \{k_{m-1}, k_m\})$.

Let $(a_\iota)_{\iota \in I}$ be a semi-circular system in (A, φ) , and $(\iota(1), \dots, \iota(m))$ be a sequence of elements of I .

For $p \in \mathcal{P}_{nc}$, put

$$V_p(a_{\iota(1)}, \dots, a_{\iota(m)}) = \varphi(a_{\iota(k_1)} a_{\iota(k_2)}) \dots \varphi(a_{\iota(k_{m-1})} a_{\iota(k_m)})$$

(the expression for $V_p(a_{\iota(1)}, \dots, a_{\iota(m)})$ depends only on p not on the choice of k_1, \dots, k_m).

According to Speicher [14], for each sequence $\rho = (\iota(1), \dots, \iota(m)) \in I^m$ one has $\varphi(a_{\iota(1)} \dots a_{\iota(m)}) = 0$ if m is odd and

$$\varphi(a_{\iota(1)} \dots a_{\iota(m)}) = \sum_{p \in \mathcal{P}_{nc}(m)} V_p(a_{\iota(1)}, \dots, a_{\iota(m)})$$

if m is even. Furthermore, for every $\iota, \kappa \in I$ one has $\varphi(a_\iota a_\kappa) = 1$ if $\iota = \kappa$ and $\varphi(a_\iota a_\kappa) = 0$ if $\iota \neq \kappa$.

This implies that

$$\varphi(a_{\iota(1)} \dots a_{\iota(m)}) = \text{card}\{p \in \mathcal{P}_{nc}(m) \mid \iota(k_1) = \iota(k_2), \dots, \iota(k_{m-1}) = \iota(k_m)\}$$

for m even.

1.3.2. We shall prove the Theorem 1 by showing that for each sequence $\iota(1), \dots, \iota(m) \in I$ one has

$$\lim_{n \rightarrow \infty} \text{tr} \left(a_{\iota(1)}^n \dots a_{\iota(m)}^n \right) = \varphi(a_{\iota(1)} \dots a_{\iota(m)})$$

In order to evaluate the quantity $\text{tr} \left(a_{\iota(1)}^n \dots a_{\iota(m)}^n \right)$ we will first prove a sequence of lemma.

Let $\varepsilon_1, \dots, \varepsilon_m$ be a sequence of elements of E' , and τ_{ε_k} be the transposition (δ, ε_k) for $1 \leq k \leq m$.

The sequence $(\varepsilon_k)_{1 \leq k \leq m}$ determines a partition of $\{1, \dots, m\}$ through the equivalence relation $k \sim l$ if and only if $\varepsilon_k = \varepsilon_l$.

Lemma 1. *If the permutation $\tau_{\varepsilon_1} \tau_{\varepsilon_2} \dots \tau_{\varepsilon_m}$ is the identity then the associated partition of $\{1, \dots, m\}$ contains only sets of cardinal greater than 2.*

Proof. Suppose that for some $\eta \in E'$, there exists exactly one k such that $\varepsilon_k = \eta$, then the image of η by the permutation $\tau_{\varepsilon_1} \tau_{\varepsilon_2} \dots \tau_{\varepsilon_m}$ is not equal to η (because $\tau_{\varepsilon_1} \dots \tau_{\varepsilon_{k-1}}$ and $\tau_{\varepsilon_{k+1}} \dots \tau_{\varepsilon_m}$ leave η fixed), so that $\tau_{\varepsilon_1} \dots \tau_{\varepsilon_m}$ cannot be the identical permutation. □

Lemma 2. *If the partition associated to $\varepsilon_1, \dots, \varepsilon_m$ is composed of sets of cardinal 2 then $\tau_{\varepsilon_1} \dots \tau_{\varepsilon_m} = e$ if and only if the partition is non-crossing.*

Proof. First suppose that the partition of $\{1, \dots, m\}$ is non-crossing. Let k be such that $\varepsilon_k = \varepsilon_{k+1}$ (such a k exists because of the properties of the

partition), then $\tau_{\varepsilon_1} \tau_{\varepsilon_2} \dots \tau_{\varepsilon_m} = \tau_{\varepsilon_1} \tau_{\varepsilon_2} \dots \tau_{\varepsilon_{k-1}} \tau_{\varepsilon_{k+2}} \dots \tau_{\varepsilon_m}$, and the partition associated to the sequence $\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_{k+1}, \dots, \varepsilon_m$ is again noncrossing and composed of sets of cardinal 2. From this observation, it is easy to prove by induction on even m that $\tau_{\varepsilon_1} \tau_{\varepsilon_2} \dots \tau_{\varepsilon_m} = e$ (since this is obviously true for $m = 2$).

