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COMMUTATORS AND INVARIANT DOMAINS FOR SCHRÖDINGER PROPAGATORS

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We present an operator-theoretic approach to the problem of invariant domains for the Schrödinger evolution equation. The results are applied to the Hamiltonian operators with time-dependent potentials and electric fields.

1. Introduction.

This paper is concerned with the problem of invariant domains for the Schrödinger evolution equation

$$i\frac{d}{dt}\varphi(t) = H(t)\varphi(t), \quad \varphi(s) = \varphi_s \tag{1}$$

where $H(t)$, $t \in \mathbb{R}$, is a family of self-adjoint operators acting on a Hilbert space $\mathcal{H}$.

It is known that under suitable conditions on $H(t)$ (see e.g. Kato [**4**], Reed-Simon [**9**] and Yajima [**11**]), there exists a unique unitary propagator $U(t,s)$ on $\mathcal{H}$, and a dense subspace $\mathcal{D}$ of $\mathcal{H}$ which is invariant under the propagator so that for each $\varphi_s \in \mathcal{D}$, $\varphi(t) = U(t,s)\varphi_s$ is strongly differentiable and satisfies (1).

The problem considered here has been studied by many authors; see Faris-Lavine [**1**], Fröhlich [**2**], Hunziker [**3**], Kuroda-Morita [**5**], Ozawa [**6**, **7**], Radin-Simon [**8**] and Wilcox [**10**]. Most of them dealt with the time-independent case $H(t) \equiv H$ in which the propagator $U(t,s) = \exp[i(s-t)H]$ is given by the usual one-parameter unitary group. In a recent paper [**7**], Ozawa investigated the space-time behavior of $U(t,s)$ for the Stark Hamiltonian $H(t) = -\Delta + E \cdot x + V(x,t)$ on $L^2(\mathbb{R}^n, dx)$. By using perturbation techniques and space-time estimates for the *free* propagator $\exp[it(-\Delta + E \cdot x)]$, Ozawa established several results on the invariance property and smoothing effect for $U(t,s)$ in certain weighted Sobolev spaces. For earlier related results in the case $E = 0$, see Kuroda-Morita [**5**].

We denote the domain of an operator A by $\mathcal{D}(A)$, and if N is positive and self-adjoint, we denote its form domain by $\mathcal{Q}(N)$. Given a positive self-adjoint operator N, we are interested in conditions on $H(t)$ for $\mathcal{Q}(N)$ or

$\mathcal{D}(N^k)$, $k = 1, 2, \ldots$, to be an invariant subspace of $U(t,s)$ for all $t, s \in \mathbb{R}$. We study this problem in a general operator-theoretic setting in Section 2. Our approach is based on the commutator theorems of Faris and Lavine [**1**] and Fröhlich [**2**]. In Section 3, we apply the abstract theorems of Section 2 to Hamiltonians of the form

$$H(t) = -\Delta + E(t) \cdot x + V(x,t)$$

with $N = p^2 + x^2$ or $N = p^2$, where p is the momentum operator $-i\nabla$. Our results are related to some of those in [**5**, **7**].

2. Abstract Theorems.

Let $H(t)$, $t \in \mathbb{R}$, be a family of self-adjoint operators acting on a Hilbert space $\mathcal{H}$. Throughout this section, we will assume that $\bigcap_t \mathcal{D}(H(t)) \supseteq \mathcal{D}$ for some dense subspace $\mathcal{D}$ of $\mathcal{H}$, and that $H(t)$ generates a unitary propagator $U(t,s)$ so that

$$i\frac{d}{dt}U(t,s)\varphi = H(t)U(t,s)\varphi \text{ for all } \varphi \in \mathcal{D}.$$

We denote by $\mathcal{B}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$ with the usual operator norm $\|\cdot\|$. For a positive self-adjoint operator N on $\mathcal{H}$ and $\epsilon > 0$, we define $N_\epsilon = N(\epsilon N + 1)^{-1}$. Note that $N_\epsilon \in \mathcal{B}(\mathcal{H})$ is positive and self-adjoint. Concerning the invariance of the form domain $\mathcal{Q}(N) = \mathcal{D}(N^{1/2})$, we prove:

Theorem 2.1. *Let N be a positive self-adjoint operator so that*
(i) $\mathcal{D}(N) \subseteq \bigcap_t \mathcal{D}(H(t))$.
(ii) $\pm i\,[H(t), N] \leq c(t)N$ *for some* $c \in L^1_{loc}(\mathbb{R})$; *that is,*
$\pm i\,\{\langle H(t)\varphi, N\varphi\rangle - \langle N\varphi, H(t)\varphi\rangle\} \leq c(t)\,\langle \varphi, N\varphi\rangle$ *for all* $\varphi \in \mathcal{D}(N)$.
Then $U(t,s)\,[\mathcal{Q}(N)] = \mathcal{Q}(N)$ *for all* t, s.

Proof. Fix s and set $\varphi(t) = U(t,s)\varphi$ for $\varphi \in \mathcal{H}$. Then we have for $\varphi \in \mathcal{D}$

$$\begin{aligned}(d/dt)\,\langle \varphi(t), N_\epsilon\varphi(t)\rangle &= \langle \varphi(t), i\,[H(t), N_\epsilon]\,\varphi(t)\rangle \\ &= \left\langle (\epsilon N + 1)^{-1}\varphi(t), i\,[H(t), N]\,(\epsilon N + 1)^{-1}\varphi(t)\right\rangle.\end{aligned}$$

The hypothesis (ii) now gives that

$$\begin{aligned}|(d/dt)\,\langle \varphi(t), N_\epsilon\varphi(t)\rangle\,| &\leq c(t)\,\left\langle (\epsilon N + 1)^{-1}\varphi(t), N(\epsilon N + 1)^{-1}\varphi(t)\right\rangle \\ &\leq c(t)\,\langle \varphi(t), N_\epsilon\varphi(t)\rangle.\end{aligned}$$

Integrating we obtain

$$\langle \varphi(t), N_\epsilon\varphi(t)\rangle \leq \langle \varphi, N_\epsilon\varphi\rangle \ \exp\left|\int_s^t c(u)du\right|.$$

Since $\mathcal{D}$ is dense in $\mathcal{H}$ and N_ϵ is bounded, this estimate holds for all $\varphi \in \mathcal{H}$. Now let $\varphi \in \mathcal{Q}(N)$. Taking $\epsilon \to 0$, we find that $\varphi(t) \in \mathcal{Q}(N)$ with

$$\|N^{1/2}\varphi(t)\|^2 \leq \|N^{1/2}\varphi\|^2 \ \exp\left|\int_s^t c(u)du\right|.$$

This shows that $\mathcal{Q}(N)$ is invariant under $U(t,s)$. Since $U(t,s)U(s,t) = I$, we conclude that $U(t,s)\left[\mathcal{Q}(N)\right] = \mathcal{Q}(N)$. □

Now for any positive integer k, we define (leaving aside the domain questions)

$$(2) \quad Z^k(t) = N^{k-1}\left[H(t), N\right]N^{-k} \quad \text{and} \quad Z_\epsilon^k(t) = N_\epsilon^{k-1}\left[H(t), N_\epsilon\right]N_\epsilon^{-k}.$$

In our applications, these operators are defined on certain dense subspaces and extend to bounded operators on $\mathcal{H}$. We also define

$$(\operatorname{ad} N)H(t) = [N, H(t)] \quad \text{and} \quad (\operatorname{ad} N)^k H(t) = \left[N, (\operatorname{ad} N)^{k-1}H(t)\right].$$

As a preparation for our next theorem and further applications, we prove the following:

Lemma 2.2.
(a) $Z_\epsilon^k(t) = (\epsilon N+1)^{-k}\sum_{j=0}^{k-1}\binom{k-1}{j}(\epsilon N)^j Z^{k-j}(t)$. *In particular, if* $Z^1(t), \ldots, Z^k(t) \in \mathcal{B}(\mathcal{H})$, *then* $Z_\epsilon^k(t) \in \mathcal{B}(\mathcal{H})$ *and* $\|Z_\epsilon^k(t)\| \leq \sum_{j=0}^{k-1}\binom{k-1}{j}\|Z^{k-j}(t)\|$.
(b) $\left\{(\operatorname{ad} N)^k H(t)\right\}N^{-k} = \sum_{j=0}^{k-1}(-1)^{j+1}\binom{k-1}{j}Z^{k-j}(t)$.

