Ruelle zeta functions for finite dynamical systems and Koyama-Nakajima's L-functions

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Abstract: A complex reflection determines an L-function which is a generalization of the Artin-Mazur zeta function associated with an element of the symmetric group. The present paper shows that the L-function is the Ruelle zeta function associated with a weighted $\mathbf{Z}$-dynamical system.

Key words: Finite dynamical systems; Ruelle zeta functions; Koyama-Nakajima's L-functions; complex reflections.

1. Introduction. Koyama and Nakajima [KN] introduce a class of L-functions associated with complex reflections, a genreralization of the Artin-Mazur zeta functions for finite dynamical systems. They define the L-functions in the form of Euler product, and give a simple determinant expression. Let σ be an element of the symmetric group S_n acting on a set $X = \{1, 2, \dots, n\}$ of n points. The pair (X, σ) forms a finite discrete dynamical system. Let m be a positive integer. An *m-periodic point* is an element $x \in X$ satisfying $\sigma^m(x) = x$. The set of m-periodic points is denoted by $\mathrm{Fix}(\sigma^m)$, and N_m denotes the cardinality of $\mathrm{Fix}(\sigma^m)$. Let u be an indeterminate. Then the formal power series

$$\exp\left(\sum_{m\geq 1} \frac{N_m}{m} u^m\right)$$

is called the *Artin-Mazur zeta function* ([AM]; see also [Y]) associated with (X, σ), or simply σ, denoted by $Z^{\mathrm{AM}}_{(X,\sigma)}(u)$, or simply $Z_\sigma(u)$. For any permutation σ, the Artin-Mazur zeta function $Z_\sigma(u)$ always has an Euler product and a determinant expression. Let $\sigma = p_1 p_2 \cdots p_r$ be the cyclic decomposition of σ, and let $\mathrm{Cyc}(\sigma) = \{p_1, p_2, \dots, p_r\}$. One can easily show that the Euler product of $Z^{\mathrm{AM}}_\sigma(u)$ is given by

$$\prod_{p\in \mathrm{Cyc}(\sigma)} \frac{1}{1 - u^{l(p)}},$$

where $l(p)$ denotes the number of elements in the cyclic domain of p. Let $M_\sigma = (\delta_{\sigma(i)j})_{i,j=1,2,\dots,n}$ be the permutation matrix representing σ, where δ denotes the Kronecker delta. It is known that $Z^{\mathrm{AM}}_\sigma(u)$ has the determinant expression

$$\frac{1}{\det(I - uM_\sigma)},$$

where I stands for the identity matrix.

Zeta functions that arise from combinatorial settings are called *combinatorial zeta functions*. In [MS], we observe that combinatorial zeta functions whose determinant expressions are of the form $1/\det(I - A)$ should be constructed as Ruelle zeta functions [R] for essentially finite dynamical systems defined for finite digraphs. Koyama and Nakajima [KN] introduce their L-functions by Euler product, and show that its determinant formula is of the form $1/\det(I - uM)$ for a square matrix M. Hence their L-functions should be expressed as Ruelle zeta functions.

Note that, if the components of M are commutative, then it is well known that $1/\det(I - uM)$ equals

$$\exp\left(\sum_{m\geq 1} \frac{\mathrm{tr}\, M^m}{m} u^m\right)$$

where $\mathrm{tr}\, X$ denotes the trace of a square matrix X. Thus, one can always reformulate $1/\det(I - uM)$ into the generating function. So the problem treated in the present article lies in constructing a dynamical system which characterizes $\mathrm{tr}\, M^m$ in terms of its m-periodic points.

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In the sequel, $|X|$ denotes the cardinality of a set X, $\mathbf{Z}$ the set of integers, $\mathbf{Z}_P$ the set of integers satisfying a property P, and R a commutative $\mathbf{Q}$-algebra with an identity element 1. For each $n \in \mathbf{Z}_{\geq 1}$, $[n]$ denotes the finite set $\{1, 2, \ldots, n\}$.