For the ‘only if’ part of the lemma, let l be the greatest number such that $\varepsilon_{m-l}, \dots, \varepsilon_m$ are all different elements of E' (one has $l \leq \frac{m}{2}$ because the partition is composed of pairs), and suppose $\varepsilon_{m-l-1} = \varepsilon_r$ with $r > m - l$, then the permutation $\tau_{\varepsilon_{m-l-1}} \tau_{\varepsilon_{m-l}} \dots \tau_{\varepsilon_m}$ does not leave ε_r fixed (because it sends ε_r to ε_{r-1}). Since the permutation $\tau_{\varepsilon_1} \tau_{\varepsilon_2} \dots \tau_{\varepsilon_{m-l-2}}$ does not contain any factor τ_{ε_r} it leaves ε_r invariant, and so the product $\tau_{\varepsilon_1} \tau_{\varepsilon_2} \dots \tau_{\varepsilon_m}$ cannot be the identity. We deduce from this that necessarily if $\tau_{\varepsilon_1} \tau_{\varepsilon_2} \dots \tau_{\varepsilon_m} = e$ then $\varepsilon_{m-l-1} = \varepsilon_{m-l}$, so that $\tau_{\varepsilon_1} \tau_{\varepsilon_2} \dots \tau_{\varepsilon_m} = \tau_{\varepsilon_1} \tau_{\varepsilon_2} \dots \tau_{\varepsilon_{m-l-2}} \tau_{\varepsilon_{m-l+1}} \dots \tau_{\varepsilon_m} = e$. Now, if the partition associated to the sequence $\varepsilon_1, \dots, \varepsilon_{m-l-2}, \varepsilon_{m-l+1}, \dots, \varepsilon_m$ is non crossing, then so is the partition associated to $\varepsilon_1, \dots, \varepsilon_{m-l-2}, \varepsilon_{m-l-1}, \varepsilon_{m-l}, \varepsilon_{m-l+1}, \dots, \varepsilon_m$ because $\varepsilon_{m-l-1} = \varepsilon_{m-l}$. The proof follows now by induction on even m . \square

1.3.3. We are now ready to finish the proof of the theorem. Let $\rho = (\iota(1), \dots, \iota(m))$ be a sequence in I . We compute the expectation $tr(a_{\iota(1)}^n \dots a_{\iota(m)}^n)$ by expanding the product. We find that

$$tr(a_{\iota(1)}^n \dots a_{\iota(m)}^n) = n^{-\frac{m}{2}} \sum_{\varepsilon_1 \dots \varepsilon_m} tr(\tau_{\varepsilon_1} \dots \tau_{\varepsilon_m}) \tag{1}$$

where the sum is taken over all sequences $\varepsilon_1, \dots, \varepsilon_m$ such that

$$\varepsilon_k \in \{e_{\iota(k)}^1, \dots, e_{\iota(k)}^n\} \text{ for } 1 \leq k \leq m.$$

We call such a sequence a ρ -sequence.

Recall that the quantity $tr(\tau_{\varepsilon_1} \dots \tau_{\varepsilon_m})$ is equal to 1 if $\tau_{\varepsilon_1} \dots \tau_{\varepsilon_m} = e$ and 0 in the other cases.

To each sequence $\varepsilon_1, \dots, \varepsilon_m$ in E' let us associate a partition of $\{1, \dots, m\}$ as in 1.3.2. Let p be a partition of $\{1, \dots, m\}$ such that in each set there are at least 2 elements and in at least one of the sets there are strictly more than 2 elements, this partition is composed of at most $\frac{m-1}{2}$ sets, so that the number of ρ -sequences $\varepsilon_1, \dots, \varepsilon_m$ whose associated partition is p , is less than $n^{\frac{m-1}{2}}$. The contribution of such sequences to the expression (1) is therefore negligible as n goes to infinity.

On the other hand, by Lemma 1 and 2, the ρ -sequences for which the associated partition contains at least one singleton, or which are crossing,

give a contribution equal to 0, so that it remains to count the contribution of the ρ -sequences whose partitions are non-crossing and composed of sets of cardinal 2.

Let $p = (\{k_1, k_2\}, \dots, \{k_{m-1}, k_m\})$ be a non-crossing partition of $\{1 \dots, m\}$ composed of pairs. If there exists an l such that $\iota(k_{2l-1}) \neq \iota(k_{2l})$ then there is no ρ -sequence $\varepsilon_1, \dots, \varepsilon_m$ such that $\varepsilon_k \in E_{\iota(k)}$ for $1 \leq k \leq m$ and whose associated partition is p . If $\iota(k_{2l-1}) = \iota(k_{2l})$ for $1 \leq l \leq \frac{m}{2}$ then for $n \geq \frac{m}{2}$ the number of sequences $\varepsilon_1, \dots, \varepsilon_m$ with associated partition p , such that $\varepsilon_k \in \{e_{\iota(k)}^1, \dots, e_{\iota(k)}^n\}$ for $1 \leq k \leq m$ is comprised between the two numbers $(n - \frac{m}{2})^{\frac{m}{2}}$ and $n^{\frac{m}{2}}$.

From this we deduce that

$$tr \left(a_{\iota(1)}^1 \dots a_{\iota(m)}^m \right) \rightarrow \text{card} \left\{ p \in \mathcal{P}_{nc}(m) \mid \iota(k_{2l-1}) = \iota(k_{2l}) \text{ for } 1 \leq l \leq \frac{m}{2} \right\}$$

as n goes to infinity. In view of 1.3.1, this proves the Theorem 1.

