Manifestations of the Parseval identity

Dedicated to Prof. Schôichi Ôta on his sixtieth birthday with compliments and friendship

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Abstract: In this paper, we make structural elucidation of some interesting arithmetical identities in the context of the Parseval identity.

In the continuous case, following Romanoff [R] and Wintner [Wi], we study the Hilbert space of square-integrable functions $L_2(0,1)$ and provide a new complete orthonormal basis—the Clausen system–, which gives rise to a large number of intriguing arithmetical identities as manifestations of the Parseval identity. Especially, we shall refer to the identity of Mikolás-Mordell.

Secondly, we give a new look at enormous number of elementary mean square identities in number theory, including H. Walum's identity [Wa] and Mikolás' identity (1.16). We show that some of them may be viewed as the Parseval identity. Especially, the mean square formula for the Dirichlet L-function at 1 is nothing but the Parseval identity with respect to an orthonormal basis constructed by Y. Yamamoto [Y] for the linear space of all complex-valued periodic functions.

Key words: Parseval identity; orthnormal basis; Dirichlet L-function.

1. The Hilbert space $L_2(0,1)$.

The purpose of the present paper is to show that some number-theoretic identities have very natural hidden structure, i.e. the Parseval identity, that is why they are to hold. To uncover such phenomena, we are to discover suitable complete orthonormal systems. Note that in the case of finite-dimensional normed vector spaces, all orthnormal systems (ONS) consisting of the dimension number of elements are complete and therefore, we immediately obtain the Parseval identities. However, in the Hilbert space $L_2(0,1)$, the space of all square-integrable functions, completeness is essential to attain the Parseval identity. This section is a sequel to [KTZ] and by incorporating the studies of Wintner [Wi] and Romanoff [R], provides a new orthonormal basis (ONB)—the Clausen system, $\{1, \alpha_{n;k}(x)\}$ ($n \not\equiv k \bmod 2$) given in Theorem 2.

The following lemma is a slight modification of Wintner's result [Wi, pp. 566–569].

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Lemma 1. *Let the system $\{\phi(kt)\}$, where we understand $\phi(0t) = 1$ and ϕ is an L_2-function of mean value 0, be an ONS and each is given by*

(1.1)
$$\phi(kt) \sim \sum_{n=1}^{\infty} (a_n \cos(2\pi knt) + b_n \sin(2\pi knt)) \quad (a_0 = 0).$$

For any function $f \in L_2(0,1)$, let

$$f(t) - c_0 \sim \sum_{k=1}^{\infty} c_k \phi(kt) \tag{1.2}$$

be its orthogonal expansion, whence in particular, $\sum_{n=1}^{\infty} c_n^2 < \infty$. Suppose the growth conditions

$$A_n = \sum_{d\,|n} |c_d a_{n/d}| = O\Big(n^{-\frac{1}{2}-\delta}\Big), \tag{1.3}$$
$$B_n = \sum_{d\,|n} |c_d b_{n/d}| = O\Big(n^{-\frac{1}{2}-\delta}\Big)$$

hold in view of the assumption

$$c_n = O\Big(n^{-\frac{1}{2}-\delta}\Big), \quad \forall \delta > 0. \tag{1.4}$$

Then the partial sums $f_n(t) = \sum_{k=1}^{n} c_k \phi(kt)$ converge to $f(t) - c_0$ l.i.m., i.e. $\int_0^1 |f - c_0 - f_n|^2 dt \to 0$ as $n \to \infty$, and the Fourier series of f is given by

(1.5)
$$f(t) - c_0 \sim \sum_{n=1}^{\infty} (\alpha_n \cos(2\pi nt) + \beta_n \sin(2\pi nt)) \ (\alpha_0 = 0),$$

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where

$$\alpha_n = \sum_{d\,|\,n} c_d a_{n/d}, \quad \beta_n = \sum_{d\,|\,n} c_d b_{n/d}. \tag{1.6}$$

Proof is essentially given in [Wi, pp. 567–568]. The main ingredients are that the sequence of partial sums $\{f_n(t)\}$ is a Cauchy sequence in view of (1.3) and that there exists a function to which the sequence $f_n(t)$ converges l.i.m., which must be $f(t) - c_0$ by assumption. The statement becomes simpler if we restrict to those f whose mean values are 0. Then we need to add the non-vanishing constant 1 to the system $\{\phi(kt)\}$, $k \in \mathbf{N}$ (cf. [Wi, pp. 564–565]).

