On radial distributions of Julia sets of Newton's method of solutions of complex differential equations

By Guowei ZHANG*) and Lianzhong YANG**)

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Abstract: In this paper we mainly investigate the radial distribution of Julia sets of Newton's method of entire solutions of some complex linear differential equations. Under certain conditions, we find the lower bound of it and also obtain some related results.

Key words: Radial distribution; Julia set; Newton's method; complex differential equation.

1. Introduction and main results. In this paper, we assume the reader is familiar with standard notations and basic results of Nevanlinna's value distribution theory; see [7,9,12,19,21]. Let f be a meromorphic function in the whole complex plane. We use $\sigma(f)$, $\mu(f)$ to denote the order and lower order of f respectively, and $\delta(a,f)(\Theta(a,f))$ to denote the (simplified) Nevanlinna deficient value of f at a; see [9,21] for the definitions. Nevanlinna theory plays an important role in the study of complex differential equations; see [12].

Let u be a solution of the second order complex linear differential equation

$$u'' + A(z)u = 0, \tag{1}$$

where $A(z)$ is a transcendental entire function with finite order. It's well known that every nontrivial solution of (1) is entire and of infinite order. We recall that for any nonconstant meromorphic function u, the Newton's method of finding the zeros of u consists of iterating the function f defined by

$$f(z) = z - \frac{u(z)}{u'(z)}. \tag{2}$$

Then zeros of u are then attracting fixed points of f, and the simple zeros of u are even super attracting fixed points of f. Note that if u satisfies (1) and f is defined by (2), then f satisfies the following Riccati equation

$$f'(z) + A(z)(f(z) - z)^2 = 0, \tag{3}$$

where $A(z)$ is defined in (1).

In this article we should also know some basic knowledge of complex dynamics of meromorphic functions; see [5,23]. We define $f^n, n \in \mathbf{N}$, denote the nth iterate of f. The Fatou set $F(f)$ of transcendental meromorphic function f is the subset of the plane $\mathbf{C}$ where the iterates f^n of f form a normal family. The complement of $F(f)$ in $\mathbf{C}$ is called the Julia set $J(f)$ of f. It is well known that $F(f)$ is open and completely invariant under f, $J(f)$ is closed and non-empty.

We denote $\Omega(\alpha,\beta) = \{z \in \mathbf{C} | \arg z \in (\alpha,\beta)\}$, where $0 < \alpha < \beta < 2\pi$. Given $\theta \in [0, 2\pi)$, if $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap J(f)$ is unbounded for any $\varepsilon > 0$, then we call the ray $\arg z = \theta$ the radial distribution of $J(f)$. Define $\Delta(f) = \{\theta \in [0, 2\pi) | \ argz = \theta$ is the radial distribution of $J(f)\}$.

Obviously, $\Delta(f)$ is closed and so measurable. We use the $mes\Delta(f)$ to denote the linear measure of $\Delta(f)$. Many important results of radial distributions of transcendental meromorphic functions have been obtained, for example [2,13–16,24]. Qiao [13] proved that $mes\Delta(f) = 2\pi$ if $\mu(f) < 1/2$ and $mes\Delta(f) \geq \pi/\mu(f)$ if $\mu(f) \geq 1/2$, where $f(z)$ is a transcendental entire function of finite lower order. Recently, Huang and Wang [10,11] considered radial distributions of Julia sets of entire solutions of linear complex differential equations. Their results are stated as follows:

Theorem A ([10]). *Let* $\{f_1, f_2, \dots, f_n\}$ *be a solution base of*

$$f^{(n)} + A(z)f = 0, \tag{4}$$

where $A(z)$ *is a transcendental entire function with*

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*) School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455000, China.

**) School of Mathematics, Shandong University, Jinan, Shandong 250100, China.

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finite order, and denote $E = f_1 f_2 \ldots f_n$. *Then* $mes\Delta(E) \geq \min\{2\pi, \pi/\sigma(A)\}$.

Theorem B ([11])**.** *Let* $A_i(z)(i = 0, 1, \ldots, n-1)$ *be entire functions of finite lower order such that* A_0 *is transcendental and* $m(r, A_i) = o(m(r, A_0))$, $(i = 1, 2, \ldots, n-1)$ *as* $r \to \infty$. *Then every non-trivial solution* f *of the equation*

$$f^{(n)} + A_{n-1} f^{(n-1)} + \ldots + A_0 f = 0 \tag{5}$$

satisfies $mes\Delta(f) \geq \min\{2\pi, \pi/\mu(A_0)\}$.