2. Relation with the gaussian matrix model.

We will now show how the permutation model introduced in the preceding paragraph is related to the gaussian matrix model of Voiculescu. For this we will firstly describe a dynamic way of obtaining a random gaussian hermitian matrix, by letting a brownian motion evolve on the space of hermitian matrices.

2.1. Brownian motion on hermitian matrices.

2.1.1. The brownian motion on hermitian $d \times d$ matrices was studied by Dyson [5] after the work of Wigner on gaussian matrices and the semi-circle law ([18]). This is a stochastic process $X(t) = (X^{jk}(t))_{1 \leq j, k \leq d}$ for $t \in \mathbb{R}_+$ such that for $1 \leq j < k \leq d$, $(X^{jk}(t))_{t \in \mathbb{R}_+}$ are independent complex brownian motions, $X^{jk}(t) = \bar{X}(t)^{kj}$, the processes $X^{jj}(t)$ are independent real brownian motions for $1 \leq j \leq d$, and the variances are given by $E[|X^{jk}(t)|^2] = t$ for $1 \leq j, k \leq d$.

Dyson investigated the stochastic process $(\lambda_1(t), \dots, \lambda_d(t))_{t \in \mathbb{R}_+}$ whose components are the eigenvalues in increasing order of the matrix $X(t)$, and showed that it behaves like “a gas of brownian particles subject to repulsive Coulomb forces”. This means, as shown by Mac-Kean [9], that this process satisfies a stochastic differential equation of the form

$$d\lambda_j(t) = dB_j(t) + \sum_{k \neq j} \frac{dt}{\lambda_j(t) - \lambda_k(t)}$$

the processes B_j being independent brownian motions, or equivalently, that it is a diffusion process with generator $\frac{1}{2}\Delta + \text{gradLog } h \nabla$, Δ being the usual

Laplacian, ∇ the gradient operator and h the function $h(\lambda) = \prod_{1 \leq k < j \leq d} (\lambda_j - \lambda_k)$ (cf [6]). If we observe that the function h is positive, harmonic on the domain $\Omega = \{\lambda \in \mathbb{R}^d \mid \lambda_j > \lambda_k \text{ for } j > k\}$ and vanishes on its boundary, (see Lemma 3 below), then this generator can be put in the form $\frac{1}{2h} \Delta(h \cdot)$ where Δ is the Laplacian on the domain Ω with Dirichlet boundary conditions. Probabilistically, this means that the process $(\lambda(t))_{t \in \mathbb{R}_+}$ is obtained from brownian motion on \mathbb{R}^d by a killing at the boundary of Ω followed by taking an h -process in the sense of Doob (cf [4]) with respect to the positive harmonic function h . Furthermore, one can show that this harmonic function is minimal, and corresponds to the unique Martin boundary point at infinity of the domain Ω .

Lemma 3. *The function h is harmonic.*

Proof. This proof comes from [1] Proposition 3. For every permutation σ of $\{1 \dots, d\}$ let $\sigma(\lambda_1, \dots, \lambda_d) = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(d)})$, then $h \circ \sigma = \text{sgn}(\sigma)h$ and every polynomial which verifies this identity is a multiple of h . Since Δ commutes with every σ , one has again $\Delta h \circ \sigma = \text{sgn}(\sigma)\Delta h$, and Δh is a polynomial of lower degree than h , so that $\Delta h = 0$. □

As a corollary of the above remark, we can give an explicit expression of the semi-group of the process $(\lambda_t)_{t \in \mathbb{R}_+}$. Let $Q_t(x, y)$ be the density of this semi-group with respect to Lebesgue measure in the cone

$$\{\lambda \in \mathbb{R}^d \mid \lambda_j > \lambda_k \text{ for } j > k\},$$

and $P_t(x, y)$ be that of the brownian semi-group in \mathbb{R}^d .

Proposition 1.

$$Q_t(x, y) = \frac{h(y)}{h(x)} \sum_{\sigma \in W_d} \text{sgn}(\sigma) P_t(x, \sigma(y)).$$

Proof. The semi-group of brownian motion killed at the boundary of the cone is given by the formula $\sum_{\sigma \in W_d} \text{sgn}(\sigma) P_t(x, \sigma(y))$, (here W_d is the Weyl group, that is the permutation group on d letters acting on the coordinates of \mathbb{R}^d) as follows from the reflection principle, and the result is a consequence of Doob's definition of h -processes. □

2.1.2. For our purposes, it will be more convenient to deal with brownian motion on traceless hermitian matrices. This is the orthogonal projection of the process $(X(t))_{t \in \mathbb{R}_+}$ on the set of $d \times d$ matrices with zero trace.

