Limit theorems for random walks under irregular conductance

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(Communicated by Masaki KASHIWARA, M.J.A., Sept. 12, 2013)

Abstract: For a general one-dimensional random walk with state-dependent transition probabilities, we present weak limits of the empirical moments of conductance along the path of the random walk. In particular we obtain remarkably simple quenched convergences under random conductance model.

Key words: Arc-sine law; empirical moment; random conductance; weak convergence.

1. Introduction. Let $X = \{X_n\}_{n=0}^{\infty}$ be a Markov chain with state space $\mathbf{Z}$ and transition probabilities

$$p_{ij} = \mathbf{E}[X_1 = j | X_0 = i] = \begin{cases} p_i & \text{if } j = i+1 \\ 1-p_i & \text{if } j = i-1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\{p_i\}_{i\in\mathbf{Z}} \subset (0,1)$. The sequence $\{p_i\}$ admits a parametrization

$$p_j = \frac{c_j}{c_{j-1}+c_j}$$

with another sequence $c_j > 0$, $j \in \mathbf{Z}$. The value c_j is interpreted as conductance of the bond connecting the states j and $j+1$. See [2] for some relations between random walks and electric networks. This is also related to the scale function and speed measure of X, which are used in [5] to show a generalized arc-sine law under conditions on the asymptotic behavior of

$$\sum_{i=\min\{0,j\}}^{\max\{0,j\}} \frac{1}{c_i}, \qquad \sum_{i=\min\{0,j\}}^{\max\{0,j\}} (c_{i-1}+c_i)$$

as $|j| \to \infty$. Random conductance model assumes $\{c_j\}_{j\in\mathbf{Z}}$ to be a realization of an exogeneous positive stationary ergodic random variables $\{C_j\}_{j\in\mathbf{Z}}$. If $\mathbf{E}[C_0] < \infty$ and $\mathbf{E}[C_0^{-1}] < \infty$, then by the above mentioned result of [5], we have a quenched arc-sine law, that is, for almost every realization of C_j,

$$\frac{1}{N}\sum_{n=0}^{N-1} 1_{\{X_n\geq 0\}} \to A \tag{1}$$

in law as $N \to \infty$, where A is a random variable with distribution function

$$F_A(a) = \begin{cases} 0 & a \leq 0 \\ \dfrac{2}{\pi}\arcsin\sqrt{a} & 0 < a < 1 \\ 1 & 1 \leq a. \end{cases} \tag{2}$$

See [1,4] for other results in the study of the random conductance model. The random conductance model is an example of random walks in random environments, for which see [6].

In this study, we are mainly interested in the limits of the empirical moments

$$\Gamma_N^{(k)} = \frac{1}{N}\sum_{n=0}^{N-1} c(X_n, X_{n+1})^k, \quad k \in \mathbf{Z}$$

as $N \to \infty$, where

$$c(i,j) = \begin{cases} c_i & \text{if } j = i+1, \\ -c_{i-1} & \text{if } j = i-1. \end{cases}$$

The value $c(i,j)$ is the signed conductance of the bond connecting the states i and j with $|i-j| = 1$. We show their weak convergence under conditions on the asymptotic behavior of

$$\sum_{i=\min\{0,j\}}^{\max\{0,j\}} c_i^k, \quad k \in \mathbf{Z}$$

as $|j| \to \infty$. As an application to the random conductance model, we obtain unexpectedly simple limits

$$\Gamma_N^{(k)} \to \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dfrac{\mathbf{E}[C_0^{1+k}]}{\mathbf{E}[C_0]} & \text{if } k \text{ is even} \end{cases} \tag{3}$$

in probability for almost every realization of $\{C_j\}_{j\in\mathbf{Z}}$. The quenched arc-sine law (1) is also obtained as a corollary.

2. Main results. Let $s(0) = 0$ and for $j \in \mathbf{N}$,

2010 Mathematics Subject Classification. Primary 60K37.

doi: 10.3792/pjaa.89.87

$$s(j) = \sum_{i=0}^{j-1} \frac{1}{c_i}, \quad s(-j) = -\sum_{i=-j}^{-1} \frac{1}{c_i}.$$

We introduce the following conditions for $k \in \mathbf{Z}$.