2. *L*-functions. The symmetric group S_n of the set $[n]$ acts on the direct product $(\mathbf{Z}/r\mathbf{Z})^n$ of n copies of the cyclic group $\mathbf{Z}/r\mathbf{Z}$ of order $r \in \mathbf{Z}_{\geq 1}$. An element τ of the semi-direct product $G(r, n) = (\mathbf{Z}/r\mathbf{Z})^n \rtimes S_n$ is called a *complex reflection*. Thus there exists a unique $\sigma \in S_n$ and a unique sequence $(s_1, \ldots, s_n)$ of nonnegative integers smaller than r, satisfying $\tau = (\xi^{s_1}, \ldots, \xi^{s_n})\sigma \in G(r, n)$ where ξ is a primitive r-th root of unity. Let $\mathrm{Dom}(p)$ denote the cyclic domain for a cyclic permutation $p \in \mathrm{Cyc}(\sigma)$. For $\tau = (\xi^{s_1}, \ldots, \xi^{s_n})\sigma \in G(r, n)$ and $p \in \mathrm{Cyc}(\sigma)$, let $\chi_\tau(p) = \xi^{\sum_{i \in \mathrm{Dom}(p)} s_i}$.

Definition 1. Let $\tau \in G(r, n)$ and u an indeterminate. Then *Koyama-Nakajima's L-function* $L_\tau(u)$ is defined by the Euler product

$$\prod_{p \in \mathrm{Cyc}(\sigma)} \frac{1}{1 - \chi_\tau(p) u^{l(p)}}.$$

For example, if $\tau = (\xi^{s_1}, \ldots, \xi^{s_5})(123)(45) \in G(r, 5)$, then $\mathrm{Cyc}(\sigma) = \{(123), (45)\}$, and $L_\tau(u) = 1/\{(1 - \xi^{s_1+s_2+s_3}u^3)(1 - \xi^{s_4+s_5}u^2)\}$. Note that if $r = 1$ then $L_\tau(u)$ is nothing but the Artin-Mazur zeta function $Z_\sigma^{\mathrm{AM}}(u)$. A complex reflection $\tau \in G(r, n)$ is associated with a square matrix

$$M_\tau := (\xi^{s_i}\delta_{\sigma(i)j})_{1 \leq i, j \leq n}.$$

For example, the matrix M_τ representing $\tau = (\xi^{s_1}, \ldots, \xi^{s_5})(123)(45)$ is

$$\begin{pmatrix} 0 & \xi^{s_1} & 0 & 0 & 0 \\ 0 & 0 & \xi^{s_2} & 0 & 0 \\ \xi^{s_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi^{s_4} \\ 0 & 0 & 0 & \xi^{s_5} & 0 \end{pmatrix}.$$

If $r = 1$, then M_τ coincides with the permutation matrix M_σ corresponding to the permutation $\sigma \in S_n$.

Let $\tau \in G(r, n)$. Koyama and Nakajima [KN] show that the L-function $L_\tau(u)$ has the determinant expression

$$\frac{1}{\det(I - uM_\tau)}.$$

If $\tau = (\xi^{s_1}, \ldots, \xi^{s_5})(123)(45)$, then one can easily confirm the identity by direct caluculation. Note that, as is mentioned in Introduction, for a square matrix A whose components lie in a commutative ring R, the form $1/\det(I - uA)$ can always be reformulated in a generating function of exponential type, that is, if we let $N_m = \mathrm{tr}\, A^m$ for each $m \in \mathbf{Z}_{\geq 1}$, then the form equals

$$\exp\left(\sum_{m \geq 1} \frac{N_m}{m} u^m\right).$$

If one wants to show this, through the argument in the algebraic closure of R, then it is enough to consider the case where A is upper-triangular, and takes logarithm of both sides for the identity. It is, however, not trivial that the quantities N_m have an interpretation in terms of dynamical systems, as in the case of Artin-Mazur zeta function.

3. Dynamical systems. Let $\sigma \in S_n$, and $X = [n]$. Then we have a finite discrete dynamical system (X, σ). This dynamical system is called a $\mathbf{Z}$-*dynamical system*. Let R be a commutative $\mathbf{Q}$-algebra with the identity element 1. A map

$$w : X \to R$$

is called a *weight map* of (X, σ), or simply a *weight* of X. Let $(X, \sigma; w)$ be a weighted $\mathbf{Z}$-dynamical system with a weight map $w : X \to R$. For $x \in X$ and $m \in \mathbf{Z}_{\geq 1}$, let $w_m(x) = \delta_{\sigma^m(x)x} \prod_{k=0}^{m-1} w(\sigma^k(x))$, and $N_m(w) = \sum_{x \in X} w_m(x)$. Let u be an indeterminate. The formal power series

$$\exp\left(\sum_{m \geq 1} \frac{N_m(w)}{m} u^m\right)$$

is called the *Ruelle zeta function* [R] for the weighted $\mathbf{Z}$-dynamical system $(X, \sigma; w)$, denoted by $Z^{\mathrm{R}}_{(X,\sigma)}(u; w)$, or simply $Z_\sigma(u; w)$. The Ruelle zeta function $Z^{\mathrm{R}}_{(X,\sigma)}(u; w)$ is an element of the ring $R[\![u]\!]$ of formal power series the coefficients of which lie in R. Note that $N_m(1)$ equals the number $|\mathrm{Fix}(\sigma^m)|$ of m-periodic points. Therefore we have $Z_\sigma(u; w) = Z_\sigma(u)$ if $w = 1$. For $p \in \mathrm{Cyc}(\sigma)$, we define $w(p) = \prod_{i \in \mathrm{Dom}(p)} w(i)$.

