38. *An Almost Periodic Function in the Mean.*

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Let x be a variable point in a measurable point set R_n of n dimensional Euclidean space. Let the function $f(t;x)$ be defined for all $x \subset R_n$ and $-\infty < t < \infty$; summable with index $p \geqq 1$ in x for all t in the sense of Lebesgue integral,[1] continuous in t; and continuous in the mean for x, that is, for any given $\varepsilon > 0$ there exists a point δ in n dimensional Euclidean space such that

$$\int_{R_n} |f(t;x+\delta)-f(t;x)|^p\,dx < \varepsilon^p$$

for all t.

Displacement numbers τ will be taken in the direction of the t-axis; these will be defined as follows:

We say τ is a displacement number of $f(t;x)$ belonging to ε if

$$\int_{R_n} |f(t+\tau;x)-f(t;x)|^p\,dx \leqq \varepsilon^p$$

uniformly for all t.

A function $f(t;x)$ is said to be almost periodic in the mean in t in any region as above if all the possible displacement numbers of $f(t;x)$ belonging to any given ε form a relatively dense set of numbers along the t-axis.

Muckenhoupt[2] and Avakian[3] have studied an almost periodic function in the mean with index 2 and applied the theory to some physical problems.

Bochner[4] has also shown that an almost periodic function in the mean with index $p \geqq 1$ has the Fourier series

$$f(t;x) \sim \sum_n A_n(x) e^{i\Lambda_n t}$$

$$\int_{R_n} |A_n(x)|^p\,dx < \infty$$

and proved "approximation theorem" for this class of an almost periodic function, whose enunciation is as follows:

For any almost periodic function in the mean with index $p \geqq 1$ there exists always a sequence of exponential polynomials

1) In this paper the integral means always the Lebesgue integral.

2) Muckenhoupt, Almost Periodic Functions and Vibrating Systems. Journ. Math. Phys., Massachusetts Institute of Technology, **8** (1929), 163.

3) Avakian, Almost Periodic Functions and the Vibrating Membrane. Journ. Math. Phys., Massachusetts Institute of Technology, **14** (1935), 350.

4) Bochner, Abstrakte fastperiodische Funktionen. Acta Mathematica, **61** (1933), 149.

$$S_m(t;\, x)=\sum_n r_n^{(m)} A_n(x)\, e^{i\Lambda_n t}$$

with rational coefficients $r_n^{(m)}$ such that

$$\lim_{m=\infty}\int_{R_n} |f(t;\, x)-S_m(t;\, x)|^p\, dx=0\,, \qquad -\infty<t<\infty$$

In this note we shall prove some structural properties of an almost periodic function in the mean with index $p>1$.

Lemma. If a and b are non-negative, then the inequality

$$(a+b)^p \leqq (1+m)^{p-1}a^p+\left(1+\frac{1}{m}\right)^{p-1} b^p$$

is true for any positive numbers m and $p>1$.

Proof. $x^p(p>1)$ is a convex function in any positive interval.

Then we have

$$\left(\frac{a+mb}{1+m}\right)^p \leqq \frac{a^p+mb^p}{1+m}\,,$$

where m is any positive number.

Putting

$$\frac{a}{1+m}=A\,, \qquad \frac{mb}{1+m}=B$$

the above inequality may be written

$$(A+B)^p \leqq (1+m)^{p-1}A^p+\left(1+\frac{1}{m}\right)^{p-1} B^p\,. \qquad \text{q. e. d.}$$

Now the following two theorems 1° and 2° may be easily proved from our definition of an almost periodic function in the mean.

1° Every function almost periodic in the mean is bounded in the mean.

2° Every function almost periodic in the mean is uniformly continuous in the mean.

3° If $f(t;\, x)$ is a function almost periodic in the mean, then the square $\{f(t;\, x)\}^2$ is also almost periodic in the mean.

Proof. Using Schwarz' inequality we have

$$\int_{R_n} \left|\left\{f(t+\tau;\, x)\right\}^2-\left\{f(t;\, x)\right\}^2\right|^p dx$$

$$=\int_{R_n} |f(t+\tau;\, x)-f(t;\, x)|^p\, |f(t+\tau;\, x)+f(t;\, x)|^p\, dx$$

$$\leqq\left[\int_{R_n} |f(t+\tau;\, x)-f(t;\, x)|^{2p}\, dx\right]^{\frac{1}{2}}\left[\int_{R_n} |f(t+\tau;\, x)+f(t;\, x)|^{2p}\, dx\right]^{\frac{1}{2}}.$$

Since the inequality

$$\int_{R_n} \varphi^2\, dx \leqq \left[\int_{R_n} \varphi\, dx\right]^2$$

is true if φ is always positive, our next step is as follows:

$$\leqq \int_{R_n} |f(t+\tau;x)-f(t;x)|^p\,dx \int_{R_n} |f(t+\tau;x)+f(t;x)|^p\,dx$$

$$\leqq 2^{p-1}\left[\int_{R_n} |f(t+\tau;x)-f(t;x)|^p\,dx\right]$$

$$\times\left[\int_{R_n} |f(t+\tau;x)|^p\,dx+\int |f(t;x)|^p\,dx\right]$$

$$\leqq 2^{p-1} M\varepsilon^p \leqq \varepsilon_1^p .$$

As a corollary of 3°, we have the following theorem.

4° The absolute value of a function almost periodic in the mean is likewise almost periodic in the mean.