The process of eigenvalues of such a brownian motion is the orthogonal projection of the process $(\lambda_t)_{t \in \mathbb{R}_+}$ on the set of elements of \mathbb{R}^d which verify $\sum_j \lambda_j = 0$. It is again obtained from brownian motion on the cone domain $\{\lambda \in \mathbb{R}^d \mid \lambda_j > \lambda_k \text{ for } j > k \text{ and } \sum_j \lambda_j = 0\}$ by a killing at the boundary of the cone, followed by an h -transform with respect to the harmonic function h restricted to this cone.

2.2. Quantum random walks.

In this section we shall describe a discrete non-commutative approximation of brownian motion on hermitian matrices, with the help of quantum random walks. More details about these processes can be found in [2, 10].

2.2.1. Let G be a compact group, and $vN(G)$ be the von Neumann algebra generated by its left regular representation. For $g \in G$ call λ_g the corresponding element of $vN(G)$.

Let ϕ be a continuous positive type function on G , with $\phi(e) = 1$ where e is the identity element in G . This function defines a state μ on $vN(G)$ characterised by $\mu(\lambda_g) = \phi(g)$, and a completely positive map $T : vN(G) \rightarrow vN(G)$ by the formula $Q(\lambda_g) = \phi(g)\lambda_g$.

Let \mathcal{W} be the von Neumann algebra generated by the GNS representation of the infinite tensor product $vN(G)^{\otimes \infty}$ with respect to the infinite product state $\omega = \mu^{\otimes \infty}$, and $j_n : vN(G) \rightarrow \mathcal{W}$ be the morphisms defined by $j_0(\lambda_g) = \text{Id}_{\mathcal{W}}$ and $j_n(\lambda_g) = \lambda_g^{\otimes n} \otimes \text{Id} \otimes \dots \otimes \text{Id} \otimes \dots$ for $n \geq 1$. The family of maps $(j_n)_{n \in \mathbb{N}}$ is called a quantum random walk. They form a dilation of the semi-group of completely positive maps $(T^n)_{n \in \mathbb{N}}$, which means that for each $n \in \mathbb{N}$ there is a conditionnal expectations E_n from \mathcal{W} onto the von Neumann subalgebra generated by the image of j_0, j_1, \dots, j_n , and these conditionnal expectations satisfy the ‘‘Markov property’’ $E_m \circ j_n = j_m \circ T^{n-m}$ for $m \leq n$.

The semi-group $(T^n)_{n \in \mathbb{N}}$ is a non commutative analogue of a Markov semi-group. It is a convolution semi-group for the Hopf-von Neumann algebra structure of $vN(G)$ and the dilation $(j_n)_{n \in \mathbb{N}}$ is the analogue of a Markov process associated to this semi-group, which is a ‘‘non-commutative process with independent increments’’ (see Schürmann [12]).

2.2.2. Let $Z(vN(G))$ be the center of $vN(G)$. By Peter-Weyl theorem, this algebra is isomorphic to $L^\infty(\Gamma)$ where Γ is the set of equivalence classes of irreducible representations of the group G . The restrictions of the maps j_n to $Z(vN(G))$ take value in a commutative von Neumann subalgebra \mathcal{X} of \mathcal{W} .

If the function ϕ is central, then the maps T^n send $Z(vN(G))$ into itself. Since they are completely positive maps, using the Peter-Weyl isomorphism $\chi : Z(vN(G)) \rightarrow L^\infty(\Gamma)$ we see that they give rise to a Markovian semi-group

$(Q^n)_{n \in \mathbb{N}}$ on the set Γ , by the formula $Q^n = \chi \circ T^n \circ \chi^{-1}$. For $n \in \mathbb{N}$, let \mathcal{X}_n be the von Neumann algebra generated by the images $j_k(Z(vN(G)))$ for $1 \leq k \leq n$, then the conditionnal expectation E_n maps \mathcal{X} onto \mathcal{X}_n . Since the algebra \mathcal{X} is abelian there is a probability space (Ω, \mathcal{F}, P) with a filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ and random variables X_n with values in Γ , adapted to the filtration \mathcal{F}_n which form a Markov process on Γ with semi-group $(Q^n)_{n \in \mathbb{N}}$, there is an isomorphism $U : L^\infty(\Omega, \mathcal{F}, P) \rightarrow \mathcal{X}$ such that $U(L^\infty(\Omega, \mathcal{F}_n, P)) = \mathcal{X}_n$, the restriction of E_n to \mathcal{X} is equal to $U \circ E[\cdot | \mathcal{F}_n] \circ U^{-1}$ ($E[\cdot | \mathcal{F}_n]$ is the conditionnal expectation on the σ -field \mathcal{F}_n), the maps j_n satisfy $U^{-1} \circ j_n \circ U = \mathfrak{X}_n$ where $\mathfrak{X}_n : L^\infty(\Gamma) \rightarrow L^\infty(\Omega, \mathcal{F}, P)$ is the morphism determined by X_n .

We can summarize this by saying that, in the case of a central function ϕ , the construction in 2.2.1 of a quantum random walk gives a (commutative) Markov process on the set of classes of irreducible representations by restriction to the center.