Proof. Part (a) is obvious for $k = 1$. The general case follows by induction on k:

$$\begin{aligned}
Z_\epsilon^{k+1}(t) &= N_\epsilon Z_\epsilon^k(t)N_\epsilon^{-1} \\
&= N_\epsilon(\epsilon N+1)^{-k}\sum_{j=0}^{k-1}\binom{k-1}{j}(\epsilon N)^j Z^{k-j}(t)N_\epsilon^{-1} \\
&= (\epsilon N+1)^{-k-1}\sum_{j=0}^{k-1}\binom{k-1}{j}(\epsilon N)^j N Z^{k-j}(t)N^{-1}(1+\epsilon N) \\
&= (\epsilon N+1)^{-k-1}\sum_{j=0}^{k-1}\binom{k-1}{j}\left\{(\epsilon N)^j Z^{k+1-j}(t) + (\epsilon N)^{j+1}Z^{k-j}(t)\right\} \\
&= (\epsilon N+1)^{-k-1}\sum_{j=0}^{k}\binom{k}{j}(\epsilon N)^j Z^{k+1-j}(t)
\end{aligned}$$

where we have used the identity $\binom{k-1}{j} + \binom{k-1}{j-1} = \binom{k}{j}$. The last statement of part (a) follows from the fact that $\|(\epsilon N+1)^{-k}(\epsilon N)^j\| \leq 1$ for $0 \leq j \leq k-1$.

Part (b) can also be proven by an induction argument. □

Theorem 2.3. *Let N be a positive self-adjoint operator, and define $Z^j(t)$ as in (2). Suppose that $Z^j(t) \in \mathcal{B}(\mathcal{H})$ with $\|Z^j(\cdot)\| \in L^1_{loc}(\mathbb{R})$ for each $j = 1, 2, \ldots, k$. Then $U(t,s)\left[\mathcal{D}(N^k)\right] = \mathcal{D}(N^k)$ for all t, s.*

Proof. As in the proof of Theorem 2.1, set $\varphi(t) = U(t,s)\varphi$ for $\varphi \in \mathcal{H}$. Then we have for $\varphi \in \mathcal{D}$

$$\begin{aligned}(d/dt)\langle N_\epsilon^k \varphi(t), N_\epsilon^k \varphi(t)\rangle &= \langle \varphi(t), i\left[H(t), N_\epsilon^{2k}\right]\varphi(t)\rangle \\ &= i\sum_{j=0}^{2k-1} \langle \varphi(t), N_\epsilon^j \left[H(t), N_\epsilon\right] N_\epsilon^{2k-j-1}\varphi(t)\rangle \\ &= 2\,\mathrm{Im}\sum_{j=0}^{k-1} \langle N_\epsilon^{k-j-1}\left[H(t), N_\epsilon\right] N_\epsilon^j \varphi(t), N_\epsilon^k \varphi(t)\rangle\end{aligned}$$

where we have used

$$\left[A, B^{2k}\right] = \sum_{j=0}^{2k-1} B^j \left[A, B\right] B^{2k-j-1}.$$

Since $Z^j(t)$ is bounded and $\|Z^j(\cdot)\| \in L^1_{loc}(\mathbb{R})$ for $1 \leq j \leq k$, Lemma 2.2 (a) implies that $Z_\epsilon^j(t)$ is bounded for $1 \leq j \leq k$ and that $2\sum_{j=1}^k \|Z_\epsilon^j(t)\| \leq$ const. $\sum_{j=1}^k \|Z^j(t)\| \equiv f_k(t)$, where $f_k \in L^1_{loc}(\mathbb{R})$ and is independent of ϵ. It follows that

$$\begin{aligned}\left|(d/dt)\|N_\epsilon^k \varphi(t)\|^2\right| &\leq 2\sum_{j=0}^{k-1} \|N_\epsilon^{k-j-1}\left[H(t), N_\epsilon\right] N_\epsilon^j \varphi(t)\| \|N_\epsilon^k \varphi(t)\| \\ &\leq 2\sum_{j=0}^{k-1} \|Z_\epsilon^{k-j}(t)\| \, \|N_\epsilon^k \varphi(t)\|^2 \\ &\leq f_k(t)\|N_\epsilon^k \varphi(t)\|^2.\end{aligned}$$

