Finite sum Cauchy identity for dual Grothendieck polynomials

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Abstract: We notice that dual Grothendieck polynomials are specializations of some vexillary Schubert polynomials. Hence they have determinantal expressions in terms of complete or elementary symmetric functions, as well as a description in terms of tableaux and Giambelli type formula. We give for them a finite sum Cauchy identity.

Key words: Cauchy identity; dual Grothendieck polynomial; Schur function.

This is a joint work while the first author (Alain Lascoux) visited Okayama University. Unfortunately he passed away recently, and the second author is responsible for the final form of this paper.

We shall need symmetric functions of a difference of alphabets, the definition of which we recall. For more considerations about λ-rings and their use in the theory of symmetric functions, see [6].

Given two sets of variables (we say alphabets) $\mathbf{A}, \mathbf{B}$, the *complete functions* $s_k(\mathbf{A}-\mathbf{B})$ are given by the generating function

$$\sum_{k=0}^{\infty} z^k s_k(\mathbf{A}-\mathbf{B}) = \prod_{b\in\mathbf{B}}(1-zb)\prod_{a\in\mathbf{A}}(1-za)^{-1}.$$

In particular, when one adds r letters specialized to 1 to one of the two alphabets, one has

$$\sum_{k=0}^{\infty} z^k s_k(\mathbf{A}-\mathbf{B}\pm r) = (1-z)^{\mp r}\prod_{b\in\mathbf{B}}(1-zb)\prod_{a\in\mathbf{A}}(1-za)^{-1}.$$

MultiSchur functions are determinants in the complete functions, which can be expressed as a Jacobi-Trudi type determinant with different alphabets in rows

$$s_\lambda(\mathbf{A}_1-\mathbf{B}_1,\dots,\mathbf{A}_n-\mathbf{B}_n) = \det(s_{\lambda_i+j-i}(\mathbf{A}_i-\mathbf{B}_i)).$$

The symmetric group $\mathfrak{S}_n$ acts on the ring of polynomials in $\mathbf{x}_n=\{x_1,\dots,x_n\}$, with coefficients in other alphabets. Specifically, the *simple transposition* s_i, $i=1,\dots,n-1$, acts by transposing x_i, x_{i+1}, fixing the other variables. The action is denoted exponentially $f\to f^{s_i}$. The *divided difference* ∂_i, acting on its left, is the operator

$$(1)\quad f(\mathbf{x}_n)\to f(\mathbf{x}_n)\,\partial_i = (f-f^{s_i})(x_i-x_{i+1})^{-1}.$$

To any permutation $\sigma\in\mathfrak{S}_n$ corresponds a divided difference ∂_σ which may be obtained as a product of ∂_i corresponding to a reduced decomposition of σ. The divided difference ∂_ω associated to the maximal permutation $\omega=[n,\dots,1]$ can also be interpreted as the *Cauchy-Jacobi operator* [6, Prop. 7.6.2]

$$f\to\sum_{\sigma\in\mathfrak{S}_n}(-1)^{\ell(\sigma)} f^\sigma\ \frac{1}{\Delta}.$$

Notice that, given n functions $f_1,\dots,f_n$ of a single variable, the image of the product $f_1(x_1)\dots f_n(x_n)$ under ∂_ω is equal to

$$\det(f_i(x_j))\Delta(\mathbf{x}_n)^{-1}.$$

Thanks to the Newton interpolation formula in one variable, one can rewrite this quotient as a single determinant (see [6]):

$$f_1(x_1)\dots f_n(x_n)\,\partial_\omega = (-1)^{\binom{n}{2}}\det(f_i(x_1)\partial_1\dots\partial_j)_{\substack{i=1\dots n\\ j=0\dots n-1}},$$

where an empty product of divided differences stands for the function evaluated in x_1.

When $f_1,\dots,f_n$ are just powers of a single variable, one obtains a determinant of complete functions of the flag of alphabets $\mathbf{x}_1\hookrightarrow\mathbf{x}_2\hookrightarrow\dots\hookrightarrow\mathbf{x}_n$ that one can transform by linear combination of columns into the usual Jacobi-Trudi determinant of complete functions of the same alphabet $\mathbf{x}_n$.