We [22] extended the above results and studied the radial distribution of Julia sets of the derivatives of entire solutions of equations (4) and (5). Indeed, we obtained the following results.

Theorem C. *Let* $A_i(z)(i = 0, 1, \ldots, n-1)$ *be entire functions of finite lower order such that* A_0 *is transcendental and* $m(r, A_i) = o(m(r, A_0))$, $(i = 1, 2, \ldots, n-1)$ *as* $r \to \infty$. *Then every non-trivial solution* f *of the equation* (5) *satisfies* $mes(\Delta(f) \cap \Delta(f^{(k)})) \geq \min\{2\pi, \pi/\mu(A_0)\}$, *where* k *is a positive integer.*

Theorem D. *Under the hypothesis of Theorem A, we have* $mes\Delta(E^{(k)}) \geq \min\{2\pi, \pi/\sigma(A)\}$, *where* k *is a positive integer.*

Based on these results, we shall do some further researches in this direction. Our main purpose of this paper is to investigate the radial distributions of the Julia sets of Newton's method of entire solutions of differential equation (1), that is, we shall study the radial distributions of Julia sets of the meromorphic solutions of Riccati equation (3). First of all, we consider the complex dynamical properties of solutions to second order linear differential equations with polynomial coefficients and obtain the following two remarks.

Remark 1.1. Suppose that $p(z)$ is a nonconstant polynomial with degree n, and u is a nontrivial solution of differential equation

$$u'' + p(z)u = 0, \tag{6}$$

then $mes\Delta(u) \geq \frac{2\pi}{n+2}$. In fact, by Lemma 2.8 in section 2 every nontrivial solution u of (6) is entire with order $1 + n/2$. Applying Lemma 2.6 to u, then there exists an angular domain $\Omega(\theta_1, \theta_2)$ with $\theta_2 - \theta_1 \geq 2\pi/(n+2)$ such that $\sigma_\theta(u) = \sigma(u)$ for any $\theta \in (\theta_1, \theta_2)$. Finally, by Lemma 2.5, we have for any $\theta \in (\theta_1, \theta_2)$, $\arg z = \theta$ is a radial distribution of $J(u)$. Thus, we get our assertion.

Remark 1.2. Suppose that u is a nontrivial solution of equation (6) and that f is the Newton's method of u, then for any $\alpha \in \hat{\mathbf{C}}$, where $\widehat{\mathbf{C}}$ is the extended complex plane, we have $\delta(\alpha, f) = 0$. Clearly, the Newton's method f of u satisfies the Riccati equation

$$f'(z) + p(z)(f(z) - z)^2 = 0, \tag{7}$$

where $p(z)$ is a nonconstant polynomial. By the work of Wittich [18, pp. 73–80], the solutions of (7) are meromorphic in the complex plane, and every non-rational solution has order of growth $\sigma(f) = 1 + n/2$, where the non-negative integer n depends on the coefficients $p(z)$ only. Since $p(z) \not\equiv 0$, then for any constant $\alpha \in \hat{\mathbf{C}}$, $p(z)(\alpha - z)^2 \not\equiv 0$. Thus, by Lemma 2.7, we have $\delta(\alpha, f) = 0$ for any $\alpha \in \hat{\mathbf{C}}$.

For the solutions of differential equations with transcendental coefficients, we have the results below.

Theorem 1.1. *Suppose that* f *is transcendental meromorphic solution of Riccati equation* (3), *where* $A(z)$ *is a transcendental entire function with finite order* $\sigma(A)$. *Then* $mes\Delta(f) \geq \min\{2\pi, \pi/\sigma(A)\}$.

Example 1.1. It's clear that entire function $f(z) = z + e^{-z}$ satisfies Riccati equation (3), where $A(z) = e^z - e^{2z}$ with $\sigma(A) = 1$, by Theorem 1.1 we have $mes\Delta(f) \geq \pi$. From [3, Section 5] we know the lines $y = (2n \pm 1)\pi, n \in \mathbf{Z}$ are in $J(f)$, thus $mes\Delta(f) = 2\pi$ which coincides with our result.

Corollary 1.1. *Suppose that* u *is a nontrivial solution of linear differential equation* (1), *where* $A(z)$ *is transcendental entire function with finite order* $\sigma(A)$, *and* u/u' *is transcendental meromorphic. Then the Newton's method* f *of* u *satisfies* $mes\Delta(f) \geq \min\{2\pi, \pi/\sigma(A)\}$.