Let

$$l_s(t) = \sum_{n=1}^{\infty} \frac{e^{2\pi itn}}{n^s}, \quad \sigma = \mathrm{Re}\, s > 1 \tag{1.7}$$

be the polylogarithm function, and for $s = k \in \mathbf{N}$, let

$$l_k^c(t) = \sum_{n=1}^{\infty} \frac{\cos(2\pi tn)}{n^k},\ l_k^s(t) = \sum_{n=1}^{\infty} \frac{\sin(2\pi tn)}{n^k}. \tag{1.8}$$

Under this notation, Wintner [Wi, Statement (II), p. 566] essentially proved that for $\nu \in \mathbf{N}$, each of

$$\{1, l_\nu^c(kt)\},\ \{l_\nu^s(kt)\}$$

forms a basis of $L_2(0, \frac{1}{2})$. We may proceed slightly further to contend.

Proposition 1. *Each of the following systems is complete in $L_2(0,1)$:*

$$\big\{1, l_{2k}^c(kt)\big\} \cup \{l_{2k-1}^s(kt)\},$$

the system of (periodic) Bernoulli polynomials, which we denote by $\{\bar{B}_k(t)\}$;

$$\big\{l_{2k-1}^c(kt)\big\} \cup \{1, l_{2k}^s(kt)\},$$

the system of Clausen functions, which we denote by $\{A_k(t)\}$.

Proof. We show that $\{1, l_\nu^c(kt)\}$ and $\{1, \cos(2\pi kt)\}$ are equivalent, i.e. that their closures coincide. Indeed, using $f(t) = \cos(2\pi kt)$ and $\phi(kt) = l_\nu^c(kt)$, in (1.2) we deduce that α_n are all 0 except for $n = k$; $\alpha_k = 1$. Denoting this function by $E(n)$, and applying the Möbius inversion formula we derive

$$c_n = \sum_{d\,|\,n} \mu\Big(\frac{n}{d}\Big)\Big(\frac{n}{d}\Big)^{-\nu} E(d). \tag{1.9}$$

Hence it follows that $c_n = O(n^{-\nu})$ and so (1.4) is satisfied and *a fortiori*, (1.3). Hence, Lemma 1 assures that $\cos(2\pi kt)$ is well approximated by finite combinations of $l_\nu^c(kt)'s$, so that it lies in the closure of $\big\{1, l_\nu^c(kt)\big\}$. The reverse inclusion being clear, we conclude that these systems are equivalent. Similarly we can prove the equivalence of $\{1, l_\nu^s(kt)\}$ and $\{1, \sin(2\pi kt)\}$. Hence we conclude the assertion in view of the completeness of the trigonometric functions. □

At this point we quote the following result from [KTZ] and [R].

Proposition 2 ([KTZ, Proposition 2] and [R])**.** *Each of the sequences $\{1, \beta_{n;k}(x)\}$ and $\{1, \alpha_{n;k}(x)\}$ $(n \not\equiv k \bmod 2)$ forms an orthonormal system of $L_2(0,1)$, where*

$$\beta_{n;k}(x) = \frac{1}{\sqrt{\binom{2k}{k}^{-1} B_{2k}\varphi_{2k}(n)}} \sum_{d\,|\,n} \mu\Big(\frac{n}{d}\Big) d^k B_k(dx - [dx]), \tag{1.10}$$

$$\alpha_{n;k}(x) = \frac{1}{\pi\sqrt{\binom{2k}{k}^{-1} B_{2k}\varphi_{2k}(n)}} \sum_{d\,|\,n} \mu\Big(\frac{n}{d}\Big) d^k A_k(dx - [dx]), \tag{1.11}$$

and where $\varphi_{2k}(n) = n^{2k} \prod_{p\,|\,n} \Big(1 - \frac{1}{p^{2k}}\Big)$ is the Jordan totient function.

Now from Proposition 1 and Proposition 2 we deduce the following result. The Bernoulli system is due to Romanoff [R] and the Clausen system is new.