Lemma 2. *For each integer $m \geq 1$, it follows that* $N_m = \displaystyle\sum_{\substack{p \in \mathrm{Cyc}(\sigma) \\ l(p) | m}} l(p) w(p)^{m/l(p)}$.

Proof. Since $w_m(x) = 0$ if $x \neq \mathrm{Fix}(\sigma^m)$, it follows that $N_m(w) = \sum_{x \in \mathrm{Fix}(\sigma^m)} \prod_{k=0}^{m-1} w(\sigma^k(x))$. One can easily see that an element $i \in X$ belongs to

$\mathrm{Fix}(\sigma^m)$ if and only if there exists $p \in \mathrm{Cyc}(\sigma)$ satisfying $i \in \mathrm{Dom}(p)$ and $l(p)|m$. Thus one has a disjoint union

$$\mathrm{Fix}(\sigma^m) = \bigsqcup_{\substack{p\in\mathrm{Cyc}(\sigma)\\ l(p)|m}} \mathrm{Dom}(p).$$

Let $x \in \mathrm{Fix}(\sigma^m)$, and suppose that $x \in \mathrm{Dom}(p)$ for $p \in \mathrm{Cyc}(\sigma)$ satisfying $l(p)|m$. It follows from the assumption that $\prod_{k=0}^{m-1} w(\sigma^k(x)) = l(p)w(p)^{m/l(p)}$. This completes the proof. □

Let $\tau = (\xi^{s_1}, \dots, \xi^{s_n})\sigma \in G(r, n)$ and $X = [n]$. Then τ defines a weighted $\mathbf{Z}$-dynamical system $(X, \sigma; w)$, where the weight map is defined by $w(i) = \xi^{s_i}$, $i \in X$. Consider the case where $n = 5$ and $\sigma = (123)(45)$. We have $\mathrm{Fix}(\sigma^1) = \emptyset$, $\mathrm{Fix}(\sigma^2) = \{4, 5\}$, $\mathrm{Fix}(\sigma^3) = \{1, 2, 3\}$, $\mathrm{Fix}(\sigma^4) = \{4, 5\}$, $\mathrm{Fix}(\sigma^5) = \emptyset$, $\mathrm{Fix}(\sigma^6) = \{1, 2, 3, 4, 5\}$, and so on. We shall examine the value of $N_m(w)$ in this case. One can readily see that $w_1(x) = 0$ for any $x \in X$ and $N_1(w) = 0$. If $m = 2$, then it follows that $w_2(1) = w_2(2) = w_2(3) = 0$, $w_2(4) = \xi^{s_4}\xi^{s_5} = \xi^{s_4+s_5}$ and $w_2(5) = \xi^{s_5}\xi^{s_4} = \xi^{s_4+s_5}$. Hence we have $N_2(w) = 2\xi^{s_4+s_5}$. Note that the coefficient 2 coincides with the number of 2-periodic points of (X, σ). Similar inspection shows that $N_3(w) = 3\xi^{s_1+s_2+s_3}$, $N_4(w) = 2\xi^{2(s_4+s_5)}$ and $N_5(w) = 0$. In the case where $m = 6$, we have $w_6(1) = w_6(2) = w_6(3) = \xi^{2(s_1+s_2+s_3)}$ and $w_6(4) = w_6(5) = \xi^{3(s_4+s_5)}$. Hence we have $N_6(w) = 3\xi^{2(s_1+s_2+s_3)} + 2\xi^{3(s_4+s_5)}$. One should notice here again that the coefficient 3 (resp. 2) coincides with the number of 3-periodic points (resp. 2-periodic points) of (X, σ).

