97. Note on Dirichlet Series. X. Remark on S. Mandelbrojt's Theorem

By Chuji TANAKA
Mathematical Institute, Waseda University, Tokyo

(Comm. by Z. SUETUNA, M.J.A., Oct. 12, 1953)

(1) **Introduction.** Let us put

(1.1) $F(s)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty).$

Let $F(s)$ be uniformly convergent in the whole plane. Then $F(s)$ defines the integral function, and for any given σ, $\operatorname{Sup}_{-\infty<t<+\infty}|F(\sigma+it)|$ has the finite value $M(\sigma)$. After J. Ritt[1] (pp. 18–19), we can define the order and type of $F(s)$ as follows:

Definition I. *The order σ of* (1.1) *is defined by*

$$\rho=\overline{\lim_{\sigma\to-\infty}}\ 1/(-\sigma).\ \log^+\log^+ M(\sigma),$$

where $\log^+ x=\mathrm{Max}(0, \log x)$. If $0<\rho<+\infty$, then the type k of (1.1) *is defined by*

$$k=\overline{\lim_{\sigma\to-\infty}}\ 1/\exp((-\sigma)\rho).\ \log^+ M(\sigma).$$

Definition II. *Let $D(r; C)$ be the curved strip which is generated by circles with radii r, and having its centres on the analytic curve C, which extends to $\Re(s)=-\infty$. Then the order $\rho(D)$ in D is defined by*

$$\rho(D)=\lim\ 1/(-\sigma).\ \log^+\log^+ M(\sigma; D),$$

where $M(\sigma; D)=\mathrm{Max}_{s\in D,\ \Re(s)=\sigma}|F(s)|$. If $0<\rho(D)<+\infty$, then the type $k(D)$ in D is defined by

$$k(D)=\overline{\lim_{\sigma\to-\infty}}\ 1/\exp((-\sigma)\rho(D)).\ \log^+ M(\sigma; D).$$

S. Mandelbrojt has proved the following.

Theorem (S. Mandelbrojt[1] p. 19). *Let* (1.1) *with* $\underline{\lim}_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=h>0$, $\overline{\lim}_{n\to+\infty} n/\lambda_n=\delta(\leqq 1/h)$ *be simply (necessarily absolutely) convergent in the whole plane. Then, in any strip: $|\Im(s)-t|\leqq\pi(\delta+\varepsilon)$ (t: arbitrary but fixed, ε: any given positive constant),* (1.1) *has the same order as in the whole plane.*

In this note, we shall generalize it as follows:

Theorem. *Let* (1.1) *with* $\underline{\lim}_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=h>0$, $\overline{\lim}_{n\to+\infty} n/\lambda_n=\delta$ $(\leqq 1/h)$ *be simply (necessarily absolutely) convergent in the whole plane. Then, in any curved strip $D(\pi(\delta+\varepsilon); C)$ (ε: any given positive constant),* (1.1) *has the same order as in the whole plane.*

If furthermore $\delta=0$, then in $D(\varepsilon; C)$, (1.1) *has the same order and type as in the whole plane.*

Remark. G. Pólya[2] (p. 627) has proved the second part of this theorem in the case of Taylor series by the very complicated method.

(2) **Lemma.** We shall establish next lemma, which is a generalization of J. J. Gergen-S. Mandelbrojt's theorem[1)3)]. (1) pp. 13–14, 3) pp. 4–6).

Lemma. *Under the same conditions as in the theorem, we have*

$$\sup_{\Re(s)=\Re(s_0)} |F(s)| \leqq A\cdot \max_{|u-s_1|=\pi(\sigma+\varepsilon)} |F(u)|$$

where (i) s_0, s_1: *two arbitrary points, but* $\Re(s_1)=\Re(s_0)-(3\delta\log(e^6/h\delta)+2\varepsilon)$,

(ii) A: *constant depending upon only* ε, δ *and* $\{\lambda_n\}$.

Proof. By $\overline{\lim\limits_{n\to+\infty}}\, n/\lambda_n=\delta<+\infty$, $\sum\limits_{n=1}^{\infty} 1/\lambda_n^2$ converges, so that, putting

$$\varphi_n(z)=\prod_{\substack{\nu=1\\ \nu\neq n}}^{\infty}(1-z^2/\lambda_\nu^2) \tag{2.1}$$

(2.1) is an integral function. Hence, by F. Carlson-A. Ostrowski's theorem[4)] (p. 267), for any given $\varepsilon(>0)$, we have

$$|\varphi_n(z)|<\exp(\pi(\delta+\varepsilon)|z|) \quad \text{for} \quad |z|>R(\varepsilon). \tag{2.2}$$

Setting $\varphi_n(z)=\sum\limits_{\nu=0}^{\infty} c_\nu^{(n)}/\nu!\,.\,z^\nu$, by Cauchy's theorem and (2.2), we get easily

$$|c_\nu^{(n)}/\nu!|<1/r^\nu.\ \exp(\pi(\delta+\varepsilon)r) \quad r=|z|.$$

Since the right-hand side takes its minimum at $r=\nu/\pi(\delta+\varepsilon)$, for sufficiently large ν, we have

$$|c_\nu^{(n)}|<\{\pi(\delta+2\varepsilon)\}^\nu.$$

Hence, there exists a constant $K_1(\varepsilon)$ such that

$$|c_\nu^{(n)}|<K_1(\varepsilon).\ \{\pi(\delta+2\varepsilon)\}^\nu \quad (\nu=1,2,\ldots). \tag{2.3}$$