A$[k,+]$:

$$\lim_{j\to\infty} \frac{c_j^{1+k}}{s(j)} = 0$$

A$[k,-]$:

$$\lim_{j\to-\infty} \frac{c_j^{1+k}}{s(j)} = 0$$

B$[k,+]$: There exists $\mu_k^+ \in (0,\infty)$ such that

$$\lim_{j\to\infty} \frac{1}{s(j)} \sum_{i=1}^{j} c_i^{1+k} = \mu_k^+$$

B$[k,-]$: There exists $\mu_k^- \in (0,\infty)$ such that

$$\lim_{j\to-\infty} \frac{1}{s(j)} \sum_{i=j}^{-1} c_i^{1+k} = -\mu_k^-.$$

We consider the limits of the following sequence.

$$\Gamma_N^{(k)+} = \frac{1}{N} \sum_{n=0}^{N-1} c(X_n, X_{n+1})^k 1_{\{X_n \geq 0\}},$$

$$\Gamma_N^{(k)-} = \frac{1}{N} \sum_{n=0}^{N-1} c(X_n, X_{n+1})^k 1_{\{X_n < 0\}}$$

and $\Gamma_N^{(k)} = \Gamma_N^{(k)+} + \Gamma_N^{(k)-}$.

Theorem 1. *Let k be odd and assume $A[-2,\pm]$ and $B[0,\pm]$. Then,*

(a) *under $A[k,+]$,*

$$\Gamma_N^{(k)+} \to 0$$

in probability as $N \to \infty$, and

(b) *under $A[k,-]$,*

$$\Gamma_N^{(k)-} \to 0$$

in probability as $N \to \infty$.

Theorem 2. *Let k be even and assume $A[-2,\pm]$ and $B[0,\pm]$ with $\mu_0^+ = \mu_0^-$. Let A be a random variable with distribution function (2). Then,*

(a) *under $B[k,+]$,*

$$\Gamma_N^{(k)+} \to \frac{\mu_k^+}{\mu_0^+} A$$

in law as $N \to \infty$,

(b) *under $B[k,-]$,*

$$\Gamma_N^{(k)-} \to \frac{\mu_k^-}{\mu_0^-}(1-A)$$

in law as $N \to \infty$, and

(c) *under the both $B[k,\pm]$,*

$$(\Gamma_N^{(k)+}, \Gamma_N^{(k)-}) \to \left(\frac{\mu_k^+}{\mu_0^+} A, \frac{\mu_k^-}{\mu_0^-}(1-A)\right)$$

in law as $N \to \infty$.

Remark 1. In fact we can prove the weak convergence of the joint distribution

$$(\Gamma_N^{(k_1)+}, \Gamma_N^{(k_1)-}, \Gamma_N^{(k_2)+}, \ldots, \Gamma_N^{(k_n)-})$$

under A$[-2,\pm]$, B$[0,\pm]$, B$[k_1,\pm]$, $\cdots$, B$[k_n,\pm]$.

Remark 2. Consider the random conductance model. Here we assume that $\{c_j\}$ is determined by a stationary sequence $\{C_j\}_{j\in\mathbf{Z}}$ which is not necessarily ergodic. Let $\mathcal{I}$ be the set of invariant events. Then by Birkhoff's ergodic theorem, almost every realization satisfies A$[-2,\pm]$ and B$[k,\pm]$ with

$$\mu_k^+ = \mu_k^- = \frac{\mathbf{E}[C_0^{1+k}|\mathcal{I}]}{\mathbf{E}[C_0^{-1}|\mathcal{I}]}$$

as soon as

$$\mathbf{E}[C_0^{-1}] < \infty \text{ and } \mathbf{E}[C_0^{1+k}] < \infty. \tag{4}$$

Since B$[k,\pm]$ implies A$[k,\pm]$ under A$[-2,\pm]$, we obtain (3) under (4) plus $\mathbf{E}[C_0] < \infty$ in the case $\mathcal{I}$ is trivial. We may have similar results under ergodicity instead of stationarity.