Theorem 1. *Each of $\{1, \beta_{n;k}(x)\}$ and $\{1, \alpha_{n;k}(x)\}$ $(n \not\equiv k \bmod 2)$ forms a complete orthonormal basis of $L_2(0,1)$.*

Now we shall give some examples of orthogonal expansions and Parseval identities based on Theorem 1. We first evaluate the orthogonal coefficient, with $\zeta(s,u)$ designating the Hurwitz zeta-function,

$$c_{n;k}(s) = \int_0^1 \zeta(s,u)\alpha_{n;k}(u)\mathrm{d}u, \tag{1.12}$$

which reduces to

$$\int_0^1 \zeta(s,u) A_k(bu)\mathrm{d}u$$

for $b \in \mathbf{N}$. As in the proof of [KTZ, Corollary 2], we may prove the following result which is a generalization of [KTZ, (20)], which in turn is a generalization of the results of [EM1, EM2].

Proposition 3. *We have for $\sigma < 0$*

(1.13)

$$\int_0^1 \zeta(s,u)A_k(bu)\mathrm{d}u$$

$$= -k!(2\pi)^{s-k}\cos\frac{\pi(s+k)}{2}b^{s-1}\Gamma(1-s)\zeta(1+k-s).$$

Rewriting (1.11) in the form

$$\alpha_{n;k}(u) = \frac{1}{a_{n;k}}\sum_{\delta|n}\mu\Big(\frac{n}{\delta}\Big)\delta^k A_k(\{\delta u\})$$

with $a_{n;k} = \pi\sqrt{2kk^{-1}B_{2k}\varphi_{2k}(n)}$, and using (1.13) we evaluate the orthogonal coefficients:

$$\begin{aligned}(1.14)\quad c_{n;k}(s) &= \frac{1}{a_{n;k}}\sum_{\delta|n}\mu\Big(\frac{n}{\delta}\Big)\delta^k\int_0^1\zeta(s,u)A_k(\delta u)\mathrm{d}u\\ &= -\frac{k!}{a_n;k}\varphi_{k+s-1}(n)(2\pi)^{s-k}\cos\frac{\pi(s+k)}{2}\\ &\quad\times\Gamma(1-s)\zeta(1+k-s).\end{aligned}$$

Example 1. *For $\sigma < 0$, we have the orthogonal expansion with respect to $\{1, \alpha_{n;k}(u)\}$*

$$\zeta(s,u) = \sum_{n,k} c_{n;k}\alpha_{n,k}(u),$$

where $c_{n;k} = c_{n;k}(s)$ is given by (1.14).

Using (1.14), we may easily write down the Parseval formula for

$$\int_0^1 \zeta(s,u)\zeta(z,u)\mathrm{d}u.$$

However, we shall not evaluate it here and refer to the Mordell-Mikolás formula given in the following remark.

Remark 1. *For $a,\ b \in \mathbf{N}$ we have*

$$\begin{aligned}(1.15)\quad &\int_0^1 \zeta(s,\{au\})\zeta(z,\{bu\})\mathrm{d}u\\ &= \Big(\frac{a}{d}\Big)^{s-1}\Big(\frac{b}{d}\Big)^{s-1}(2\pi)^{s+z-2}\cos\frac{\pi(s-z)}{2}\\ &\quad\times\Gamma(1-s)\Gamma(1-z)\zeta(2-s-z),\end{aligned}$$

valid for $\max\{\sigma,0\}+\max\{\mathrm{Re}z,0\}<1$ (*cf.* [Mi1, p. 158]), *where $d=(m,n)$ the g.c.d. of m, n. Indeed,* (1.15) *is equivalent to*

(1.16)

$$\int_0^1 \zeta(s,u)\zeta(z,u)\mathrm{d}u = 2(2\pi)^{s+z-2}\Gamma(1-s)\Gamma(1-z)$$
$$\times\cos\Big(\frac{\pi}{2}(s-z)\Big)\zeta(2-s-z),$$

which is [Mi1, (5.1)]. *Mikolás* [Mi2] *gave the special case of* (1.15) *with $s = z$.*

Proof of (1.15) is easily accessible if we make use of the results of Mikolás [Mi1, Mi2;] and Mordell [Mo], where an argument similar to Mordell's has been given also by Romanoff [R].

We may obtain an enormous amount of new intriguing identities and the following are illustrative examples which will be treated more thoroughly elsewhere.