In the value distribution theory of entire functions, Brück's conjecture [6] is still an open question, and it has close relation to the following nonhomogeneous complex linear differential equation,

$$f' + e^{p(z)} f = 1, \tag{8}$$

where $p(z)$ is an entire function. It's known that every nontrivial solution of (8) is of infinite order; see [8]. Using the same method in the proof of Theorem 1.1, we can obtain a result about the radial distributions of the Julia sets of solutions of (8).

Theorem 1.2. *Suppose that* f *is a transcendental entire solution of equation* (8), *then* $mes\Delta(f) \geq \min\{2\pi, \pi/\sigma(e^{p(z)})\}$.

2. Preliminary lemmas. We recall the Nevanlinna characteristic in an angle firstly; see [7]. We set

$$\Omega(\alpha,\beta,r)=\{z:z\in\Omega(\alpha,\beta),|z|<r\};$$
$$\Omega(r,\alpha,\beta)=\{z:z\in\Omega(\alpha,\beta),|z|\geq r\}$$

and denote by $\overline{\Omega}(\alpha,\beta)$ the closure of $\Omega(\alpha,\beta)$. Let $g(z)$ be meromorphic on the angle $\overline{\Omega}(\alpha,\beta)$, where $\beta-\alpha\in(0,2\pi]$. Following [7], we define

$$A_{\alpha,\beta}(r,g)=\frac{w}{\pi}\int_1^r\left(\frac{1}{t^w}-\frac{t^w}{r^{2w}}\right)\times\{\log^+|g(te^{i\alpha})|+\log^+|g(te^{i\beta})|\}\frac{dt}{t};$$

$$B_{\alpha,\beta}(r,g)=\frac{2w}{\pi r^w}\int_\alpha^\beta\log^+|g(re^{i\theta})|\sin w(\theta-\alpha)d\theta;$$

$$C_{\alpha,\beta}(r,g)=2\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^w}-\frac{|b_n|^w}{r^{2w}}\right)\sin w(\beta_n-\alpha),$$

where $w=\pi/(\beta-\alpha)$, and $b_n=|b_n|e^{i\beta_n}$ are poles of $g(z)$ in $\overline{\Omega}(\alpha,\beta)$ appearing according to their multiplicities. The Nevanlinna angular characteristic is defined as

$$S_{\alpha,\beta}(r,g)=A_{\alpha,\beta}(r,g)+B_{\alpha,\beta}(r,g)+C_{\alpha,\beta}(r,g).$$

In particular, we denote the order of $S_{\alpha,\beta}(r,g)$ by

$$\sigma_{\alpha,\beta}(g)=\limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,g)}{\log r}.$$

Set $M(r,\Omega(\alpha,\beta),g)=\sup_{\alpha\leq\theta\leq\beta}|g(re^{i\theta})|$, then we define the sectorial, respectively radial, order of growth for entire function $g(z)$, as

$$\sigma_{\theta,\varepsilon}(g)=\limsup_{r\to\infty}\frac{\log^+\log^+M(r,\Omega(\theta-\varepsilon,\theta+\varepsilon),g)}{\log r},$$
$$\sigma_\theta(g)=\lim_{\varepsilon\to0}\sigma_{\theta,\varepsilon}(g).$$

Equivalently, for $\theta\in[0,2\pi)$, $\sigma_\theta(g)$ can also be defined as

$$\sigma_\theta(g)=\limsup_{r\to\infty}\frac{\log^+\log^+|g(re^{i\theta})|}{\log r}.$$

The following lemma shows some estimates for the logarithmic derivative of functions being analytic in an angle. Before this, we recall the definition of an R-set; for reference, see [12]. Set $B(z_n,r_n)=\{z:|z-z_n|<r_n\}$. If $\sum_{n=1}^\infty r_n<\infty$ and $z_n\to\infty$, then $\cup_{n=1}^\infty B(z_n,r_n)$ is called an R-set. Clearly, the set $\{|z|:z\in\cup_{n=1}^\infty B(z_n,r_n)\}$ is of finite linear measure.