Integrating we obtain

$$\|N_\epsilon^k \varphi(t)\| \leq \|N_\epsilon^k \varphi\| \, \exp\left|\frac{1}{2}\int_s^t f_k(u)du\right|.$$

We can now pass to the same argument as in the proof of Theorem 2.1 to conclude that $U(t,s)\left[\mathcal{D}(N^k)\right] = \mathcal{D}(N^k)$. □

3. Applications.

In this section we want to give some applications of the results of Section 2 to the Schrödinger equation

$$(3) \qquad i\frac{d}{dt}\varphi(t) = H(t)\varphi(t); \qquad \varphi(s) = \varphi_s$$

where $H(t)$ is the time-dependent Hamiltonian acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n, dx)$.

We first consider Hamiltonians of the form

$$H(t) = -\Delta + E(t) \cdot x + V(x,t).$$

We will restrict attention to electric fields $E(t) : \mathbb{R} \to \mathbb{R}^n$ and potentials $V(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ obeying :

$$\begin{cases} \text{(i)} & E(t) \text{ is differentiable.} \\ \text{(ii)} & |\nabla_x V(x,t)| \le f(t)(|x|+1) \text{ for some continuous function } f. \\ \text{(iii)} & \text{the mapping } t \mapsto (x^2+1)^{-1}\frac{\partial V}{\partial t}(x,t) \in L^\infty(\mathbb{R}^n, dx) \text{ is continuous.} \end{cases}$$

As for N, we take $N = p^2 + x^2$, where $p = -i\nabla$. Note that the operator $N \ge 1$ and is self-adjoint on $\mathcal{D}(N) = \mathcal{D}(p^2) \cap \mathcal{D}(x^2)$. By Theorem 4 of Faris-Lavine **[1]**, condition (ii) implies that $H(t)$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$, the space of C^∞-functions on $\mathbb{R}^n$ rapidly decreasing at infinity, with domain $\mathcal{D}(H(t)) \supseteq \mathcal{D}(N)$. We remark that by the construction of the form domain, $\mathcal{Q}(N) = \mathcal{D}(|p|) \cap \mathcal{D}(|x|)$. Also, one can prove that $\mathcal{D}(N^k) = \mathcal{D}(p^{2k}) \cap \mathcal{D}(x^{2k})$ by integration by parts.

Given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ the space of all bounded linear operators with domain $\mathcal{X}$ and range in $\mathcal{Y}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, where each α_j is a nonnegative integer, and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we put $|\alpha| = \alpha_1 + \dots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\nabla^\alpha = (\frac{\partial}{\partial x})^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. Let $B_\infty^m(\mathbb{R}^n)$ be the space of all m-times continuously differentiable functions φ on $\mathbb{R}^n$ with bounded derivatives $(\frac{\partial}{\partial x})^\alpha \varphi$ for $0 < |\alpha| \le m$. Our result is:

Theorem 3.1. *Let $H(t) = -\Delta + E(t) \cdot x + V(x,t)$, where $E(t)$ and $V(x,t)$ obey conditions* (i)-(iii) *above, and let $N = p^2 + x^2$. Then there exists a unique unitary propagator $U(t,s)$, $t, s \in \mathbb{R}$, so that:*

(a) *for each $\varphi_s \in \mathcal{D}(N)$, $\varphi(t) = U(t,s)\varphi_s$ is strongly differentiable and satisfies* (3).