3. Proofs. Let $\mathbf{G} = \{s(j)\}_{j\in\mathbf{Z}} \subset \mathbf{R}$. Consider a deformed Markov chain $\hat{X}_n = s(X_n)$ with state space $\mathbf{G}$. Since for all $j \in \mathbf{Z}$,

$$\mathbf{P}[\hat{X}_{n+1} - \hat{X}_n = 1/c_j | \hat{X}_n = s(j)] = \frac{c_j}{c_{j-1} + c_j},$$

$$\mathbf{P}[\hat{X}_{n+1} - \hat{X}_n = -1/c_{j-1} | \hat{X}_n = s(j)] = \frac{c_{j-1}}{c_{j-1} + c_j},$$

the Markov chain $\hat{X}_n$ is a martingale. Next we embed $\hat{X} = \{\hat{X}_n\}$ to a Brownian motion W as follows. Let $\epsilon > 0$, $\tau_0^\epsilon = 0$, $W_0 = \hat{X}_0$ and define a sequence of stopping times τ_n^ϵ as

$$\tau_{n+1}^\epsilon = \inf\{t > \tau_n^\epsilon;\ W_t \in \mathbf{G}^\epsilon \setminus \{W_{\tau_n^\epsilon}\}\},$$
$$\mathbf{G}^\epsilon = \{\epsilon s(j)\}_{j\in\mathbf{Z}}.$$

Put $W_n^\epsilon = \epsilon^{-1} W_{\tau_n^\epsilon}$ for brevity. Then by the optional sampling theorem, $\{\hat{X}_n\}$ and $\{W_n^\epsilon\}$ have an identical law for any $\epsilon > 0$. Moreover, by definition,

$$\hat{X}_{n+1} - \hat{X}_n = \frac{1}{c(X_n, X_{n+1})}.$$

So $\Gamma_N^{(k)+}$ and $\Gamma_N^{(k)-}$ have the same distributions as

$$\hat{\Gamma}_N^{(k)+} := \frac{1}{N}\sum_{n=0}^{N-1}(W_{n+1}^{\epsilon} - W_n^{\epsilon})^{-k} 1_{\{W_n^{\epsilon}\geq 0\}},$$

$$\hat{\Gamma}_N^{(k)-} := \frac{1}{N}\sum_{n=0}^{N-1}(W_{n+1}^{\epsilon} - W_n^{\epsilon})^{-k} 1_{\{W_n^{\epsilon}< 0\}}$$

respectively for any $\epsilon > 0$. Notice that $s(j) \to \pm\infty$ as $j \to \pm\infty$ under B$[0, \pm]$. Therefore $\mathbf{G}^{\epsilon}$ does not have accumulation points and so, we have $\tau_n^{\epsilon} \to \infty$ a.s. as $n \to \infty$. From this we conclude that $\hat{X}$ is recurrent since so is the Brownian motion. By the recurrence property we assume without loss of generality that $W_0 = \hat{X}_0 = 0$. Let

$$N_{\epsilon} = \max\{n \geq 0;\ \tau_n^{\epsilon} \leq 1\}.$$

Lemma 1. *Let k be odd. Then,*

(a) *under $A[-2, \pm]$, $B[0, \pm]$ and $A[k, +]$,*

$$\epsilon^2 \max_{0\leq m\leq N_{\epsilon}} \left|\sum_{n=0}^{m}(W_{n+1}^{\epsilon} - W_n^{\epsilon})^{-k} 1_{\{W_n^{\epsilon}\geq 0\}}\right| \to 0$$

in probability as $\epsilon \to 0$, and

(b) *under $A[-2, \pm]$, $B[0, \pm]$ and $A[k, -]$,*

$$\epsilon^2 \max_{0\leq m\leq N_{\epsilon}} \left|\sum_{n=0}^{m}(W_{n+1}^{\epsilon} - W_n^{\epsilon})^{-k} 1_{\{W_n^{\epsilon}< 0\}}\right| \to 0$$

in probability as $\epsilon \to 0$.