Example 2. *We have*

$$ie^{\frac{\pi s}{2}i}\Big(e^{\frac{\pi iz}{2}}(-1)^j - e^{-\frac{\pi iz}{2}}\Big)\zeta(j+1-z)\zeta(1-z+s)$$
$$= j!\sin\pi s\Big[\frac{\zeta^2(k+s)\zeta^2(k+2s-j)\zeta^2(k+1-z)}{\zeta^2(2k)\zeta(2k+3s-j)} - \frac{\zeta^2(k+1+s)\zeta^2(k+1+2s-j)\zeta^2(k+2-z)}{\zeta^2(2k+2)\zeta(2k+2+3s-j)}\Big]$$

Example 3. *Let*

$$\Psi(v,\ u) = \Psi(\omega) = \sqrt{v}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\rho(n)K_{i\kappa}(2\pi|n|v)e^{2\pi inu}$$

be a Maass wave form corresponding to the eigen value $1/4+\kappa^2$ of the automorphic Laplacian, where K signifies the modified Bessel function. Then

$$\frac{\zeta^2(k+2-z)}{\zeta^2(2k+2)}\sum_{n=1}^{\infty}\frac{d(n)K_i\kappa(2v)}{n^{k+1}}\rho(n)$$
$$= \sum_{n=1}^{+\infty}\frac{d(n)K_i\kappa(2\pi nv)}{n^{1-z}}\rho(n).$$

Example 4. *Let*

$$\psi_0(v,\ u) = \psi_0(\omega) = \sqrt{v}\sum_{n\equiv 0\ (24)}T(n)K_0\Big(\frac{2\pi|n|v}{24}\Big)e^{\frac{2\pi inu}{24}},$$

where $T(n)$ is the coefficients defined by [Coh]. *Then*

$$i(24)^{1-z-k}\sin\frac{\pi z}{2}\sum_{n\equiv 0\ (24),\ n>0}\frac{T(n)K_0\Big(\frac{2\pi nv}{24}\Big)}{n^{1-z}}$$

$$= \cos\frac{\pi z}{2}\frac{\zeta^2(k+1-z)}{\zeta^2(2k)}\sum_{n\equiv 0\ (24),\ n>0}\frac{T(n)K_0\Big(\frac{2\pi nv}{24}\Big)d(\frac{n}{24})}{n^k}$$

$$+ i\sin\frac{\pi z}{2}\frac{\zeta^2(k+2-z)}{\zeta^2(2k+2)}\sum_{n\equiv 0(24),\ n>0}\frac{T(n)K_0\Big(\frac{2\pi nv}{24}\Big)d(\frac{n}{24})}{n^{k+1}}.$$

2. Mean square identity and Yamamoto's basis. In the remaining of the paper we will establish a new look at discrete mean square result as Parseval identities.

Let χ be a Dirichlet character modulo N, with $N > 1$. Let $L(s, \chi)$ denote the associated Dirichlet L-function defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \mathrm{Re}\, s > 1.$$

For non-principal χ, the series defining $L(s, \chi)$ is convergent for $\sigma > 0$ and we may speak of the value $L(1, \chi)$, which is related to the class number of a quadratic field.

In the special case of $N = p$, an odd prime, H. Walum [Wa] discovered an intriguing identity

$$\sum_{\chi(-1)=-1} |L(1, \chi)|^2 = \frac{\varphi(p)(p-1)(p-2)}{12p^2} \pi^2, \tag{2.1}$$

where the sum is over all odd Dirichlet characters χ modulo a prime p, and $\varphi(n)$ is the Euler function. Walum's identity (2.1) stimulated some authors to generalize it to the case of composite modulus and the product of two L-functions; cf. [Lou1, Lou2, Lou3], which were later generalized by [LZ] and this last result was recently generalized and elucidated fully in [KMZ]. However, except for the last paper, other authors used ad hoc methods and one cannot see the reason why such identities are to hold.

We push forward the structural study on such sums in [KMZ] and establish the following structural elucidation of the generalization of (2.1) to be proved in §3.