Grothendieck polynomials are the classes, in K-theory, of the structure sheaves of Schubert subvarieties of the flag variety [5]. In the case of Graßmannian permutations, it have been defined in [4, Th. 4.4] as image of

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$\prod_{i+j\leq \text{diagram of } \lambda+\rho} \left(1-\frac{y_j}{x_i}\right)$ under $x^\rho \partial_\omega$ where $\rho = [n-1,\dots,0]$. Specializing all y_j to 1 and changing variables $1-\frac{1}{x_i} \to x_i$, the Grothendieck polynomial becomes $G_\lambda = x^{\lambda+\rho}(1-x_2)\cdots(1-x_n)^{n-1}\partial_\omega$. The change of variables transforming $x^\rho\partial_\omega$ into $(1-x_2)\cdots(1-x_n)^{n-1}\partial_\omega$. Thus

$$\begin{aligned} G_\lambda(\mathbf{x}_n) &= x^{\lambda+[n-1,\dots,0]}(1-x_2)\dots(1-x_n)^{n-1}\ \partial_\omega \\ &= (-1)^{\binom{n}{2}}\left(\prod_{i=1}^n s_{\lambda_i+n-1}(x_i-i+1)\right)\partial_\omega. \end{aligned}$$

Therefore, one has

$$(2) \qquad G_\lambda(\mathbf{x}_n) = (-1)^{\binom{n}{2}}\, s_{\lambda_1,\lambda_2+1,\dots,\lambda_n+n-1}(\mathbf{x}_n, \mathbf{x}_n-1,\dots,\mathbf{x}_n-n+1)$$

expression given by Lenart [9].

One finds in the literature [1–3,11], still in the Graßmannian case, *dual Grothendieck polynomials* g_λ. They first appear in an implicit form in Buch's paper ([2]). Buch gives a Pieri-type formula for the coproduct on Grothendieck polynomials. By duality this formula may be used to uniquely define a basis, which are the dual Grothendieck polynomials. So this defines dual Grothendieck polynomials as the dual basis to Grassmannian Grothendieck polynomials. The determinantal definition of dual Grothendieck polynomial, as well as its equivalence with the duality definition, is due to Shimozono and Zabrocki ([11]). The Cauchy identity and the duality formulation, are equivalent.

Let us define the dual Grothendieck polynomials, using ∂_ω, by

$$g_\lambda(\mathbf{x}_n) = \left(\prod_{i=1}^n s_{\lambda_i+n-i}(x_i+i-1)\right)\partial_\omega.$$

Thus a dual Grothendieck polynomial is still a discrete Wronskian that one identifies with a multiSchur function (in the case of an increasing or decreasing sequence of alphabets, one also uses the term *flagged Schur function* cf. [12]):

$$(3) \qquad g_\lambda(\mathbf{x}_n) = s_\lambda(\mathbf{x}_n, \mathbf{x}_n+1,\dots,\mathbf{x}_n+n-1).$$

However, the Schubert polynomial $Y_{0^{n-1},\lambda}(\mathbf{x};\mathbf{0})$ is equal to ([6])

$$s_\lambda(\mathbf{x}_n, \mathbf{x}_{n+1},\dots,\mathbf{x}_{2n-1}).$$

Therefore, $g_\lambda(\mathbf{x}_n)$ *is equal to the specialization* $x_{n+1} = 1 = x_{n+2} = \cdots = x_{2n-1}$ *of the Schubert polynomial* $Y_{0^{n-1},\lambda}(\mathbf{x};\mathbf{0})$.

Explicitly, one has for any integers k, r,

$$s_k(\mathbf{x}_n + r) = \sum_{i=0}^k \binom{r+i-1}{i} s_{k-i}(\mathbf{x}_n)$$

the flagged Schur function (3) being equal to the determinant

$$(4) \qquad g_\lambda(\mathbf{x}_n) = \det(s_{\lambda_i+j-i}(\mathbf{x}_n+i-1)).$$

Schubert polynomials $Y_v(\mathbf{x};\mathbf{y})$ satisfy a symmetry property exchanging $\mathbf{x}$ and $\mathbf{y}$. In the present case, with μ the partition conjugate to λ, this property reads

$$Y_{0^{n-1}\lambda}(\mathbf{x};\mathbf{0}) = (-1)^{|\lambda|} Y_{0^{n-1}\mu}(\mathbf{0};\mathbf{x}),$$

the second function being a determinant in the elementary symmetric functions of $\mathbf{x}_{n+r}$, that one specializes to $\mathbf{x}_n + r$ as before. Thus, simplifying signs, one has

$$(5) \qquad g_\lambda(\mathbf{x}_n) = \det(e_{\mu_i+j-i}(\mathbf{x}_n+\mu_i-1)).$$