Next we wish to prove:

5° If τ_1 is a displacement number for $f(t;x)$ belonging to ε_1, then there exists a $\delta>0$ depending on ε_1 and an $\varepsilon_2>\varepsilon_1$ such that if $|\tau_2-\tau_1|<\delta$ then τ_2 is a displacement number belonging to ε_2.

Proof. Let δ be chosen according to 2° such that for any two points t' and t'', $|t'-t''|<\delta$, we have

$$\int_{R_n} |f(t';x)-f(t'';x)|^p\,dx \leqq \frac{\varepsilon_2^p-\varepsilon_1^p}{2(1+m)^{p-1}} ,$$

where m is any positive number large enough such that

$$\left(1+\frac{1}{m}\right)^{p-1}\varepsilon_1^p < \frac{\varepsilon_1^p+\varepsilon_2^p}{2} .$$

Then using the above lemma, we get

$$\int_{R_n} |f(t+\tau_2;x)-f(t;x)|^p\,dx$$

$$=\int_{R_n} |f(t+\tau_2;x)-f(t+\tau_1;x)+f(t+\tau_1;x)-f(t;x)|^p\,dx$$

$$\leqq (1+m)^{p-1}\int_{R_n} |f(t+\tau_2;x)-f(t+\tau_1;x)|^p\,dx$$

$$+\left(1+\frac{1}{m}\right)^{p-1}\int_{R_n} |f(t+\tau_1;x)-f(t;x)|^p\,dx$$

$$\leqq (1+m)^{p-1}\frac{\varepsilon_2^p-\varepsilon_1^p}{2(1+m)^{p-1}}+\left(1+\frac{1}{m}\right)^{p-1}\varepsilon_1^p < \frac{\varepsilon_2^p-\varepsilon_1^p}{2}+\frac{\varepsilon_1^p+\varepsilon_2^p}{2}=\varepsilon_2^p .$$

From 5° we can easily deduce:

6° Any two functions almost periodic in the mean with index p are simultaneously almost periodic in the mean with index p.

7° The sum and difference of any two functions almost periodic in the mean with index p is likewise almost periodic in the mean with index p.

On the other hand, using 3°, 7° and the equality

$$f(t;x)\,g(t;x)=\frac{1}{4}\left[\left(f(t;x)+g(t;x)\right)^2-\left(f(t;x)-g(t;x)\right)^2\right]$$

we have:

8° The product of any two functions almost periodic in the mean with index p is likewise almost periodic in the mean with index p.

9° Let $f(t;x)$ and $g(t;x)$ be any two almost periodic functions with index p such that

$$\underset{-\infty<t<\infty}{l.\ b.}\ |g(t;x)|=m>0 \qquad \text{for all } x\subset R_n,$$

then $f(t;x)/g(t;x)$ is likewise almost periodic in the mean with index p.

10° An almost periodic function in the mean with index p is likewise almost periodic in the mean with index $p'(p>p'>1)$.

Proof. Using the well-known inequality

$$\left[\frac{1}{mR_n}\int_{R_n}|f(t;x)|^{p'}dx\right]^{\frac{1}{p'}}\leqq\left[\frac{1}{mR_n}\int_{R_n}|f(t;x)|^{p}dx\right]^{\frac{1}{p}},$$

where mR_n means the measure of R_n, we get immediately the relations

$$\int_{R_n}|f(t;x)|^{p'}dx<\infty$$

and

$$\left[\frac{1}{mR_n}\int_{R_n}|f(t+\tau;x)-f(t;x)|^{p'}dx\right]^{\frac{1}{p'}}$$
$$\leqq\left[\frac{1}{mR_n}\int_{R_n}|f(t+\tau;x)-f(t;x)|^{p}dx\right]^{\frac{1}{p}}\leqq\varepsilon_1.$$

The theorem 10° gives us the modifications of the theorems 7°, 8° and 9°.

The sum, difference and product of two almost periodic functions in the mean whose indices are p and q respectively are likewise almost periodic in the mean with index min (p, q).

Let $f(t;x)$ and $g(t;x)$ be any two almost periodic functions in the mean whose indices are p and q respectively such that

$$\underset{-\infty<t<\infty}{l.\ b.}\ |g(t;x)|=m>0 \qquad \text{for all } x\subset R_n,$$

then $f(t;x)/g(t;x)$ is likewise almost periodic in the mean with index min (p, q).

11° Let $f_m(t;x)$ be a sequence of an almost periodic functions in the mean with index p converging in the mean with index p to a function $f(t;x)$ uniformly in all t. Then $f(t;x)$ is necessarily almost periodic in the mean with index p.

Proof. From the definition of $f_m(t;x)$, we have

$$\left[\int_{R_n}|f_m(t+\tau;x)-f_m(t;x)|^{p}dx\right]^{\frac{1}{p}}\leqq\frac{\varepsilon}{3}. \tag{1}$$

Now there must exist an integer $N\left(\frac{\varepsilon}{3}\right)$ such that for all $m \geqq N\left(\frac{\varepsilon}{3}\right)$ the following two inequalities hold:

$$\left[\int_{R_n} |f(t; x)-f_m(t; x)|^p \, dx\right]^{\frac{1}{p}} \leqq \frac{\varepsilon}{3}, \tag{2}$$

$$\left[\int_{R_n} |f(t+\tau; x)-f_m(t+\tau; x)|^p \, dx\right]^{\frac{1}{p}} \leqq \frac{\varepsilon}{3}. \tag{3}$$

Using Minkowski's inequality and (1), (2), (3), we have

$$\left[\int_{R_n} |f(t+\tau; x)-f(t; x)|^p \, dx\right]^{\frac{1}{p}} \leqq \varepsilon.$$