(b) *$U(t,s)$ leaves $\mathcal{Q}(N)$ and $\mathcal{D}(N)$ invariant.*

If, in addition, $V(\cdot,t) \in B^{2k}_\infty(\mathbb{R}^n)$ *with* $\|(\frac{\partial}{\partial x})^\alpha V(x,\cdot)\|_\infty \in L^1_{loc}(\mathbb{R})$ *for* $0 < |\alpha| \le 2k$, *then* $U(t,s)$ *leaves* $\mathcal{D}(N^k)$ *invariant.*

Proof. To prove the existence of the propagator, we define for $\varphi \in \mathcal{D} \equiv \mathcal{D}(N)$, $\|\varphi\|_\mathcal{D} = \|\varphi\| + \|p^2\varphi\| + \|x^2\varphi\|$. Then $(\mathcal{D}, \|\cdot\|_\mathcal{D})$ forms a Banach space which is continuously and densely embedded in $\mathcal{H}$. From (ii), we have $|V(x,t)| \le \frac{1}{2}f(t)x^2 + f(t)|x| + |V(0,t)|$. It follows by the continuity of E, V and f that on any compact interval $[-T,T]$, there are constants a and b so that $|E(t)\cdot x + V(x,t)| \le ax^2 + b$ for all $t \in [-T,T]$. Since

$$\|p^2\varphi\|^2 + \|cx^2\varphi\|^2 \le \|(p^2+cx^2)\varphi\|^2 + 2cn\|\varphi\|^2 \quad \text{for } \varphi \in \mathcal{D},$$

we see that if $c > a$, then $E(t)\cdot x + V(x,t)$ is (p^2+cx^2)-bounded with relative bound less than one. Thus, by the Kato-Rellich theorem, $H(t) + cx^2$ is self-adjoint on $\mathcal{D}$ for all $t \in [-T,T]$. Now, take $S(t) = H(t) + cx^2 + i$. Then $S(t) \in \mathcal{B}(\mathcal{D},\mathcal{H})$ is an isomorphism with $S(t)H(t)S(t)^{-1} = H(t) + G(t)$, where $G(t) = 2ci(p\cdot x + x\cdot p)S(t)^{-1} \in \mathcal{B}(\mathcal{H})$. By (i) and (iii), the mapping $t \mapsto S(t) \in \mathcal{B}(\mathcal{D},\mathcal{H})$ is strongly differentiable. Also, a simple computation gives that

$$\begin{aligned}
\|G(t) - G(u)\|_{\mathcal{B}(\mathcal{H})} &\le \|G(t)\|_{\mathcal{B}(\mathcal{H})}\|H(t) - H(u)\|_{\mathcal{B}(\mathcal{D},\mathcal{H})}\|S(u)^{-1}\|_{\mathcal{B}(\mathcal{H},\mathcal{D})}\\
\|H(t) - H(u)\|_{\mathcal{B}(\mathcal{D},\mathcal{H})} &\le |E(t) - E(u)|\\
&\quad + \|(x^2+1)^{-1}[V(x,t) - V(x,u)]\|_{L^\infty(\mathbb{R}^n,dx)}.
\end{aligned}$$

Thus, by (i) and (iii), the mapping $t \mapsto H(t) \in \mathcal{B}(\mathcal{D},\mathcal{H})$ and $t \mapsto G(t) \in \mathcal{B}(\mathcal{H})$ are norm continuous. It follows from a classical result of Kato ([**4**], Theorem I) that there exists a unique unitary propagator $U(t,s)$ leaving $\mathcal{D}$ invariant so that (a) holds.

Next, we show that $U(t,s)$ leaves $\mathcal{Q}(N)$ invariant. We have seen that $\mathcal{D}(H(t)) \supseteq \mathcal{D}(N)$ for all t. So by Theorem 2.1, it suffices to show that $\pm i[H(t),N] \le c(t)N$ for some locally integrable function $c(t)$. We compute

$$\begin{aligned}
\pm i\,[H(t),\ N]&\\
&= \pm i\left\{[p^2,x^2] + [E(t)\cdot x, p^2] + [V(x,t),p^2]\right\}\\
&= \pm\left\{2(p\cdot x + x\cdot p) - 2E(t)\cdot p - (p\cdot\nabla_x V(x,t) + \nabla_x V(x,t)\cdot p)\right\}\\
&\le 2(p^2+x^2) + p^2 + |E(t)|^2 + p^2 + |\nabla_x V(x,t)|^2\\
&\le \left\{4 + |E(t)|^2 + 4f(t)^2\right\}N
\end{aligned}$$

as required, where we have used condition (ii) and the fact that $N \ge 1$.