Any vexillary Schubert polynomial $Y_v(\mathbf{x};\mathbf{0})$ is equal to the Demazure character K_v of the same index, and thus possesses the same description in terms of tableaux satisfying a flag property (see [8]). Explicitly, $Y_{0^{n-1},\lambda}(\mathbf{x};\mathbf{0})$ is equal to the sum of all tableaux of shape λ in $1,\dots,2n-1$, the letter $2n-i$, for $i=1,\dots,n-1$, being permitted in rows $1,\dots,i$ only (counting from top).

Therefore, the function g_λ is obtained from this set of tableaux, giving weight x_i to i, for $i \leq n$, and weight 1 to i, for $i > n$:

$$(6) \qquad g_\lambda(\mathbf{x}_n) = Y_{0^{n-1}\lambda}(x_1,\dots,x_n,1,1,\dots;\mathbf{0}).$$

This coincides with the description of g_λ in terms of *elegant fillings* given in [3].

As flagged Schur function has Giambelli type determinantal formula ([10]), we can express g_λ as a determinant using hook partitions.

If $\lambda = (a_1, a_2,\dots,a_r|b_1,b_2,\dots,b_r)$: Frobenius notation, then

Proposition 1.

$$g_\lambda(\mathbf{x}_n) = \det\left(g^{(i,j)}_{(a_i|b_j)}(\mathbf{x}_n)\right)_{r\times r}$$

where

$$g^{(i,j)}_{(a|b)}(\mathbf{x}_n) := \sum_{p=0}^a\sum_{q=0}^b \binom{p+i-2}{p}\binom{q+j-2}{q} g_{(a-p|b-q)}(\mathbf{x}_n)$$

$$+\sum_{t=2}^{\min(i,j)}\binom{a+i-t}{a}\binom{b+j-t}{b}.$$

Proof. Flagged Schur function $s_\lambda(\mathbf{x}_n, \mathbf{x}_{n+1}, \ldots, \mathbf{x}_{2n-1})$ has Giambelli type formula ([10]),

$$s_\lambda(\mathbf{x}_n, \mathbf{x}_{n+1}, \ldots, \mathbf{x}_{2n-1}) = \det(s_{(a_i|b_j)}(\mathbf{x}_{n+i-1}, \mathbf{x}_{n+j}, \mathbf{x}_{n+j+1}, \ldots, \mathbf{x}_{2n-1})).$$

Actually the formula in [10] is a special case but we can use Bazin's formula [6] to prove the formula above. Then it is enough to prove

$$s_{(a_i|b_j)}(\mathbf{x}_{n+i-1}, \mathbf{x}_{n+j}, \mathbf{x}_{n+j+1}, \ldots, \mathbf{x}_{2n-1})|_{x_{n+1}=x_{n+2}=\cdots=1} = g^{(i,j)}_{(a_i|b_j)}(\mathbf{x}_n).$$

In tableaux formula $s_{(a_i|b_j)}(\mathbf{x}_{n+i-1}, \mathbf{x}_{n+j}, \mathbf{x}_{n+j+1}, \ldots, \mathbf{x}_{2n-1}) = \sum_T x^T$ cf. [12], we consider the number $T(1,1)$ of a tableau T of shape $(a_i|b_j)$ which satisfies the flag condition i.e., $1 \le T(1,1) \le n+i-1$ and $T(1,1) \le n+j-1$. If $T(1,1) > n$ then by the specialization $x_{n+1} = x_{n+2} = \cdots = 1$, the product becomes $x^T = 1$. If we set $t = T(1,1) - n + 1$, the number of such tableaux is $\binom{a_i+i-t}{a_i}\binom{b_j+j-t}{b_j}$ which explains the second term. For the case $T(1,1) \le n$, we can sum over the tableaux according to the length of the parts with entries greater than n. The tableaux with the number of such box in the first row is p and the number of such box in the first column is q form $g_{(a_i-p|b_j-q)}(\mathbf{x}_n)$ and the multiplicity of it is $\binom{p+i-2}{p}\binom{q+j-2}{q}$. □

Adding one extra letter y to the argument of a symmetric function is a way to obtain a combinatorial description of this function. For example, in the case of a Schur function, one obtains that the letter y occupies an *horizontal strip*. In the case of an Hall-Littlewood polynomial, one obtains a combinatorial description in terms of plane partitions [7]. This is also the case for the dual Grothendieck polynomial, as shows the next proposition.