Lemma 2. *Let k be even. Then,*

(a) *under $A[-2, \pm]$, $B[0, \pm]$ and $B[k, +]$,*

$$\epsilon^2 \sum_{n=0}^{N_{\epsilon}}(W_{n+1}^{\epsilon} - W_n^{\epsilon})^{-k} 1_{\{W_n^{\epsilon}\geq 0\}} \to \mu_k^{+}\int_0^1 1_{\{W_t\geq 0\}}\mathrm{d}t$$

in probability as $\epsilon \to 0$, and

(b) *under $A[-2, \pm]$, $B[0, \pm]$ and $B[k, -]$,*

$$\epsilon^2 \sum_{n=0}^{N_{\epsilon}}(W_{n+1}^{\epsilon} - W_n^{\epsilon})^{-k} 1_{\{W_n^{\epsilon}< 0\}} \to \mu_k^{-}\int_0^1 1_{\{W_t< 0\}}\mathrm{d}t$$

in probability as $\epsilon \to 0$.

The proofs of these lemmas follow the same lines as in [3]. We therefore give only a sketch of them in Appendix.

Proof of Theorem 1. Let $\nu_1 = \min\{\mu_0^+, \mu_0^-\}$ and $\nu_2 = \max\{\mu_0^+, \mu_0^-\}$. By Lemma 2 with $k = 0$, we have for any $\delta \in (0, \nu_1)$ and $\delta' > 0$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$,

$$\mathbf{P}[\Omega_{\epsilon}] > 1 - \delta', \quad \Omega_{\epsilon} := \{\nu_1 - \delta \leq \epsilon^2 N_{\epsilon} \leq \nu_2 + \delta\}.$$

Let $\epsilon_1(N) = \sqrt{(\nu_1 - \delta)/N}$ and $\epsilon_2(N) = \sqrt{(\nu_2 + \delta)/N}$. Then for N with $\epsilon_2(N) \leq \epsilon_0$, we have

$$\mathbf{P}[\hat{\Omega}_N] > 1 - \delta', \quad \hat{\Omega}_N := \{N_{\epsilon_2(N)} \leq N \leq N_{\epsilon_1(N)}\}.$$

Notice that on $\hat{\Omega}_N$,

$$|\hat{\Gamma}_N^{(k)+}| \leq \frac{1}{N_{\epsilon_2(N)}} \sup_{0\leq m\leq N_{\epsilon_1(N)}} |m\hat{\Gamma}_m^{(k)+}|.$$

Since

$$\epsilon_1(N)^2 \sup_{0\leq m\leq N_{\epsilon_1(N)}} |m\hat{\Gamma}_m^{(k)+}| \to 0$$

in probability as $N \to \infty$ by Lemma 1 and

$$\epsilon_1(N)^2 N_{\epsilon_2(N)} \to \frac{\nu_1 - \delta}{\nu_2 + \delta}(\mu_0^+\hat{A} + \mu_0^-(1 - \hat{A}))$$

in probability as $N \to \infty$ by Lemma 2, where

$$\hat{A} = \int_0^1 1_{\{W_t\geq 0\}}\mathrm{d}t,$$

we conclude $\hat{\Gamma}_N^{(k)+} \to 0$ in probability, which implies $\Gamma_N^{(k)+} \to 0$ in probability since they have a common distribution. We obtain $\Gamma_N^{(k)-} \to 0$ in probability by the same argument. □

Proof of Theorem 2. Put $\mu = \mu_0^+ = \mu_0^-$. By Lemma 2, we have

$$\epsilon^2 N_{\epsilon} \to \mu$$

in probability as $\epsilon \to 0$. By the same argument as above, we have that for all $\delta \in (0, \mu)$ and $\delta' > 0$, there exists $\epsilon_0 > 0$ such that for all N with $\epsilon_2(N) \leq \epsilon_0$,

$$\mathbf{P}[\hat{\Omega}_N] > 1 - \delta', \quad \hat{\Omega}_N := \{N_{\epsilon_2(N)} \leq N \leq N_{\epsilon_1(N)}\},$$

where $\epsilon_1(N) = \sqrt{(\mu - \delta)/N}$ and $\epsilon_2(N) = \sqrt{(\mu + \delta)/N}$. On $\hat{\Omega}_N$, we have