Putting $\Phi_n(z)=\sum\limits_{\nu=0}^{\infty} c_\nu^{(n)}/z^{\nu+1}$, by (2.3), $\Phi_n(z)$ is convergent for $|z|>\pi\delta$. On account of H. Cramer-A. Ostrowski's theorem[4)] (pp. 49–52), we have

$$\begin{aligned} a_n\varphi_n(\lambda_n)\exp(-\lambda_n s) &= \sum_{\nu=1}^{\infty} a_\nu\varphi_n(\lambda_\nu)\exp(-\lambda_\nu s)\\ &= 1/2\pi i\,.\oint_{|u|=\pi(\delta+3\varepsilon)} F(s-u)\,\Phi_n(u)\,du\,, \end{aligned}$$

so that, by (2.3)

$$\begin{aligned} &|a_n\varphi_n(\lambda_n)\cdot\exp(-\lambda_n s)|\\ &\leqq \max_{|s-u|=\pi(\delta+3\varepsilon)}|F(u)|.\ 1/2\pi.\ \oint_{|u|=\pi(\delta+3\varepsilon)}\Big\{\sum_{\nu=0}^{\infty}|c_\nu^{(n)}/u^{\nu+1}|\Big\}|du|\\ &< \max_{|s-u|=\pi(\delta+3\varepsilon)}|F(u)|\cdot K_1(\varepsilon).\ \sum_{\nu=0}^{\infty}\{(\delta+2\varepsilon)/(\delta+3\varepsilon)\}^\nu. \end{aligned}$$

Therefore, replacing ε by $\varepsilon/3$, we can put

(2. 4) $$|a_n \varphi_n(\lambda_n) \exp(-\lambda_n s)| \leqq C \operatorname*{Max}_{|s-u|=\pi(\delta+\varepsilon)} |F(u)|,$$

where C: a constant depending upon only ε and δ.

On the other hand, by F. Carlson-A. Ostrowski's theorem[4] (p. 267) for any given $\varepsilon(>0)$, and sufficiently large λ_n, we get

$$|1/\varphi_n(\lambda_n)| < \exp\{(3\delta \log(e^6/h\delta) + \varepsilon)\lambda_n\}.$$

Accordingly, there exists a constant $K_2(\varepsilon)$ such that

(2. 5) $$|1/\varphi_n(\lambda_n)| < K_2(\varepsilon) \exp\{(3\delta \underset{(n=1,2,\dots)}{\log}(e^6/h\delta) + \varepsilon)\lambda_n\}.$$

By (2. 4), in which we put $s=s_1(\Re(s_1)=\Re(s_0)-(3\delta\log(e^6/h\delta)+2\varepsilon))$, we obtain

$$|a_n| \exp(-\lambda_n \Re(s_0)) \cdot \exp\{\lambda_n(3\delta \log(e^6/h\delta) + 2\varepsilon)\}$$
$$\leqq |1/\varphi_n(\lambda_n)| \cdot C. \operatorname*{Max}_{|u-s_1|=\pi(\delta+\varepsilon)} |F(u)|,$$

so that

$$|a_n| \exp(-\lambda_n \Re(s_0)) \leqq \exp(-\lambda_n \varepsilon) \cdot K_2 \cdot C. \operatorname*{Max}_{|u-s_1|=\pi(\delta+\varepsilon)} |F(u)|.$$

Hence,

(2. 6) $$\operatorname*{Sup}_{\Re(s)=\Re(s_0)} |F(s)|$$
$$\leqq \sum_{n=1}^{\infty} |a_n| \exp(-\lambda_n \Re(s_0))$$
$$\leqq \{\sum_{n=1}^{\infty} \exp(-\lambda_n \varepsilon)\}. K_2 \cdot C \cdot \operatorname*{Max}_{|u-s_1|=\pi(\delta+\varepsilon)} |F(u)|.$$

By G. Valiron's theorem[4] (p. 4) and $\lim_{n\to\infty} \log n/\lambda_n=0$, the simple convergence-abscissa σ_s of $\sum_{n=1}^{\infty} \exp(-\lambda_n s)$ is given by

$$\sigma_s = \overline{\lim_{n\to\infty}}\ 1/\log\lambda_n.\ \log 1=0,$$

so that $\sum_{n=1}^{\infty} \exp(-\lambda_n \varepsilon) < +\infty$. Thus, by (2. 6) we get

$$\operatorname*{Sup}_{\Re(s)=\Re(s_0)} |F(s)| \leqq A \operatorname*{Max}_{|u-s_1|=\pi(\delta+\varepsilon)} |F(u)|.$$

q.e.d.