Proposition 2. *Let λ be a partition. Then*

$$g_\lambda(\mathbf{x}_n + y) = \sum_{\mu\subseteq\lambda} y^{c(\lambda/\mu)}\, g_\mu(\mathbf{x}_n), \tag{7}$$

where $c(\lambda/\mu)$ is the number of non-empty columns of the skew diagram of λ/μ.

Proof. Let A_λ be the set of semistandard tableaux with shape λ and entries in $1 < 2 < \cdots < n < y < n+1 < n+2 \cdots$ such that $n+i$ is in the row i or above (in French display) for each $i = 1, 2, \cdots$. Let B_λ be the set of semistandard tableaux with a shape μ which is contained in λ and entries in $1 < 2 < \cdots < n < n+1 < n+2 \cdots$ such that $n+i$ is in the $i+1$-st row or above for each $i = 1, 2, \cdots$. Then by the tableaux expression of flagged Schur function it is sufficient to construct a bijection $\phi : A_\lambda \to B_\lambda$ such that the number of y's in $T \in A_\lambda$ is equal to the number of columns in λ/μ where μ is the shape of $\phi(T)$. We say a number $n+i$ in a tableau $T \in A_\lambda$ is in the extremal row if it is in the i-th row of T from the bottom. We define a map ϕ as follows. We use the restricted jue de taquin, which means that we can swap empty box and a neighbouring number as usual jue de taquin but one can move the number $n+i$ up to i-th row. Given a tableau $T \in A_\lambda$, we fix elements in extremal rows in T and replace all y's to z's. Then proceed the restricted jue de taquin to move z's as if they are vacant boxes. When z is in the right most position, remove the z and the above boxes of the same column (if there are). When all these procedures are terminated, the resulting tableau is $\phi(T)$. From the construction it is easy to see that it satisfies the required properties. □

Iterating this addition, one recovers the description of dual Grothendieck polynomials in terms of reverse plane partitions given by Lam and Pylyavskyy [3] (we combine two of their results, to account for the choice of a finite alphabet).

Corollary 3. *For any plane partition T, define x^T to be the product $\prod_i x_i^{T(i)}$, where $T(i)$ is the number of columns containing at least one entry equal to i. Let λ be a partition. Then*

$$g_\lambda(\mathbf{x}) = \sum_T x^T.$$

To obtain a Cauchy formula, we need more properties of multi-Schur functions.

Functions $f(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n)$, which are symmetrical in $x_1, \ldots, x_p$ and $x_{p+1}, \ldots, x_n$ separately, can be symmetrized using the divided difference

$$\partial_{p|q} := (\partial_p \cdots \partial_{n-1})(\partial_{p-1} \cdots \partial_{n-2}) \ldots (\partial_1 \ldots \partial_q).$$

In fact $\partial_{p|q}$ is a factor of ∂_ω, and one has for such functions

$$f(x_1, \ldots, x_n)\partial_{p|q} = g(x_1, \ldots, x_n)\partial_\omega,$$

where $g(x_1, \ldots, x_n)$ is any function such that $g(x_1, \ldots, x_n)\partial_{p,\ldots,1,n,\ldots,p+1} = f(x_1, \ldots, x_n)$.

In particular, one has the following lemmas.