2.2.3. Let d be a positive integer, we specialize the preceding discussion to the case where $G = SU(d)$ and ϕ is the normalised character of the basic representation of $SU(d)$ (i.e. $\phi(g) = tr(g)$ where $g \in SU(d)$ is considered as a $d \times d$ matrix). By the Cartan-Weyl theory of representations of compact semi-simple Lie groups, the set Γ of classes of irreducible representations of G can be identified with the set of dominant weights of the Lie algebra $su(d)$. A dominant weight of $su(d)$ can be identified with an integer vector $\lambda = (\lambda_1, \dots, \lambda_d)$ such that $\lambda_1 \leq \dots \leq \lambda_d$ modulo an integer multiple of the vector $\xi = (1, \dots, 1)$, or equivalently with its projection on the orthogonal of ξ in \mathbb{R}^d .

The Markov chain on Γ which has been associated to ϕ in 2.2.2 can now be described in the following way (see [2]). First consider the random walk on \mathbb{Z}^d whose jumps have a law which is uniform on the vectors $e_i - e_{i+1}$, $i = 1, \dots, d-1$ and $e_d - e_1$ (where (e_1, \dots, e_d) is the canonical basis of \mathbb{Z}^d). Since the jumps are orthogonal to the vector ξ this random walk gives rise to a random walk on the orthogonal of ξ . Now kill this random walk at its first exit time of the Weyl chamber $\{\lambda \in \mathbb{Z}^d \mid \lambda_1 \leq \dots \leq \lambda_d\}$ modulo $\xi\mathbb{Z}$, and then take the h -transform associated with the harmonic function $h(\lambda) = \prod_{1 \leq j \leq k} (\lambda_j + j - \lambda_k - k)$, the resulting process has the same transition kernel as the Markov chain obtained by restriction of the quantum random walk $(j_n)_{n \in \mathbb{N}}$ to the center of $vN(SU(d))$, when Γ is identified with the set of dominant weights. (Remark: the integer $h(\lambda)$ is the dimension of the irreducible representation with highest weight λ and it is a harmonic function of λ for the transition operator of the random walk killed at the boundary of the cone.)

It is not difficult to prove, using standard weak convergence arguments

(see Pitman [11] for the case of $SU(2)$), that the process $\left(\frac{\lambda_{[nt]}}{\sqrt{n}}\right)_{t \in \mathbb{R}_+}$ (where $[x]$ denotes the integer part of x) converges in law (on the usual Skorokhod space of Jacod, Shyriaev [7]) towards the law of the motion of eigenvalues of the brownian motion on traceless hermitian matrices, which was described in Section 2.1.2. We shall state below (Sect. 2.2.4) a result which extends this convergence to include the whole quantum random walk, at the cost of obtaining only convergence of moments.

2.2.4. The finite dimensionnal representations of $SU(d)$ and $sl(d, \mathbb{C})$ are in one to one correspondance, so we will use the Lie algebra $sl(d, \mathbb{C})$ instead of the compact group $SU(d)$ in what follows.

Let $(E_{jk})_{1 \leq j, k \leq d}$ be a family of matrix units in $M_d(\mathbb{C})$ and let $H^{jk} = E_{jk}$, for $1 \leq j \neq k \leq d$ and $H^{jj} = E_{jj} - \frac{1}{d} \text{Id}$ for $1 \leq j \leq d$, then $(H^{jk})_{1 \leq j, k \leq d, (j,k) \neq (d,d)}$ form a basis of the Lie algebra $sl(d, \mathbb{C})$.

Let $\mathcal{U}(d)$ be the universal envelopping algebra of $su(d)$. An element P of the center $Z(\mathcal{U}(d))$ of $\mathcal{U}(d)$ acts by a constant $P(\lambda)$ on the space of an irreducible finite dimensionnal representation of $sl(n, \mathbb{C})$ with highest weight λ , and $P(\lambda)$ is a symmetric polynomial function of λ on $\{\lambda \in \mathbb{R}^d \mid \sum_{j=1}^d \lambda_j = 0\}$. In this way we identify $Z(\mathcal{U}(d))$ with an algebra of symmetric polynomials.

If P is a polynomial function, and $\alpha \in \mathbb{R}$, let P^α be the polynomial function obtained by scaling the variables by a factor α .

Let γ be a non zero finite dimensionnal representation of $sl(d, \mathbb{C})$, of dimension r , and $\gamma^{\otimes n}$ the Lie algebra tensor product representation

$$\left(\gamma^{\otimes n}(u) = \sum_{k=1}^n \text{Id}^{\otimes(k-1)} \otimes \gamma(u) \otimes \text{Id}^{\otimes(n-k)} \right).$$

The representations $\gamma^{\otimes n}$ are extended as representations of the algebra $\mathcal{U}(d)$.

Let $v = \text{tr}(\gamma(H^{jk})\gamma(H^{kj}))$ for $1 \leq j < k \leq d$, then the value of v does not depend on (k, j) and $v = \frac{d}{d-1} \text{tr}(\gamma(H^{jj})\gamma(H^{jj}))$ for $1 \leq j \leq d$ (here tr is the normalized trace on $M_r(\mathbb{C})$).