(3) **Proof of theorem.**

(I) Let (1. 1) be of order ρ. Then, by definition, there exists at least one sequence $\{\sigma_m\}$ such that

(3. 1)
(i) $\lim_{m\to+\infty} \sigma_m = -\infty$

(ii) $\rho = \lim_{m\to+\infty} 1/(-\sigma_m).\ \log^+\log^+ M(\sigma_m)$.

Let us define two points s_0, s_1 on C such that

(i) $s_0=\sigma_m+it_m$,

(ii) $\Re(s_1)=\Re(s_0)-(r(\delta)+\varepsilon)$, $r(\delta)=3\delta\log(e^6/h\delta)$.

By lemma, in which we replace ε by $\varepsilon/2$, we get

$$M(\sigma_m) \leqq A. \operatorname*{Max}_{|u-s_1|=\pi(\delta+\varepsilon/2)} |F(u)| = A. \underset{|s_1'-s_1|=\pi(\delta+\varepsilon/2)}{|F(s_1')|}.$$

Therefore, putting $\Re(s_1')=\sigma_m'$, we get

$$M(\sigma_m) \leqq A\ |F(s_1')| \leqq A\ M(\sigma_m'; D),$$

where $M(\sigma_m'; D) = \underset{\Re(s)=\sigma_m',\, s\in D}{\mathrm{Max}} |F(s)|$, so that, by (3.1)

$$\rho = \lim_{m\to+\infty} 1/(-\sigma_m) \,.\, \log^+\log^+ M(\sigma_m) \tag{3.2}$$

$$\leqq \overline{\lim_{m\to+\infty}}\, 1/(-\sigma_m') \,.\, \log^+\log^+ M(\sigma_m'; D).\ \overline{\lim_{m\to+\infty}}\, (\sigma_m'/\sigma_m)\,.$$

Since $|\sigma_m - \sigma_m'| \leqq r(\delta) + \varepsilon + \pi(\delta + \varepsilon/2)$, we have evidently

$$\lim_{m\to+\infty} \sigma_m'/\sigma_m = 1\,.$$

Hence, by (3.2),

$$\rho \leqq \overline{\lim_{m\to+\infty}}\, 1/(-\sigma_m').\ \log^+\log^+ M(\sigma_m'; D) \leqq \overline{\lim_{\sigma\to-\infty}}\, 1/(-\sigma).\ \log^+\log^+ M(\sigma; D)\,.$$

Since the opposite inequality is evident, the equality holds, which proves the first part of theorem.

(II) Let (1.1) with $\delta=0$ be of order ρ $(0<\rho<+\infty)$, and of type k. Then, by definition, there exists at least one sequence $\{\sigma_m\}$ such that

$$\text{(3.3)}\quad \begin{cases} \text{(i)} & \lim_{m\to+\infty} \sigma_m = -\infty \\ \text{(ii)} & k = \lim_{m\to+\infty} 1/\exp((-\sigma_m)\rho)\,.\, \log^+ M(\sigma_m)\,. \end{cases}$$

We define two points s_0, s_1 on C such that

(i) $s_0 = \sigma_m + it_m$,

(ii) $\Re(s_1) = \Re(s_0) - \varepsilon'/\pi$. $(0<\varepsilon'<\varepsilon)$

Applying lemma, in which we replace ε by $\varepsilon'/2\pi$, we get

$$M(\sigma_m) \leqq A \underset{|u-s_1|=\varepsilon'/2}{\mathrm{Max}} |F(u)| = A \underset{|s_1'-s_1|=\varepsilon'/2}{|F(s_1')|}\,,$$

so that, putting $\sigma_m' = \Re(s_1')$, $M(\sigma_m) \leqq A.\, M(\sigma_m'; D)$.

Hence, by (3.3)

$$k = \lim_{m\to+\infty} 1/\exp((-\sigma_m)\rho).\ \log^+ M(\sigma_m)$$

$$\leqq \overline{\lim_{m\to+\infty}}\, 1/\exp((-\sigma_m')\rho).\ \log^+ M(\sigma_m'; D)\ \overline{\lim_{m\to+\infty}}\, \exp((\sigma_m - \sigma_m')\rho)$$

$$\leqq \overline{\lim_{\sigma\to-\infty}}\, 1/\exp((-\sigma)\rho).\ \log^+ M(\sigma; D).\ \exp(\varepsilon'(1/\pi + 1/2)\rho)$$

$$= k(D).\ \exp(\varepsilon'(1/\pi + 1/2)\rho)\,.$$

Letting $\varepsilon' \to 0$,

$$k \leqq k(D)\,.$$

Since the opposite inequality is evident, the equality holds, which proves the second part of theorem.

References

1) S. Mandelbrojt: "Séries lacunaires", Actualités scientifiques et industrielles, Paris, 305 (1936).

2) G. Pólya: „Untersuchungen über Lücken und Singularitäten von Potenzreihen", Math. Zeit., **29** (1929).

3) J. J. Gergen-S. Mandelbrojt: "On entire functions defined by a Dirichlet series", Amer. Jour. Math., **53** (1931).

4) V. Bernstein: "Leçons sur les progrès récents de la théorie des séries de Dirichlet", Collection Borel (1933).