Lemma 2.1 ([10]). *Let* $z=re^{i\psi},r_0+1<r$ *and* $\alpha\leq\psi\leq\beta$, *where* $0<\beta-\alpha\leq2\pi$. *If* $g(z)$ *is meromorphic in* $\Omega(r_0,\alpha,\beta)$ *with* $\sigma_{\alpha,\beta}(g)<\infty$, *then there exist* $K>0$ *and* $M>0$ *depending only on* g *and* $\Omega(r_0,\alpha,\beta)$, *and not depending on* z, *such that*

$$\left|\frac{g'(z)}{g(z)}\right|\leq Kr^M(\sin k(\psi-\alpha))^{-2}$$

for all $z\in\Omega(r_0,\alpha,\beta)$ *outside an* R*-set* D, *where* $k=\pi/(\beta-\alpha)$.

Lemma 2.2 ([20, 23]). *Let* $f(z)$ *be a transcendental meromorphic function with lower order* $\mu(f)<\infty$ *and order* $0<\sigma(f)\leq\infty$. *Then, for any positive number* λ *with* $\mu(f)\leq\lambda\leq\sigma(f)$ *and any set* H *of finite measure, there exists a sequence* $\{r_n\}$ *satisfies*

(1) $r_n\notin H,\lim_{n\to\infty}r_n/n=\infty$;
(2) $\liminf_{n\to\infty}\log T(r_n,f)/\log r_n\geq\lambda$;
(3) $T(r,f)<(1+o(1))(2t/r_n)^\lambda T(r_n/2,f),t\in[r_n/n,nr_n]$;
(4) $t^{-\lambda-\varepsilon_n}T(t,f)\leq2^{\lambda+1}r_n^{-\lambda-\varepsilon_n}T(r_n,f),1\leq t\leq nr_n$, $\varepsilon_n=(\log n)^{-2}$.

Such $\{r_n\}$ is called a sequence of Pólya peaks of order λ outside H. The following lemma, which is related to Pólya peaks, is called the spread relation; see [1].

Lemma 2.3 ([1]). *Let* $f(z)$ *be a transcendental meromorphic function with positive order and finite lower order, and has a deficient value* $a\in\widehat{\mathbf{C}}$. *Then, for any sequence of Pólya peaks* $\{r_n\}$ *of order* $\lambda>0$, $\mu(f)\leq\lambda\leq\sigma(f)$, *and any positive function* $\Upsilon(r)\to0$ *as* $r_n\to\infty$, *we have*

$$\liminf_{r_n\to\infty}mesD_\Upsilon(r_n,a)\geq\min\left\{2\pi,\frac{4}{\lambda}\arcsin\sqrt{\frac{\delta(a,f)}{2}}\right\},$$

where

$$D_\Upsilon(r,a)=\left\{\theta\in[0,2\pi):\log^+\frac{1}{|f(re^{i\theta})-a|}>\Upsilon(r)T(r,f)\right\},\quad a\in\mathbf{C}$$

and

$$D_\Upsilon(r,\infty)=\{\theta\in[0,2\pi):\log^+|f(re^{i\theta})|>\Upsilon(r)T(r,f)\}.$$

We call W a hyperbolic domain if $\widehat{\mathbf{C}}\backslash W$ contains at least three points, where $\widehat{\mathbf{C}}$ is the extended complex plane. For an $a\in\mathbf{C}\backslash W$, define $C_W(a)=\inf\{\lambda_W(z)|z-a|:\forall z\in W\}$, where $\lambda_W(z)$ is the hyperbolic density on W. It is well known that if every component of W is simply connected, then $C_W(a) \geq 1/2$; see [23, p. 84]. For a finite number $a \in J(f)$, if there is a component U in $F(f)$ such that $C_U(a) > 0$, then we call $C_{F(f)}(a) > 0$, where $f(z)$ is a transcendental meromorphic function in $\mathbf{C}$.

Lemma 2.4 ([24, Lemma 2.2]). *Let $f(z)$ be analytic in $\Omega(r_0, \theta_1, \theta_2)$, U be a hyperbolic domain, and $f : \Omega(r_0, \theta_1, \theta_2) \to U$. If there exists a point $a \in \partial U \backslash \{\infty\}$ such that $C_U(a) > 0$, then there exists a constant $d > 0$ such that, for sufficiently small $\varepsilon > 0$, we have*

$$|f(z)| = O(|z|^d), \ z \to \infty, \ z \in \Omega(r_0, \theta_1 + \varepsilon, \theta_2 - \varepsilon).$$

Lemma 2.5. *Let $f(z)$ be a transcendental entire function. If $\sigma_\theta(f) = \sigma(f)$, then $\arg z = \theta$ is a radial distribution of the Julia set of f.*