$$\frac{1}{N_{\epsilon_1(N)}}\sum_{n=0}^{N_{\epsilon_2(N)}}(W_{n+1}^{\epsilon} - W_n^{\epsilon})^{-k} 1_{\{W_n^{\epsilon}\geq 0\}} \leq \hat{\Gamma}_N^{(k)+} \leq \frac{1}{N_{\epsilon_2(N)}}\sum_{n=0}^{N_{\epsilon_1(N)}}(W_{n+1}^{\epsilon} - W_n^{\epsilon})^{-k} 1_{\{W_n^{\epsilon}\geq 0\}}$$

and

$$\frac{1}{N_{\epsilon_1(N)}}\sum_{n=0}^{N_{\epsilon_2(N)}}(W_{n+1}^{\epsilon} - W_n^{\epsilon})^{-k} 1_{\{W_n^{\epsilon}\geq 0\}} \to \frac{\mu_k^+(\mu - \delta)}{\mu(\mu + \delta)}\hat{A},$$

$$\frac{1}{N_{\epsilon_2(N)}}\sum_{n=0}^{N_{\epsilon_1(N)}}(W_{n+1}^{\epsilon} - W_n^{\epsilon})^{-k} 1_{\{W_n^{\epsilon}\geq 0\}} \to \frac{\mu_k^+(\mu + \delta)}{\mu(\mu - \delta)}\hat{A}$$

in probability as $N \to \infty$. Since δ can be arbitrarily small, we conclude

$$\hat{\Gamma}_N^{(k)+} \to \frac{\mu_k^+}{\mu}\hat{A}$$

in probability as $N \to \infty$. It is well-known that the distribution function of $\hat{A}$ is given by (2). Since $\hat{\Gamma}_N^{(k)+}$ and $\Gamma_N^{(k)+}$ have a common distribution, we obtain the result. The convergence of $\Gamma_N^{(k)-}$ is obtained in the same manner. □

4. Appendix. Here we give a sketch of the proofs for Lemmas 1 and 2. See [3] for the detailed estimates. First we recall a simple consequence from the Lenglart inequality:

Lemma 3. *Let $\{\mathcal{F}_n^\epsilon\}_{n=0}^\infty$ be a filtration and $\{U_n^\epsilon\}_{n=0}^\infty$ be an adapted sequence for each $\epsilon > 0$. If*

$$\sum_{n=0}^{\infty} \mathbf{E}[|U_{n+1}^\epsilon|^2|\mathcal{F}_n^\epsilon] \to 0$$

in probability as $\epsilon \to 0$, then,

$$\sup_{0\le m<\infty}\left|\sum_{n=0}^{m}(U_{n+1}^\epsilon - \mathbf{E}[U_{n+1}^\epsilon|\mathcal{F}_n^\epsilon])\right| \to 0$$

in probability as $\epsilon \to 0$.

From this lemma, it suffices to treat

$$\epsilon^2 \sum_{n=0}^{m} G_n^\epsilon(k,\pm),$$

where

$$G_n^\epsilon(k,+) = \mathbf{E}[(W_{n+1}^\epsilon - W_n^\epsilon)^{-k}|\mathcal{F}_{\tau_n^\epsilon}]1_{\{W_n^\epsilon\ge 0\}},$$
$$G_n^\epsilon(k,-) = \mathbf{E}[(W_{n+1}^\epsilon - W_n^\epsilon)^{-k}|\mathcal{F}_{\tau_n^\epsilon}]1_{\{W_n^\epsilon< 0\}}$$

and $\{\mathcal{F}_t\}$ is a filtration to which W is adapted. Further, using again the same lemma, we conclude that if there exist sequences $\{H_n^\epsilon\}$ and $\{K_n^\epsilon\}$ adapted to $\{\mathcal{F}_{\tau_n^\epsilon}\}$ such that

$$G_n^\epsilon(k,+) = \mathbf{E}[H_{n+1}^\epsilon|\mathcal{F}_{\tau_n^\epsilon}] + K_n^\epsilon, \tag{5}$$

$$\epsilon^4 \sum_{n=0}^{N_\epsilon} \mathbf{E}[|H_{n+1}^\epsilon|^2|\mathcal{F}_{\tau_n^\epsilon}] \to 0, \tag{6}$$

$$\epsilon^2 \max_{0\le m\le N_\epsilon}\left|\sum_{n=0}^{m} H_{n+1}^\epsilon\right| \to 0, \tag{7}$$