Theorem 2. *As a Parseval identity with respect to the orthonormal basis* (ONB) $\mathrm{X}_0(N)$ *given in Lemma* 3, *we have the identity*

$$\sum_{d|N} \frac{1}{\varphi(\frac{N}{d})} \sum_{\chi \text{ odd, mod } \frac{N}{d}} |L(1, \chi)|^2 \tag{2.2}$$

$$= \left(\frac{\pi}{2N}\right)^2 \sum_{0 \not\equiv a \bmod N} \left(\cot \frac{\frac{a}{(a,N)}}{N} \pi\right)^2.$$

Our approach depends on Y. Yamamoto's [Y] method for the inner product linear space $C(N)$ of all complex-valued periodic functions f of period N with $N \in \mathbf{Z}^+$. We assemble some basic results below.

$$C(N) = \{f | f : \{\mathbf{Z} \to \mathbf{C};\ f(n+N) = f(n)\}.$$

The inner product of $\varphi_1, \varphi_2 \in C(N)$ is defined by

$$(\varphi_1, \varphi_2) = \sum_{a \bmod N} \varphi_1(a)\overline{\varphi_2(a)}, \tag{2.3}$$

where $\bar{S}$ means the complex conjugation of S.

Let χ be a Dirichlet character mod u, then $\chi \in C(ku)$ for $k = 1, 2, \ldots$. For a positive integer d we define $\chi^{(d)} \in C(du)$ by

$$\chi^{(d)}(n) = \begin{cases} \chi\left(\frac{n}{d}\right), & \text{if } d|n, \\ 0, & \text{if } \ d|n. \end{cases} \tag{2.4}$$

Let

$$X(N) = \{\chi^{(d)} | \chi \text{ Dirichlet character mod } u,\ du = N\}.$$

In [Y, Proposition 1.1] Yamamoto showed the following result:

Lemma 2. $\mathrm{X}(N)$ *is an orthogonal system* (OS) *of* $C(N)$.

It is immediate to deduce the following result from Lemma 2.

Lemma 3. *Let* $N > 2$. *Then*

$$\mathrm{X}_0(N) = \left\{ \frac{1}{\sqrt{\varphi(\frac{N}{d})}} \chi^{(d)} \middle| d | N \right\} \tag{2.5}$$

is an ONB of $C(N)$.

We note that the normality of the system in Lemma 3 entails the fact that the Euler function $\varphi(n)$ is the number of natural numbers $\le n$ coprime to n.

If $\{\varphi_1, \cdots, \varphi_n\}$ is an ONB of an inner product space, then for any $c_1, \cdots, c_n \in \mathbf{C}$, we have (a finite form of) the Parseval identity

$$\left\| \sum_{k=1}^{n} c_k \varphi_k \right\|^2 = \sum_{k=1}^{n} |c_k|^2 \leftrightarrow ||f||^2 = \sum_{\chi} |c_\chi|^2, \tag{2.6}$$

for $f \in C(N)$.

We shall also use the following lemma [Vista, Lemma 8.3] in an essential way.

Lemma 4. *For the sum over odd characters* mod q, *we have*

$$\sum_{\chi(-1)=-1} \chi(n) = \begin{cases} 0 & \textit{if } n \not\equiv \pm 1 \pmod q \\ \frac{\varphi(q)}{2} & \textit{if } n \equiv 1 \pmod q \\ -\frac{\varphi(q)}{2} & \textit{if } n \equiv -1 \pmod q. \end{cases}$$

Remark 2. *Lemma* 4 *refers to the difference between two groups,* {all characters} *and* {even characters}. *The orthogonality of these characters give rise to the difference* {odd characters} *which looks like having orthogonality, although it is not a group.*

From now on we shall speak about the orthogonality of odd characters in the above interpretation.

3. Proof of Theorem 2.

Proof. We want to find a function f whose Fourier expansion is

$$\sum_{\chi_0 \in \mathrm{X}_0(N)} c_{\chi_0} \chi_0,$$

with

$$c_{\chi_0} = \begin{cases} \frac{1}{\sqrt{\varphi(\frac{N}{d})}} L(1, \overline{\chi}), & \text{if } \chi(-1) = -1, \\ 0, & \text{if } \chi(-1) = 1, \end{cases} \tag{3.1}$$

where $\chi \bmod N/d$ corresponds to χ_0 as in (2.4).