Finally, we prove the last statement of the theorem. Let

$$\Gamma \equiv L^1_{loc}(\mathbb{R}, dt; \mathcal{B}(\mathcal{H})).$$

By Theorem 2.3, it suffices to show that if $V(\cdot,t) \in B^{2k}_{\infty}(\mathbb{R}^n)$ with $\| \left(\frac{\partial}{\partial x}\right)^{\alpha} V(x,\cdot)\|_{\infty} \in L^1_{loc}(\mathbb{R})$ for $0 < |\alpha| \leq 2k$, then

$$Z^j = N^{j-1}\left[H(\cdot), N\right] N^{-j} \in \Gamma$$

for $1 \leq j \leq k$. We prove this inductively. Let $D = p\cdot x + x\cdot p$ be the dilation operator. Since

$$\begin{aligned} Z^1(t) &= [H(t), N]\, N^{-1} \\ &= -2i\left\{D - E(t)\cdot p - \nabla_x V(x,t)\cdot p + \frac{i}{2}\Delta_x V(x,t)\right\} N^{-1}, \end{aligned}$$

the case $k = 1$ follows easily from the closed graph theorem and the hypotheses on E and V. Now consider the case of general $k \geq 2$. By the induction hypothesis, we have $Z^j \in \Gamma$ for $1 \leq j \leq k-1$. So, we need only prove that $Z^k \in \Gamma$. By Lemma 2.2(b), it is sufficient to prove that $\left\{(\operatorname{ad} N)^k H(\cdot)\right\} N^{-k} \in \Gamma$. We compute on $\mathcal{S}(\mathbb{R}^n)$:

$$(\operatorname{ad} N)^2 H(t) = 4\left\{\begin{array}{c} 2(p^2 - x^2) + E(t)\cdot x + \nabla_x V(x,t)\cdot x + \frac{1}{4}\Delta_x^2 V(x,t) \\ -\sum_{j=1}^n \left(\nabla_x \frac{\partial V}{\partial x_j}(x,t)\right)\cdot pp_j + i\nabla_x\left(\Delta_x V(x,t)\right)\cdot p \end{array}\right\}$$

where we have used the following basic identities:

$$\begin{gathered} [N, D] = 4i(x^2 - p^2), \quad [N, E(t)\cdot p] = 2iE(t)\cdot x, \quad [N, E(t)\cdot x] = -2iE(t)\cdot p, \\ [p^2, W(x)] = -2i\nabla W\cdot p - \Delta W, \quad [x^2, \nabla W(x)\cdot p] = 2i\nabla W\cdot x, \\ [p^2, \nabla W(x)\cdot p] = -2i\sum_{j=1}^n \left(\nabla \frac{\partial W}{\partial x_j}\right)\cdot pp_j - \nabla\left(\Delta W\right)\cdot p. \end{gathered}$$

By repeated application of these formulas, we find that $(\operatorname{ad} N)^k H(t)$ is a linear combination of operators of the form:

$$p^2 - x^2 (\text{or } D), \quad E(t)\cdot x \text{ (or } E(t)\cdot p) \text{ and } \left[\left(\frac{\partial}{\partial x}\right)^{\alpha} V(x,t)\right] x^{\beta} p^{\gamma}$$

where $0 < |\alpha| \leq 2k$, $|\beta| \leq k/2$ and $|\gamma| \leq k$. Since $x^{\beta}p^{\gamma}N^{-k}$ is bounded on $\mathcal{H}$ so long as $|\beta| \leq k$ and $|\gamma| \leq k$, the hypotheses of E and V now imply that $\left\{(\operatorname{ad} N)^k H(\cdot)\right\} N^{-k} \in \Gamma$. This completes the proof. □

Corollary 3.2. *In Theorem* 3.1, *if* $V(\cdot,t)$ *is a* C^{∞}*-function on* $\mathbb{R}^n$ *with bounded derivatives and* $\| \left(\frac{\partial}{\partial x}\right)^{\alpha} V(x,\cdot)\|_{\infty} \in L^1_{loc}(\mathbb{R})$ *for all* $\alpha \neq 0$, *then* $U(t,s)$ *leaves* $\mathcal{S}(\mathbb{R}^n)$ *invariant.*