Lemma 4. *Let* $n = p + q$, $v \in \mathbf{N}^q$, $\mathbf{x}_n = \{x_1, \ldots, x_n\}$, $\mathbf{B}$ *be arbitrary. Then*

$$s_{v+p^q}(x_{p+1} + \cdots + x_n - \mathbf{B})\partial_{p|q} = (-1)^{pq}\, s_v(\mathbf{x}_n - \mathbf{B}). \tag{8}$$

Lemma 5. *Let* $k, n \in \mathbf{N}$, $\mu \in \mathbf{N}^{n-1}$, $\mathbf{A}, \mathbf{B}$ *be arbitrary. Then*

$$s_{k+n-1}(x_1 \pm \mathbf{A}) s_\mu(x_2 + \cdots + x_n \pm \mathbf{B})\, \partial_1 \ldots \partial_{n-1} = s_{k;\mu}(\mathbf{x}_n \pm \mathbf{A}; \mathbf{x}_n \pm \mathbf{B}). \tag{9}$$

Multi-Schur functions are a little more delicate to use than Schur functions of a difference of alphabets. The following lemma gives an example where passing to the inverse variables simplify the function.

Lemma 6. *Let* $n = p + q$, $\mathbf{A}$ *of cardinality* n, $\mathbf{B}$ *of cardinality* p, $\nu \in \mathbf{N}^q$, r *be an integer such that* $\nu \le r^q$. *Put* $X = \{a^{-1} : a \in \mathbf{A}\}$, $Y = \{b^{-1} : b \in \mathbf{B}\}$. *Then*

$$s_{r^p;\nu+p^q}(\mathbf{A}; \mathbf{A} - \mathbf{B}) = s_{r^n}(\mathbf{A}) s_{q^p}(\mathbf{B})\, s_{r^q/\nu}(X - Y). \tag{10}$$

Lemma 7. *Let* $n = p + q$, $r \in \mathbf{N}$, *and* $\bullet \in \mathbf{N}^p$. *Let* $\mathbf{A}, \mathbf{B}$ *be arbitrary, and* Z *be of cardinality* p. *Denote* $Z^\vee = \{z^{-1} : z \in Z\}$. *Then*

$$\sum_{\nu \le r^q} s_{r^p;\nu+p^q}(\mathbf{A}; \mathbf{A} - Z) s_{\bullet;\nu}(\mathbf{B}; \mathbf{B} + Z^\vee) = s_{q^p}(Z) \sum_{\nu \le r^q} s_{r^p,\nu}(\mathbf{A}) s_{\bullet,\nu}(\mathbf{B}), \tag{11}$$

sum over all increasing partitions ν *with* q *parts not bigger than* r.

Proof. The two sides of the equation involve only Schur functions $s_\lambda(\mathbf{A})$, $s_\lambda(\mathbf{B})$ with $\ell(\lambda) \le n$. Thus the equation is true if it is true for $\mathbf{A} = \{x_1^{-1}, \ldots, x_n^{-1}\} = X^\vee$. However, according to (10), the left-hand side rewrites into

$$\begin{aligned} s_{r^n}(\mathbf{A}) s_{q^p}(Z) &\sum_{\nu \le r^q} s_{r^q/\nu}(X - Z^\vee) s_{\bullet;\nu}(\mathbf{B}; \mathbf{B} + Z^\vee) \\ &= s_{r^n}(\mathbf{A}) s_{q^p}(Z) s_{\bullet;r^q}(\mathbf{B}; \mathbf{B} + X) \\ &= s_{r^n}(\mathbf{A}) s_{q^p}(Z) \sum_{\nu \le r^q} s_{r^q/\nu}(X) s_{\bullet,\nu}(\mathbf{B}). \end{aligned}$$

Going back to the alphabet $\mathbf{A}$, one obtains that the sum is equal to the right-hand side of (11). □

We introduce a function $F(x_1, \ldots, x_n; y_1, \ldots, y_n; r, k)$ which will play the role of a Cauchy kernel.

Proposition 8. *Given* $n, r \in \mathbf{N}$, $k \in \mathbf{Z}$, $\omega = [n, \ldots, 1]$, $\mathbf{x}_n = \{x_1, \ldots, x_n\}$, $\mathbf{y}_n = \{y_1, \ldots, y_n\}$, *let*

$$\begin{aligned} F(\mathbf{x}_n; \mathbf{y}_n; r, k) = (-1)^{\binom{n}{2}} &\times \sum_{\lambda \le r^n} \prod_{i=1}^{n} s_{\lambda_i+n-1}(x_i - i + 1) \\ &\times s_{\lambda_i+n-i}(y_i + i - 1 + k)\, \partial^x_\omega \partial^y_\omega. \end{aligned}$$