Let (\mathcal{Y}, ω) be the infinite product von Neumann algebra $M_r(\mathbb{C})^{\otimes \infty}$ (with respect to the normalized trace on $M_r(\mathbb{C})$) with the infinite product state ω , and define $s_n : \mathcal{U}(d) \rightarrow \mathcal{Y}$ by $s_n(u) = \gamma^{\otimes n}(u) \otimes \text{Id} \otimes \dots$.

Theorem 2. *The law of the family of random variables on (\mathcal{Y}, ω)*

$$\left(\frac{1}{\sqrt{nv}} s_{[nt]}(H^{jk}), s_{[nt]}(P^{\frac{1}{\sqrt{nv}}}) \right)_{t \in \mathbb{R}_+, 1 \leq j, k \leq d, P \in Z(\mathcal{U}(d))}$$

converges as n goes to infinity towards the law of

$$((X^{jk}(t)), P(\lambda(t)))_{t \in \mathbb{R}_+, 1 \leq j, k \leq d, P \in Z(\mathcal{U}(d))}$$

where $(X(t) = (X^{jk}(t))_{1 \leq j, k \leq d})_{t \in \mathbb{R}_+}$ is a brownian motion on hermitian traceless matrices as defined in 2.1.2, and $\lambda(t)$ is its eigenvalue process.

2.2.5. We shall not need the full strength of this result in the sequel, so we will only prove a weaker statement, namely, the following Proposition 2. The Theorem 2 can be proved by similar arguments.

Proposition 2. *The law of the family of random variables on (\mathcal{Y}, ω) , $(\frac{1}{\sqrt{nv}} s_n(H^{jk}))_{1 \leq j, k \leq d}$ converges as n goes to infinity towards the law of $(X(1) = (X(1)^{jk}))_{1 \leq j, k \leq d}$ with the notations of Theorem 2.*

Thus, the $X^{jk}(1)$ for $j \neq k$ are complex, centered, independent gaussian variables such that $X^{jk}(1) = \bar{X}^{kj}(1)$, and $E[|X^{jk}(1)|^2] = 1$, and the variables $X^{jj}(1)$ for $1 \leq j \leq d$ are real, centered, gaussian variables with covariance $E[X^{jj}(1)X^{kk}(1)] = -\frac{1}{d}$ if $j \neq k$ and $= 1 - \frac{1}{d}$ if $j = k$.

The proof of this result is essentially the same as the Proposition of 4) in [3] and relies on the following two simple lemma.

Lemma 4. *Let $L \in sl(2, \mathbb{C})$ be a self adjoint element, then the law of $\frac{1}{\sqrt{n}} s_n(L)$ converges towards the law of a centered gaussian random variable of variance $tr(\gamma(L)^2)$.*

Proof. One has

$$s_n(L) = \gamma(L) \otimes Id \otimes Id \dots + I \otimes \gamma(L) \otimes Id \otimes \dots + \dots Id^{\otimes n-1} \otimes \gamma(L) \otimes Id \otimes \dots$$

The proof of the lemma is just an application of the usual central limit theorem to the commuting, self-adjoint centered random variables $\gamma(L) \otimes Id \otimes Id \dots, Id \otimes \gamma(L) \otimes Id \otimes \dots, \dots, Id^{\otimes n-1} \otimes \gamma(L) \otimes Id \otimes \dots$. □

Using a polarization argument, we have the following corollary of Lemma 4.

Corollary. *For any sequence $H^{j_1 k_1}, \dots, H^{j_m k_m}$ the symmetrized moments*

$$\omega \left(\frac{1}{n!} \sum_{\sigma} \prod_{q=1}^m \frac{1}{\sqrt{n}} s_n(H^{j_{\sigma(q)} k_{\sigma(q)}}) \right)$$

(the sum is over permutations of $\{1, \dots, m\}$) have a limit as n goes to infinity which is $E[X^{j_1 k_1}(1) \dots X^{j_m k_m}(1)]$.

We now only have to prove the following lemma to finish the proof.

Lemma 5. *For any sequence $H^{j_1 k_1}, \dots, H^{j_m k_m}$, one has*

$$\lim_{n \rightarrow \infty} \omega \left(\prod_{q=1}^m \frac{1}{\sqrt{n}} s_n(H^{j_{\sigma(q)} k_{\sigma(q)}}) - \prod_{q=1}^m \frac{1}{\sqrt{n}} s_n(H^{j_q k_q}) \right) = 0.$$

Proof. Let σ be the transposition $(l, l + 1)$. One has the following identity in $\mathcal{U}(d)$.