Proof. If $J(f) = \mathbf{C}$, it's obvious. If $J(f) \neq \mathbf{C}$, suppose that $\arg z = \theta$ is not a radial distribution of $J(f)$. Since the complement of $\Delta(f)$ is open set, then there exists an open interval $(\alpha, \beta) \in [0, 2\pi)$ satisfy $(\alpha, \beta) \cap \Delta(f) = \emptyset$ and $\Omega(r, \alpha, \beta) \cap J(f) = \emptyset$. Thus, there exists an unbounded Fatou component U of $F(f)$ such that $\Omega(r, \alpha, \beta) \subset U$. By the above Lemma 2.4, we have $|f(z)| = O(|z|^d)$, $z \in \Omega(r, \alpha, \beta)$, where d is a positive constant. This contradicts to $\sigma_\theta(f) = \sigma(f) = \limsup_{r\to\infty} \frac{\log^+ |f(re^{i\theta})|}{\log r} = \infty$ since f is transcendental. $\square$

Lemma 2.6 ([17, Corollary 2.3.6]). *If $g(z)$ is an entire function with $0 < \sigma(g) < \infty$, then there exists an angular domain $\Omega(\theta_1, \theta_2)$ with $\theta_2 - \theta_1 \geq \pi/\sigma(g)$ such that $\sigma_\theta(g) = \sigma(g)$ for any $\theta \in (\theta_1, \theta_2)$.*

Lemma 2.7 ([12, Theorem 9.1.12]). *Let $f(z)$ be a meromorphic solution of Riccati differential equations*

$$f' = a_0(z) + a_1(z)f + a_2(z)f^2 \tag{9}$$

with meromorphic coefficients such that $T(r, a_i) = S(r, f)$ holds for $i = 0, 1, 2$. Then $\delta(\alpha, f) = 0$ for $\alpha = \infty$ and for all $\alpha \in \mathbf{C}$ such that $a_0(z) + \alpha a_1(z) + \alpha^2 a_2(z) \not\equiv 0$. If $a_0(z) + \alpha a_1(z) + \alpha^2 a_2(z) \equiv 0$, then $\Theta(\alpha, f) = 1$.

Lemma 2.8 ([12, Proposition 5.1]). *All nontrivial solution f of differential equation $f'' + p(z)f = 0$, where $p(z)$ is a polynomial with degree n, has the order of growth $\sigma(f) = \frac{n+2}{2}$.*

Lemma 2.9 ([4, Corollary]). *If g is an entire map and N_g is its Newton's method, then $J(N_g)$ is connected.*

Since the Julia set is closed, it is connected if and only if all the Fatou components are simply connected.

3. Proof of Theorem 1.1. The arguments of this proof is referred to that in [11], but should make some essential modifications since we are treating with nonlinear case. In fact, we choose a new $\Upsilon(r)$ function and properly estimate the module of item $\frac{f}{(f-z)^2}$. In the following, we shall obtain the assertion by reduction to contradiction. Assume that

$$mes\Delta(f) < \nu = \min\{2\pi, \pi/\sigma(A)\} \tag{10}$$

and set

$$\zeta := \nu - mes\Delta(f) > 0. \tag{11}$$

Since $\Delta(f)$ is closed, clearly $S = (0, 2\pi)\backslash\Delta(f)$ is open, so it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)$ $(i = 1, 2, \cdots, m)$ satisfying $[\alpha_i, \beta_i] \subset S$ and $mes(S \backslash \cup_{i=1}^m I_i) < \zeta/4$. For the angular domain $\Omega(\alpha_i, \beta_i)$, it is easy to see that

$$(\alpha_i, \beta_i) \cap \Delta(f) = \emptyset, \quad \Omega(r, \alpha_i, \beta_i) \cap J(f) = \emptyset$$

for sufficiently large r. This implies that, for each $i = 1, 2, \cdots, m$, there exist corresponding r_i and unbounded Fatou component U_i of $F(f)$ such that $\Omega(r, \alpha_i, \beta_i) \subset U_i$. Since the poles of f are in the set $J(f)$; see [5, Section 2.1], then f does not have poles in $\Omega(r, \alpha_i, \beta_i)$. Thus the mapping $f : \Omega(r, \alpha_i, \beta_i) \to f(U_i)$ is analytic. Moreover, $F(f)$ is invariant under f, by Lemma 2.9, all components of $f(U_i)$ are simply connected, then applying Lemma 2.4 to f in every $\Omega(r, \alpha_i, \beta_i)$, there exists a positive constant d such that, for $z \in \cup_{i=1}^m \Omega(r, \alpha_i + \varepsilon, \beta_i - \varepsilon)$, we have

$$|f(z)| = O(|z|^d), \tag{12}$$

as $|z| \to \infty$, ε is sufficiently small.