in probability as $\epsilon \to 0$, then

$$\epsilon^2 \max_{0\le m\le N_\epsilon}\left|\sum_{n=0}^{m} G_n^\epsilon(k,+) - \sum_{n=0}^{m} K_n^\epsilon\right| \to 0$$

in probability as $\epsilon \to 0$. Of course we have the same conclusion for $G_n^\epsilon(k,-)$ as well as $G_n^\epsilon(k,+)$. For $G_n^\epsilon(k,+)$ with odd k, we take $K_n^\epsilon = 0$ and

$$H_{n+1}^\epsilon = (\Psi_k(W_{n+1}^\epsilon) - \Psi_k(W_n^\epsilon))1_{\{W_n^\epsilon\ge 0\}},$$

where

$$\Psi_k(z) = -2\int_0^z \int_0^y \psi_k(x)\mathrm{d}x\mathrm{d}y,$$

$$\psi_k(x) = (g_k(i+1) - g_k(i))\left\{3\frac{x - s(i)}{s(i+1) - s(i)} - 1\right\} + g_k(i), \quad \text{for } x \in [s(i), s(i+1)), \quad i \in \mathbf{Z} \tag{8}$$

and $g_k(i)$, $i \in \mathbf{Z}$ is defined as

$$\sum_{m=0}^{-k-2}(s(i+1) - s(i))^m (s(i-1) - s(i))^{-k-2-m}$$

if $k \le -2$ and

$$-\sum_{m=0}^{k}(s(i+1) - s(i))^{-1-m}(s(i-1) - s(i))^{-k-1+m}$$

otherwise. Here $g_{-1}(i)$ is understood as 0. We have constructed these functions so that

$$G_n^\epsilon(k,+) = g_k(s^{-1}(W_n^\epsilon))G_n^\epsilon(-2)1_{\{W_n^\epsilon\ge 0\}}$$

and

$$\begin{aligned}\int_{s(i)}^{s(i+1)}&(s(i+1) - z)(\psi_k(z) - g_k(i))\mathrm{d}z\\ &= \int_{s(i)}^{s(i+1)}(z - s(i))(\psi_k(z) - g_k(i+1))\mathrm{d}z\\ &= 0.\end{aligned}$$

Then by the Itô-Tanaka formula, we have

$$G_n^\epsilon(k,+) = \mathbf{E}[H_{n+1}^\epsilon|\mathcal{F}_n^\epsilon],$$

which implies (5). Further, a direct calculation gives

$$\begin{aligned}2\int_{s(i)}^{s(i+1)}&\psi_k(z)\mathrm{d}z\\ &= (s(i+1) - s(i))(g_k(i+1) + g_k(i))\\ &= -(s(i) - s(i+1))^{-k-1} + (s(i+1) - s(i))^{-k-1}\\ &\quad - \sum_{m=0}^{-k-3}(s(i+2) - s(i+1))^{m+1}\\ &\quad \times (s(i) - s(i+1))^{-k-2-m}\\ &\quad + \sum_{m=0}^{-k-3}(s(i+1) - s(i))^{m+1}\\ &\quad \times (s(i-1) - s(i))^{-k-2-m}\end{aligned}$$

when $k \le -2$. We have a similar expression for non-negative k as well. Notice that the first two terms cancel if k is odd and the last two terms form a telescopic sum in i. This implies $\Psi_k'(x) = o(x)$ as $x \to \infty$ for odd k under A$[-2, \pm]$ and A$[k, +]$. This gives (6) and

$$\epsilon^2 \max_{0 \le m \le N_\epsilon} |\Psi_k(W^\epsilon_{m+1})| \to 0$$

in probability. Noting that

$$\sum_{n=0}^{m} H^\epsilon_{n+1} = \Psi_k(s(-1)) \sum_{n=0}^{m} 1_{\{W^\epsilon_n \ge 0, W^\epsilon_{n+1} < 0\}} + \Psi_k(W^\epsilon_{m+1}),$$

we conclude (7) with the aid of Lévy's downcrossing theorem. The same argument remains valid for $G^\epsilon_n(k, -)$ with odd k by using

$$K^\epsilon_n = 0, \quad H^\epsilon_{n+1} = (\Psi_k(W^\epsilon_{n+1}) - \Psi_k(W^\epsilon_n))1_{\{W^\epsilon_n < 0\}}.$$

Thus we obtain Lemma 1.