For any $a \not\equiv 0 \pmod N$, one has

$$\begin{aligned} f(a) &= \sum_{\chi_0} c_{\chi_0} \chi_0(a) \\ &= \sum_{d|N} \frac{1}{\sqrt{\varphi(\frac{N}{d})}} \sum_{\chi\ odd} L(1, \overline{\chi}) \frac{1}{\sqrt{\varphi(\frac{N}{d})}} \chi^{(d)}(a) \\ &= \sum_{\substack{d|N \\ d|a}}^{\infty} \frac{1}{\varphi(\frac{N}{d})} \sum_{\chi\ odd} L(1, \overline{\chi}) \chi\left(\frac{a}{d}\right). \end{aligned}$$

But noting that χ is a Dirichlet character $\bmod N/d$, we see that $\chi(\frac{a}{d}) \neq 0$ only if $(\frac{a}{d}, \frac{N}{d}) = 1$, i.e. $(a, N) = d$. Hence only one value (a, N) of d is possible, and

$$f(a) = \frac{1}{\varphi(\frac{N}{(a,N)})} \sum_{\substack{\chi\ odd \\ \mathrm{mod}\, \frac{N}{(a,N)}}} L(1, \overline{\chi}) \chi\left(\frac{a}{(a, N)}\right). \tag{3.2}$$

We appeal to the following form of the Dirichlet class number formula [Vista, (8.30), p. 174]

$$L(1, \overline{\chi}) = \frac{\pi}{2N} \sum_{0 \not\equiv b \bmod N} \overline{\chi}(b) \cot \frac{b}{N} \pi \tag{3.3}$$

which is valid for all χ not necessarily primitive, to rewrite (3.2) as

$$\begin{aligned} f(a) = \frac{\pi}{2N} \frac{1}{\varphi(\frac{N}{(a,N)})} &\sum_{b \not\equiv 0 \bmod N} \cot \frac{b}{N} \pi \\ \times &\sum_{\substack{\chi\ odd \\ \mathrm{mod}\, \frac{N}{(a,N)}}} \overline{\chi}(b) \chi\left(\frac{a}{(a, N)}\right). \end{aligned}$$

By the *orthogonality of odd characters*, this becomes

$$f(a) = \frac{\pi}{2N} \cot \frac{\frac{a}{(a,N)}}{N} \pi. \tag{3.4}$$

Next, the Parseval identity for f as

$$||f||^2 = \sum_{\chi_0} |c_{\chi_0}|^2 = \sum_{d|N} \frac{1}{\varphi(\frac{N}{d})} \sum_{\substack{\chi\ odd \\ \mathrm{mod}\, \frac{N}{(d)}}} |L(1, \chi)|^2. \tag{3.5}$$

Hence, by (3.4)

$$\begin{aligned} ||f||^2 &= \sum_{a \not\equiv 0 \bmod N} |f(a)|^2 \\ &= \left(\frac{\pi}{2N}\right)^2 \sum_{a \not\equiv 0 \bmod N} \left(\cot \frac{\frac{a}{(a,N)}}{N} \pi\right)^2. \end{aligned} \tag{3.6}$$

Equating (3.5) and (3.6), we complete the proof of Theorem 2. □

Corollary 1. *For $N = p$ an odd prime,* (2.2) *reduces to* (2.1).

Indeed, (2.2) reads

$$\sum_{d|p} \frac{1}{\varphi(\frac{p}{d})} \sum_{\substack{\chi\ odd \\ \mathrm{mod}\, \frac{p}{(d)}}} |L(1, \chi)|^2 = \left(\frac{\pi}{2p}\right)^2 \sum_{a=1}^{p-1} \left(\cot \frac{a}{p} \pi\right)^2. \tag{3.7}$$

Applying the inverse Eisenstein formula [Vista, (8.41), p. 178]

$$\cot \frac{a}{p} \pi = 2i \sum_{k=1}^{p-1} \bar{B}_1\left(\frac{k}{p}\right) e^{2\pi i \frac{a}{p} k},$$

where $\bar{B}_1(x) = B_1(x - [x])$ with $[x]$ designating the integral part of x and $B_1(x)$ denotes the 1st Bernoulli polynomial, we conclude the assertion.

Remark 3. *The proof hinges on the pseudo-group structure of the set of all odd characters and the form of the class number formula* (3.3) *valid for all odd characters, in the spirit of* [HKT] (*cf.* [Vista, Chapter 8]). *The ordinary form of the class number formula for an odd primitive character* mod N *is*

$$L(1, \chi) = -\frac{\pi i}{G(\overline{\chi})} \sum_{a=1}^{N} \overline{\chi}(a) \bar{B}_1\left(\frac{a}{N}\right). \tag{3.8}$$

Although we cannot apply the above argument to (3.8) *for a general modulus, we can give a structural proof of Walum's formula for the prime modulus case, all the non-principal characters being primitive.*

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