Then

$$F(\mathbf{x}_n; \mathbf{y}_n; r, k) = \sum_{\lambda \le r^n} s_\lambda(\mathbf{x}_n) s_\lambda(\mathbf{y}_n + k). \tag{12}$$

Proof. Factorizing $\partial_\omega = \partial_{\omega'} \partial_1 \ldots \partial_{n-1}$, one rewrites the function as

$$\begin{aligned} \sum_{\lambda_1 \le r} &\Bigg(\sum_{\mu \le \lambda_1^{n-1}} \prod_{i=1}^{n-1} (x_{i+1} - 1) s_{\mu_i+n-2}(x_{i+1} - i + 1) \\ &\times s_{\mu_{i-1}+n-1-i}(y_{i+1} + i - 1 + k + 1)\, \partial^x_{\omega'} \partial^y_{\omega'} \Bigg) \\ &\times s_{\lambda_1+n-1}(x_1) s_{\lambda_1+n-1}(y_1 + k)\, \partial^x_1 \ldots \partial^x_{n-1} \partial^y_1 \ldots \partial^y_{n-1}. \end{aligned}$$

By induction on n, the operator $\partial^x_{\omega'} \partial^y_{\omega'}$ produces the function $F(x_2, \ldots, x_n; y_2, \ldots, y_n; r, k+1)$ times the factor $(x_2 - 1) \ldots (x_n - 1)$ which commutes with it. The sum to compute has become

$$\begin{aligned} \sum_{\lambda_1 \le r} \sum_{\mu \le \lambda_1^{n-1}} &(s_{\lambda_1+n-1}(x_1) s_\mu(x_2 + \cdots + x_n) \\ &\times (x_2 - 1) \ldots (x_n - 1)) \\ &\times (s_{\lambda_1+n-1}(y_1 + k) s_\mu(y_2 + \cdots + y_n + k + 1)) \\ &\times \partial^x_1 \ldots \partial^x_{n-1} \partial^y_1 \ldots \partial^y_{n-1}. \end{aligned}$$

The symmetrization formula (9) gives

$$\sum_{\lambda_1 \le r} \sum_{\mu \le \lambda_1^{n-1}} s_{\lambda_1;\mu+1^{n-1}}(\mathbf{x}_n; \mathbf{x}_n - 1) \times s_{\lambda_1;\mu}(\mathbf{y}_n + k; \mathbf{y}_n + k + 1).$$

Formula (11) disposes of the shift by ± 1 in final the function is equal to

$$\sum_{\lambda \le r^n} s_\lambda(\mathbf{x}_n) s_\lambda(\mathbf{y}_n + k) = F(\mathbf{x}_n; \mathbf{y}_n; r, k)$$

and the proposition is proved. □

For $k = 0$, the function $F(\mathbf{x}_n; \mathbf{y}_n; r, 0)$ is a sum of products of Graßmannian Grothendieck polynomials times dual Grothendieck polynomials, and one obtains the following finite Cauchy identity as a corollary of the preceding proposition.

Theorem 9. *Let* n, r *be two positive integers. Then*

$$\sum_{\lambda \le r^n} G_\lambda(\mathbf{x}_n)\, g_\lambda(\mathbf{y}_n) = \sum_{\lambda \le r^n} s_\lambda(\mathbf{x}_n)\, s_\lambda(\mathbf{y}_n). \tag{13}$$

Using the involution on symmetric functions $f \to f^{\sim}$ which exchanges elementary and complete functions, one can also write

$$\sum_{\lambda \leq r^n} G_\lambda(\mathbf{x}_n)\,(g_\lambda(\mathbf{y}_n))^{\sim} = \sum_{\lambda \leq r^n} s_\lambda(\mathbf{x}_n)\, s_{\lambda^{\sim}}(\mathbf{y}_n) = \prod_{i \leq n, j \leq n} (1 + x_i y_j).$$

Letting n ends towards infinity gives the Cauchy formula due to Lam and Pylyavskyy [3].

$$\sum_{\lambda} G_\lambda(\mathbf{x}_\infty)\, g_\lambda(\mathbf{y}_\infty) = \sum_{\lambda} s_\lambda(\mathbf{x}_\infty)\, s_\lambda(\mathbf{y}_\infty) = \prod_{i,j} (1 - x_i y_j)^{-1}.$$

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