$$\begin{aligned} \prod_{q=1}^m H^{j_{\sigma(q)} k_{\sigma(q)}} - \prod_{q=1}^m H^{j_q k_q} &= \prod_{q=1}^{l-1} H^{j_q k_q} [H^{j_l k_l}, H^{j_{l+1} k_{l+1}}] \prod_{q=l+2}^m H^{j_q k_q} \\ &= \prod_{q=1}^{l-1} H^{j_q k_q} \left(\sum_{1 \leq u, v \leq d} c_{u,v}^{j_l k_l, j_{l+1} k_{l+1}} H^{uv} \right) \prod_{q=l+2}^m H^{j_q k_q} \end{aligned}$$

where $c_{u,v}^{j k, j' k'}$ for $1 \leq u, v, j, k, j', k' \leq d$ are the structure constants of the Lie algebra $sl(d, \mathbb{C})$. We thus find that

$$\begin{aligned} &\omega \left(\prod_{q=1}^m \frac{1}{\sqrt{n}} s_n(H^{j_{\sigma(q)} k_{\sigma(q)}}) - \prod_{q=1}^m \frac{1}{\sqrt{n}} s_n(H^{j_q k_q}) \right) \\ &= \frac{1}{\sqrt{n}} \left\{ \sum_{1 \leq u, v \leq d} c_{u,v}^{j_l k_l, j_{l+1} k_{l+1}} \omega \left(\prod_{q=1}^{l-1} \frac{1}{\sqrt{n}} s_n(H^{j_q k_q}) \frac{1}{\sqrt{n}} s_n(H^{uv}) \prod_{q=l+2}^m \frac{1}{\sqrt{n}} s_n(H^{j_q k_q}) \right) \right\} \end{aligned}$$

now, using the corollary of Lemma 4 and an easy induction argument (compare [3]) we conclude that the expression between $\{ \}$ remains bounded as n goes to infinity, and so have the proof of Lemma 5. \square

2.3. A two parameter family of matrices.

2.3.1. With the notations of 2.2.4, let us consider the $dr^n \times dr^n$ matrix

$$M_{d,\gamma}^n = \sum_{1 \leq j, k \leq d} E_{jk} \otimes \frac{1}{\sqrt{nd}} s_n(H^{jk}).$$

By Proposition 1, when n goes to infinity, the law of this matrix, considered as a non-commutative random variable in $(M_{dr^n}(\mathbb{C}), tr)$ converges to that of a hermitian gaussian traceless matrix with variance $\frac{1}{d}$ (namely the law of $X(\frac{1}{d})$ as in Theorem 2).

Furthermore, if $(\gamma_\iota)_{\iota \in I}$ is a family of commuting representations of $sl(d, \mathbb{C})$ on $M_r(\mathbb{C})$, the law of the family of matrices $(M_{d,\gamma_\iota}^n)_{\iota \in I}$ converges towards the law of a family $(M_{d,\iota})_{\iota \in I}$ of $d \times d$ hermitian gaussian traceless matrices with

independent entries and variance $\frac{1}{d}$. When d goes to infinity, by Voiculescu's theorem [17] (here the matrices are traceless, but this is a minor modification with respect to [17], which is negligible as d goes to infinity), the law of the family of random variables $(M_{d,\iota})_{\iota \in I}$ converges towards the law of a semi-circular system.

2.3.2. Instead of making n go to infinity first we will make d go to infinity first. Let us now take γ to be the contragredient representation, then $r = d$, and one gets the following matrices

$$M_d^n = \frac{1}{\sqrt{n}} \left(\sum_{1 \leq j, k \leq d} \sum_{m=1}^n E_{jk} \otimes \text{Id}^{\otimes m-1} \otimes E_{kj} \otimes \text{Id}^{\otimes n-m} - \frac{1}{d} \text{Id} \right).$$

These matrices operate on the Hilbert space $(\mathbb{C}^d)^{\otimes n+1}$. The permutation group of $\{0, 1, \dots, n\}$ operates on $(\mathbb{C}^d)^{\otimes n+1}$ by $\sigma(x_0 \otimes \dots \otimes x_n) = x_{\sigma(0)} \otimes \dots \otimes x_{\sigma(n)}$, and a simple computation shows that the matrix $\sum_{1 \leq j, k \leq d} E_{jk} \otimes \text{Id}^{\otimes m-1} \otimes E_{kj} \otimes \text{Id}^{\otimes n-m}$ is just the matrix t_m^d of the transposition $(0, m)$ acting on $(\mathbb{C}^d)^{\otimes n+1}$. We conclude that M_d^n is equal to $\frac{1}{\sqrt{n}} (t_1^d + \dots + t_n^d - \frac{1}{d} \text{Id})$.

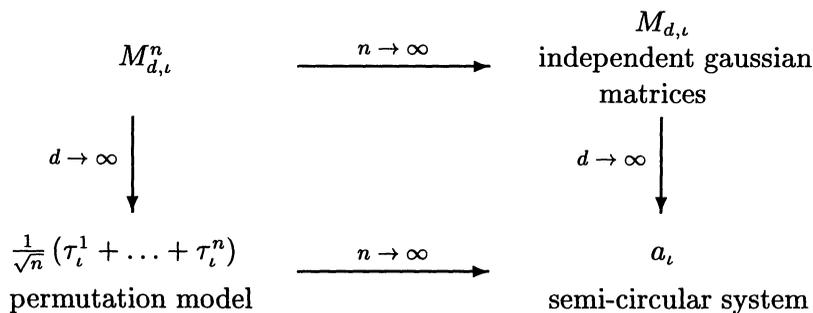
For $1 \leq m \leq n$ let τ_m be the image of the transposition $(0, m)$ in the von Neumann algebra generated by the left regular representation of the symmetric group of $\{0, 1, \dots, n\}$. This von Neumann algebra is a non commutative probability space, when equipped with the canonical trace.