4. Foata-Zeilberger. A theorem of Foata and Zeilberger [FZ] gives an Euler product to the form $1/\det(I - M)$ in a general setting. Let $X = \{1, 2, \dots, n\}$ be a finite alphabet of n letters, totally ordered by $1 < 2 < \cdots < n$. Let X^* denote the free monoid generated by X. An element of X^* is called a *word* on X. The monoid X^* is also totally ordered by the lexicographical order induced from the total order $<$ on X. Let $w = i_1 i_2 \cdots i_r \in X^*$ be a word. A set $\mathrm{Re}(w) = \{i_1 i_2 \cdots i_r, i_2 i \cdots, i_r i_1, \dots, i_r i_1 \cdots i_{r-1}\}$ of r words is called the *cyclic rearrangement class* of w. Remark that a cyclic rearrangement class of a word is a multiset in general. A word $w \in X^*$ is called a *Lyndon word* if w is the unique minimum in $\mathrm{Re}(w)$. The set of Lyndon words in X^* is denoted by $L = L(X)$. For example, in the case where $X = \{1, 2, 3\}$, $1212 \notin L, 1312 \notin L$ and $1213 \in L$. The Lyndon words are the "primes" of the monoid X^* in the following sense. The factorization of w stated in Proposition 3 is called the *Lyndon factorization* of w. A proof of the following will be found in [L].

Proposition 3 (The Lyndon Factorization Theorem, LFT)**.** *Let w be a word on X. Then there exists a unique non-increasing sequence $(l_{k_1}, l_{k_2}, \dots, l_{k_r})$ of Lyndon words on X satisfying $w = l_{k_1} l_{k_2} \dots l_{k_r}$.*

Let $[L] = \{[l] \mid l \in L\}$ be a set of commutative variables which is in one-to-one correspondence with L, and let A denote the $\mathbf{Z}$-algebra $\mathbf{Z}[\![[l] \mid l \in L]\!]$ of formal power series generated by $[L]$. Let $M = (m_{ij})_{i,j\in X}$ be a square matrix whose components lie in a commutative $\mathbf{Q}$-algebra R. Note that the size of the matrix M is $n \times n$. Let B be the $\mathbf{Z}$-algebra $\mathbf{Z}[\![m_{ij} \mid i, j \in X]\!]$ of formal power series generated by the components of M. For a word $w = i_1 i_2 \cdots i_r \in X^*$, we denote the element $m_{i_1 i_2} m_{i_2 i_3} \cdots m_{i_r i_1} \in B$ by $\mathrm{circ}_M(w)$:

$$\mathrm{circ}_M(w) = m_{i_1 i_2} m_{i_2 i_3} \cdots m_{i_r i_1}.$$

Since the algebras A and B are commutative, we can define a ring homomorphism in the following manner:

$$\varphi_M : A \to B : [l] \mapsto \mathrm{circ}_M(l).$$

Let Λ be an element of A defined by $\Lambda = \prod_{l\in L}(1 - [l])$. Note that Λ is invertible in the algebra A: $\Lambda^{-1} = \prod_{l\in L}(1 + [l] + [l]^2 + \cdots) \in A$. Foata-Zeilberger's thorem states that the form $1/\det(I - M)$ is the image of Λ^{-1} by φ_M.

Proposition 4 (Foata-Zeilberger)**.** $\varphi_M(\Lambda) = \det(I - M)$.

Since Λ is invertible and φ_M is an algebra homomorphism, it follows that $\varphi_M(\Lambda^{-1}) = 1/\det(I - M)$. Thus we have

$$\frac{1}{\det(I - M)} = \prod_{l\in L} \frac{1}{1 - \mathrm{circ}_M(l)},$$

that is, the form $1/\det(I - M)$ can always be reformulated into an Euler product.

5. Main results. Let $X = [n]$ and $(X, \sigma; w)$ a $\mathbf{Z}$-dynamical system. Suppose $p \in \mathrm{Cyc}(\sigma)$. Let $w(p) = \prod_{i\in\mathrm{Dom}(p)} w(i)$. The Ruelle zeta function $Z_\sigma(u; w)$ has the following Euler product.

Theorem 5. $Z_\sigma(u; w) = \displaystyle\prod_{p\in\mathrm{Cyc}(\sigma)} \frac{1}{1 - w(p)u^{l(p)}}$.