Proof. The corollary follows immediately from the fact that

$$\mathcal{S}(\mathbb{R}^n) = \cap_{k=1}^{\infty}\mathcal{D}(N^k).$$

□

In the remainder of this section, we want to give an application to Hamiltonians of the form

$$H(t) = -\Delta + V(x,t).$$

We will assume potentials $V(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ obeying:

$$\begin{cases} \text{(i)} & \text{for each } t, V(\cdot,t) \text{ is } \Delta\text{-bounded with relative bound less than one.} \\ \text{(ii)} & \text{the mapping } t \mapsto \frac{\partial V}{\partial t}(x,t) \in L^\infty(\mathbb{R}^n, dx) \text{ is continuous.} \end{cases}$$

Notice that condition (i) and the Kato-Rellich theorem imply that $H(t)$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$ with domain $\mathcal{D}(H(t)) = \mathcal{D}(\Delta)$. Corresponding to Theorem 3.1, we have:

Theorem 3.3. *Let $H(t) = -\Delta + V(x,t)$, where $V(x,t)$ obeys conditions* (i) *and* (ii) *above. Then there is a unique unitary propagator $U(t,s)$, $t,s \in \mathbb{R}$, leaving $\mathcal{D}(\Delta)$ invariant so that for each $\varphi_s \in \mathcal{D}(\Delta)$, $\varphi(t) = U(t,s)\varphi_s$ is strongly differentiable and satisfies* (3). *Moreover,*

(a) *If $|\nabla_x V(x,t)| \le f(t)$ for some continuous f, then $U(t,s)$ leaves $\mathcal{Q}(-\Delta)$ invariant.*

(b) *If $V(\cdot,t) \in B^{2k}_\infty(\mathbb{R}^n)$ with $\|(\frac{\partial}{\partial x})^\alpha V(x,\cdot)\|_\infty \in L^1_{loc}(\mathbb{R})$ for $0 < |\alpha| \le 2k$, then $U(t,s)$ leaves $\mathcal{D}(\Delta^k)$ invariant.*

Proof. The proof of the existence statement closely parallels the proof given in Theorem 3.1 except that we choose $\mathcal{D} = \mathcal{D}(\Delta)$, $S(t) = H(t)+i$ and define $\|\varphi\|_{\mathcal{D}} = \|\varphi\| + \|p^2\varphi\|$ so that $S(t)H(t)S(t)^{-1} = H(t)$. Then one proves that the mapping $t \mapsto S(t) \in \mathcal{B}(\mathcal{D},\mathcal{H})$ is strongly differentiable and that the mapping $t \mapsto H(t) \in \mathcal{B}(\mathcal{D},\mathcal{H})$ is norm continuous as before. To prove (a) and (b), we take $N = -\Delta + 1$. In case (a), since

$$\begin{aligned} \pm i\,[H(t), N] &= \mp \{p \cdot \nabla_x V(x,t) + \nabla_x V(x,t) \cdot p\} \\ &\le p^2 + |\nabla_x V(x,t)|^2 \le \{1 + f(t)^2\}\, N, \end{aligned}$$

Theorem 2.1 implies that $U(t,s)$ leaves $\mathcal{Q}(N) = \mathcal{Q}(-\Delta)$ invariant. In case (b), the computations similar to those used in Theorem 3.1 show that $(\operatorname{ad} N)^k H(t)$ is a linear combination of operators of the form: $\left[(\frac{\partial}{\partial x})^\alpha V(x,t)\right] p^\gamma$, where $0 < |\alpha| \le 2k$ and $|\gamma| \le k$. Thus by hypothesis, we have

$$\left\{(\operatorname{ad} N)^k H(\cdot)\right\} N^{-k} \in L^1_{loc}(\mathbb{R}, dt; \mathcal{B}(\mathcal{H})).$$

Again, following the proof of Theorem 3.1, we conclude that $U(t,s)$ leaves $\mathcal{D}(N^k) = \mathcal{D}(\Delta^k)$ invariant. □

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