For even k, we replace g_k in (9) with

$$\hat{g}_k(i) = g_k(i) - \mu_k^+ 1_{\{i \ge 0\}} - \mu_k^- 1_{\{i < 0\}}$$

to define $\hat{\psi}_k$ and $\hat{\Psi}_k$. Then we get, for example,

$$\begin{aligned} G^\epsilon_n(k, +) &= (\mathbf{E}[\hat{\Psi}_k(W^\epsilon_{n+1}) - \hat{\Psi}_k(W^\epsilon_n) | \mathcal{F}_{\tau^\epsilon_n}] + \mu_k^+ G^\epsilon_n(-2)) \\ &\quad \times 1_{\{W^\epsilon_n \ge 0\}}. \end{aligned}$$

Therefore we take

$$\begin{aligned} H^\epsilon_{n+1} &= (\hat{\Psi}_k(W^\epsilon_{n+1}) - \hat{\Psi}_k(W^\epsilon_n))1_{\{W^\epsilon_n \ge 0\}}, \\ K^\epsilon_n &= \mu_k^+ G^\epsilon_n(-2) 1_{\{W^\epsilon_n \ge 0\}} \end{aligned}$$

to get (5). To see (6) and (7), observe that

$$\begin{aligned} 2\int_{s(i)}^{s(i+1)} \hat{\psi}_k(z)\mathrm{d}z &= (s(i+1) - s(i))(\hat{g}_k(i+1) + \hat{g}_k(i)) \\ &= 2(s(i+1) - s(i))^{-k-1} - 2\mu_k^+(s(i+1) - s(i)) \\ &\quad - \sum_{m=0}^{-k-3} (s(i+2) - s(i+1))^{m+1} \\ &\quad \times (s(i) - s(i+1))^{-k-2-m} \\ &\quad + \sum_{m=0}^{-k-3} (s(i+1) - s(i))^{m+1} \\ &\quad \times (s(i-1) - s(i))^{-k-2-m} \end{aligned}$$

when $k \le -2$. We have a similar expression for non-negative k as well. Notice that the first two terms become negligible after summing up in i by B$[k, +]$ since

$$\frac{1}{s(i+1) - s(i)} = c_i.$$

The last two terms form a telescopic sum in i as before. Thus we obtain $\hat{\Psi}'(x) = o(x)$ as $x \to \infty$ under A$[k, +]$, which follows from B$[k, +]$ and A$[-2, \pm]$. This is enough to conclude (6) and (7). Under A$[-2, \pm]$,

$$\sup_{n \ge 0} |\tau^\epsilon_{n+1} \wedge 1 - \tau^\epsilon_n \wedge 1| \to 0$$

and so by Lemma 3,

$$\begin{aligned} \epsilon^2 \sum_{n=0}^{N_\epsilon} K^\epsilon_n &= \mu_k^+ \epsilon^2 \sum_{n=0}^{N_\epsilon} G^\epsilon_n(-2) 1_{\{W^\epsilon_n \ge 0\}} \\ &= \mu_k^+ \sum_{n=0}^{N_\epsilon} 1_{\{W^\epsilon_n \ge 0\}} \mathbf{E}[\tau^\epsilon_{n+1} - \tau^\epsilon_n | \mathcal{F}_{\tau^\epsilon_n}] \\ &\to \mu_k^+ \int_0^1 1_{\{W_t \ge 0\}} \mathrm{d}t \end{aligned}$$

in probability. The same argument remains valid for $G^\epsilon_n(k, -)$ by taking

$$\begin{aligned} H^\epsilon_{n+1} &= (\hat{\Psi}_k(W^\epsilon_{n+1}) - \hat{\Psi}_k(W^\epsilon_n))1_{\{W^\epsilon_n < 0\}}, \\ K^\epsilon_n &= \mu_k^- G^\epsilon_n(-2) 1_{\{W^\epsilon_n < 0\}}. \end{aligned}$$

Thus we prove Lemma 2.

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