Lemma 6. *When d goes to infinity the family of random variables $(t_m^d)_{1 \leq m \leq n}$ converges in law towards the law of the family $(\tau_m)_{1 \leq m \leq n}$.*

Proof. It is enough to prove that for any non trivial permutation σ the normalized trace of its action on the space $(\mathbb{C}^d)^{\otimes n+1}$ goes to zero as d goes to infinity. But this trace is just $\frac{d^{c(\sigma)}}{d^{n+1}}$ where $c(\sigma)$ is the number of cycles of σ , as can be verified using the basis $(e_{i_1} \otimes \dots \otimes e_{i_n})_{1 \leq i_1, \dots, i_n \leq d}$ (where $(e_k)_{1 \leq k \leq d}$ is the canonical basis of \mathbb{C}^d) to compute the trace, and so the lemma is proved, since $c(\sigma) < n + 1$ for a non-identical permutation σ . \square

There is no difficulty in constructing a family $M_{d,\iota}^n$ for $\iota \in I$ if I is finite, just take the Hilbert space $\mathbb{C}^d \otimes \left((\mathbb{C}^d)^{\otimes n} \right)^{\otimes I}$ and construct a copy of the matrix M_d^n on the component $\mathbb{C}^d \otimes (\mathbb{C}^d)^{\otimes n}$ corresponding to $\iota \in I$. The family of random variables $(M_{d,\iota}^n)_{\iota \in I}$ will converge in law as d goes to infinity, towards the law of the family $(a_{\iota \in I}^n)$ with the notations of Theorem 1.

We can summarize the convergence schemes that we have just seen in the following diagram



References

- [1] P. Bérard and G. Besson, *Spectres et groupes cristallographiques II*, Ann. Inst. Fourier, **30** (3) (1980), 237-248.
- [2] Ph. Biane, *Quantum random walk on the dual of $SU(n)$* , Prob. Th. and Rel. Fields, **89** (1991), 117-129.
- [3] ———, *Some properties of quantum Bernoulli random walks*, Quantum probability and applications 6, L.Accardi, W. von Waldenfels ed., World scientific, 1991, 193-204.
- [4] J.L. Doob, *Classical potential theory and its probabilistic counterpart*, Springer Verlag, Berlin, Heidelberg, New York, 1984.
- [5] F.J. Dyson, *A brownian motion model for the eigenvalues of a random matrix*, Journ. Math. Phys., **3** (1962), 1191-1198.
- [6] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, North holland, Kodansha, 1981.
- [7] J. Jacod and A.N. Shyriaev, *Limit theorems for stochastic processes*, Springer, 1987.
- [8] G. Kreweras, *Sur les partitions non croisées d'un cycle* Discrete Math., **1** (1972).
- [9] H.P. Mac Kean, *Stochastic integrals*, Academic press, New York, 1969.
- [10] K.R. Parthasarathy, *A generalized Biane's process*, Séminaire de probabilités XXIV, Springer Lecture Notes in Mathematics, **1426** (1990), 345-348.
- [11] J.W. Pitman *One dimensionnal brownian motion and the three dimensionnal Bessel process*, Adv. Appl. Prob., **7** (1975), 511-526.
- [12] M. Schürmann, *Quantum stochastic processes with independent additive increments*, Jour. Multivariate Anal., **38** (1991), 15-35.
- [13] G. Skandalis, *Algèbres de von Neumann de groupes libres et probabilités non commutatives [d'après Voiculescu et al.]*, Séminaire Bourbaki, exposé 764, Novembre 1992.
- [14] R. Speicher, *A new example of independence and white noise*, Prob. Th. and Rel. Fields, **84** (1990), 141-159.
- [15] D. Voiculescu, *Symmetries of some reduced free product C^* -algebras*, Operator algebras and their connection with topology and ergodic theory. Lect. Notes in Math. Springer, 1985, 556-588.
- [16] ———, *Circular and semi-circular systems and free product factors*, Operator algebras, Unitary representations, envelopping algebras, and invariant theory, Progress

in Mathematics, Volume 92, Birkhäuser, Boston, 1990.

- [17] ———, *Limit laws for random matrices and free products*, *Invent. Math.*, **104** (1991), 201-220.
- [18] E.P. Wigner, *Characteristic vectors of bordered matrices with infinite dimensions*, *Ann. Math.*, **62** (1955), 548-564.

Received July 29, 1993.

C.N.R.S., LABORATOIRE DE PROBABILITÉS
TOUR 56-66, 3^e ÉTAGE
4, PLACE JUSSIEU, 75252, PARIS CEDEX 05
FRANCE