Proof. By Lemma 2, we have

$$\sum_{m\geq 1}\frac{N_m}{m}u^m=\sum_{m\geq 1}\sum_{\substack{p\in\mathrm{Cyc}(\sigma)\\ l(p)|m}}\frac{l(p)}{m}w(p)^{m/l(p)}u^m.$$

Since the sum in the right-hand side ranges all over $\mathrm{Cyc}(\sigma)$, it follows that this equals

$$\begin{aligned}&\sum_{p\in\mathrm{Cyc}(\sigma)}\sum_{k\geq 1}\frac{1}{k}w(p)^k u^{kl(p)}\\ &\quad=-\sum_{p\in\mathrm{Cyc}(\sigma)}\log(1-w(p)u^{l(p)}).\end{aligned}$$

Therefore we have

$$\begin{aligned}Z_\sigma(u;w)&=\exp\left(\sum_{m\geq 1}\frac{N_m}{m}u^m\right)\\ &=\prod_{p\in\mathrm{Cyc}(\sigma)}\frac{1}{1-w(p)u^{l(p)}}.\end{aligned}$$

□

We assign a matrix $M_\sigma(w)$ to the weighted $\mathbf{Z}$-dynamical sysytem $(X,\sigma;w)$, defined by $M_\sigma(w)=(w(i)\delta_{\sigma(i)j})_{i,j\in X}$. In the case where $X=[5]$ and $\sigma=(123)(45)$, the matrix $M_\sigma(w)$ is

$$\begin{pmatrix}0&w(1)&0&0&0\\0&0&w(2)&0&0\\w(3)&0&0&0&0\\0&0&0&0&w(4)\\0&0&0&w(5)&0\end{pmatrix},$$

where $w(i)\in R$ for $i\in X$. One can observe that the reciprocal of the determinant $\det(I-uM_\sigma(w))$ coincides with the Euler product $(1-w((123))u^3)^{-1}(1-w((45))u^2)^{-1}$, that is, by Theorem 5, we have a determinant expression

$$Z_{(123)(45)}(u;w)=\frac{1}{\det(I-uM_{(123)(45)}(w))}.$$

This identity actually holds in a general case.

Theorem 6. *For any $\sigma\in S_n$, it holds that*

$$Z_\sigma(u;w)=\frac{1}{\det(I-uM_\sigma(w))}.$$

Proof. Let $X=[n]$. We know that

$$\frac{1}{\det(I-uM_\sigma(w))}=\prod_{l\in L(X)}\frac{1}{1-\mathrm{circ}_{uM_\sigma(w)}(l)}.$$

Since each cyclic permutation can be regarded as an element of X^*, the set $\mathrm{Cyc}(\sigma)$ is considered to be a subset of X^*. Suppose that $l=i_1i_2\cdots i_r$ is a Lyndon word on X. By the definition of $M=M_\sigma(w)$, it follows that $\mathrm{circ}_{uM}(l)\neq 0$ if and only if $l\in\mathrm{Cyc}(\sigma)$. Hence we have

$$\prod_{l\in L(X)}\frac{1}{1-\mathrm{circ}_{uM}(l)}=\prod_{p\in\mathrm{Cyc}(\sigma)}\frac{1}{1-\mathrm{circ}_{uM}(p)}.$$

Since $\mathrm{circ}_{uM}(p)=\mathrm{circ}_M(p)u^{l(p)}$, the right hand side equals

$$\prod_{p\in\mathrm{Cyc}(\sigma)}\frac{1}{1-\mathrm{circ}_M(p)u^{l(p)}}.$$

Note that $\mathrm{circ}_M(p)=w(p)$. Then by Theorem 5, the assertion follows. □

Let $\tau\in G(r,n)$ and ξ a primitive r-th root of unity. Since $G(r,n)=(\mathbf{Z}/r\mathbf{Z})^n\rtimes S_n$, there exists a unique sequence $(s_1,\ldots,s_n)$ of nonnegative integers smaller than r and a unique element σ of S_n satisfying $\tau=(\xi^{s_1},\ldots,\xi^{s_n})\sigma$. Then, as in Section 3, τ settles the weighted $\mathbf{Z}$-dynamical system $(X,\sigma;w)$, where the weight map w is defined by $w(i)=\xi^{s_i}$. On the dynamical system $(X,\sigma;w)$, one has $M_\sigma(w)=M_\tau$. Thus we have an expression of Koyama-Nakajima's L-functions $L_\tau(u)$ in the form of generating functions of exponential type, founded on a dynamical systematic setting.

Corollary 7. *Let $(X,\sigma;w)$ be the weighted $\mathbf{Z}$-dynamical system determined by $\tau\in G(r,n)$, say $\tau=(\xi^{s_1},\ldots,\xi^{s_n})\sigma$ as in Section 2. Then we have $L_\tau(u)=Z^R_{(X,\sigma)}(u;w)$, that is, Koyama-Nakajima's L-function associated with τ is the Ruelle zeta function for the weighted $\mathbf{Z}$-dynamical system $(X,\sigma;w)$.*

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