Applying Lemma 2.3 to $A(z)$, we have a Pólya peak $\{r_j\}$ of order $\sigma(A)$ with all $r_j \notin H$. Since $A(z)$ is transcendental entire function, it follows that the Nevanlinna deficient $\delta(\infty, A) = 1$. By Lemma 2.3, for the Pólya peak $\{r_j\}$, we have

$$\liminf_{r_j\to\infty} mes(D_\Upsilon(r_j, \infty)) \geq \pi/\sigma(A), \tag{13}$$

where the function $\Upsilon(r)$ is defined by

$$\Upsilon(r) = \sqrt{\frac{\log r}{m(r, A)}} \tag{14}$$

and $m(r, A)$ is the proximation function of $A(z)$. Obviously, $\Upsilon(r)$ is positive and $\lim_{r\to\infty} \Upsilon(r) = 0$.

For the sake of simplicity, we denote $D_\Upsilon(r_j,\infty)$ by $D(r_j)$ in the following part.

Therefore, for sufficiently large j, we have

$$mes(D(r_j)) > \frac{\pi}{\sigma(A)} - \frac{\zeta}{4}. \tag{15}$$

Clearly,

$$\begin{aligned} mes(D(r_j)\cap S) &= mes(D(r_j)\backslash(\Delta(f)\cap D(r_j))) \\ &\geq mes(D(r_j)) - mes\Delta(f) \\ &> \frac{3\zeta}{4} > 0. \end{aligned} \tag{16}$$

Then, for each j, we have

$$\begin{aligned} mes(D(r_j)\cap(\cup_{i=1}^m I_i)) &= mes(D(r_j)\cap S) \\ - mes(D(r_j)\cap(S\backslash \cup_{i=1}^m I_i)) &> \frac{3\zeta}{4} - \frac{\zeta}{4} = \frac{\zeta}{2}. \end{aligned} \tag{17}$$

Thus, there exists an open interval $I_{i_0} = (\alpha,\beta) \subset \cup_{i=1}^m I_i \subset S$ such that, for infinitely many j,

$$mes(D(r_j)\cap(\alpha,\beta)) > \frac{\zeta}{2m}. \tag{18}$$

Without loss of generality, we can assume that (18) holds for all j. It follows from the definition of $D(r_j)$ and (18) that

$$\begin{aligned} \int_{F_j} \log^+|A(r_je^{i\theta})|d\theta &\geq mes(F_j)\Upsilon(r_j)m(r_j,A) \\ &\geq \frac{\zeta}{4m}\Upsilon(r_j)m(r_j,A), \end{aligned} \tag{19}$$

where $\theta \in F_j := (\alpha+2\varepsilon, \beta-2\varepsilon)\cap D(r_j)$. On the other hand, by Lemma 2.4 and combining (3) and (18) leads to

$$\begin{aligned} &\int_{F_j} \log^+|A(r_je^{i\theta})|d\theta \\ &\leq \int_{F_j}\left(\log^+\left|\frac{f'(r_je^{i\theta})}{f(r_je^{i\theta})}\right| \right.\\ &\quad \left. + \log^+\left|\frac{f(r_je^{i\theta})}{(f(r_je^{i\theta})-r_je^{i\theta})^2}\right|\right)d\theta + O(1) \\ &\leq \int_{F_j}\left(\log^+\left|\frac{f'(r_je^{i\theta})}{f(r_je^{i\theta})}\right| + 3\log^+|f(r_je^{i\theta})| \right.\\ &\quad \left. + 2\log^+ r_j\right)d\theta + O(1) \\ &\leq mes(F_j)O(\log r_j) \\ &\leq mes(F_j)c_0\Upsilon^2(r_j)m(r_j,A), \end{aligned} \tag{20}$$

where c_0 is a positive constant. From (19) and (20), we have

$$1 \leq c_0\Upsilon(r_j), \tag{21}$$

which contradicts to the fact $\Upsilon(r_j)\to 0$ as $j\to\infty$. Thus, we complete the proof.

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