150. Supplementary Note on Free Algebraic Systems

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In the previous note,[1] we have defined the free P-algebraic systems which are the generalization of the free A-algebraic systems in the sense of K. Shoda.[2] And we have shown the necessary and sufficient condition for the existence of the free P-algebraic systems, and the other results. In this note, we shall show a relation between the free algebraic system in the sense of G. Birkhoff[3] and the free A-algebraic system (Theorem 1), and shall show a necessary and sufficient condition for the existence of the Birkhoff's free algebraic systems[4] (Theorem 2).

Let V be a system of single-valued compositions—hereafter an algebraic system always means an algebraic system with respect to V. An algebraic system $\mathfrak{A}$ is said to satisfy the composition-identity $p(x_1,\cdots,x_r)=q(x_1,\cdots,x_r)$, if $p(a_1,\cdots,a_r)=q(a_1,\cdots,a_r)$ for every r elements $a_1,\cdots,a_r$ in $\mathfrak{A}$. Let A be a family consisting of composition-identities. If an algebraic system $\mathfrak{A}$ satisfies all the composition-identities in A, we say that $\mathfrak{A}$ satisfies A or that $\mathfrak{A}$ is an A-algebraic system. We denote by $\boldsymbol{K}(A)$ the class consisting of all the A-algebraic systems, and denote by $F(\{a_\mu \mid \mu\in M\}, A)$ the free A-algebraic system with its free generator system $\{a_\mu \mid \mu\in M\}$.

Let $\boldsymbol{K}$ be an arbitrary class consisting of algebraic systems. An algebraic system contained in the class $\boldsymbol{K}$ is called a $\boldsymbol{K}$-algebraic system. A composition-identity $p(x_1,\cdots,x_r)=q(x_1,\cdots,x_r)$ is called a defining composition-identity of $\boldsymbol{K}$, if every $\boldsymbol{K}$-algebraic system $\mathfrak{A}$ satisfies the composition-identity $p(x_1,\cdots,x_r)=q(x_1,\cdots,x_r)$. A $\boldsymbol{K}$-algebraic system $\mathfrak{F}$ with a generator system $\{f_\mu \mid \mu\in M\}$ is called a strongly free $\boldsymbol{K}$-algebraic system with its free generator system $\{f_\mu \mid \mu\in M\}$, and is denoted by $SF(\{f_\mu \mid \mu\in M\}, \boldsymbol{K})$, if, for every set $\{a_\mu \mid \mu\in M\}$ of elements in any $\boldsymbol{K}$-algebraic system $\mathfrak{A}$, the mapping $f_\mu\rightarrow a_\mu$ can be extended to a homomorphism of $\mathfrak{F}$ into $\mathfrak{A}$.

Theorem 1. *Let $\boldsymbol{K}$ be a class of algebraic systems. If there exists a strongly free $\boldsymbol{K}$-algebraic system $SF(\{f_\mu \mid \mu\in M\}, \boldsymbol{K})$, then $SF(\{f_\mu \mid \mu\in M\}, \boldsymbol{K})$ is isomorphic to a free A-algebraic system $F(\{a_\mu \mid$*

1) T. Fujiwara: Note on free algebraic systems, Proc. Japan Acad., **32** (1956).

2) K. Shoda: Allgemeine Algebra, Osaka Math. J., **1** (1949).

3) G. Birkhoff: Lattice Theory, Amer. Math. Soc. Coll. Publ., **25** (1948).

4) In this note, a free algebraic system in the sense of G. Birkhoff is simply called a Birkhoff's free algebraic system or a strongly free algebraic system.

$\mu\in M\}, A)$ by the mapping $a_\mu\to f_\mu$, where A is the family of all the defining composition-identities of the class $\boldsymbol{K}$.

Proof. Clearly $SF(\{f_\mu\mid\mu\in M\},\boldsymbol{K})$ is an A-algebraic system, since every $\boldsymbol{K}$-algebraic system is an A-algebraic system. Hence there exists a homomorphism φ of $F(\{a_\mu\mid\mu\in M\}, A)$ onto $SF(\{f_\mu\mid\mu\in M\},\boldsymbol{K})$, which is an extension of the mapping $a_\mu\to f_\mu$. On the other hand, let a relation $p(f_{\mu_1},\cdots,f_{\mu_r})=q(f_{\mu_1},\cdots,f_{\mu_r})$ hold in $SF(\{f_\mu\mid\mu\in M\},\boldsymbol{K})$. Then the composition-identity $p(x_1,\cdots,x_r)=q(x_1,\cdots,x_r)$ is contained in the family A, because it is obtained from the definition of the strongly free $\boldsymbol{K}$-algebraic system that every $\boldsymbol{K}$-algebraic system satisfies the composition-identity $p(x_1,\cdots,x_r)=q(x_1,\cdots,x_r)$. Hence there exists a homomorphism ψ of $SF(\{f_\mu\mid\mu\in M\},\boldsymbol{K})$ onto $F(\{a_\mu\mid\mu\in M\}, A)$, which is an extension of the mapping $f_\mu\to a_\mu$. Thus, it is easy to see from the existence of the two homomorphisms φ and ψ that $SF(\{f_\mu\mid\mu\in M\},\boldsymbol{K})$ is isomorphic to $F(\{a_\mu\mid\mu\in M\}, A)$ by the mapping $a_\mu\to f_\mu$. This completes the proof.

Let $\boldsymbol{K}$ be an arbitrary class of algebraic systems, and let $\mathfrak{A}$ be an algebraic system. If, for each generator system $\{a_\mu\mid\mu\in M\}$ of $\mathfrak{A}$, there exists a $\boldsymbol{K}$-algebraic system $\mathfrak{B}$ generated by its generator system $\{b_\mu\mid\mu\in M\}$ with the same suffix-set M such that the mapping $b_\mu\to a_\mu$ can be extended to a homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$, we say that $\mathfrak{A}$ is in contact with $\boldsymbol{K}$. The class consisting of all the algebraic systems in contact with $\boldsymbol{K}$ is called the closure of $\boldsymbol{K}$, and denoted by $\overline{\boldsymbol{K}}$. If $\boldsymbol{K}=\overline{\boldsymbol{K}}$, we say that $\boldsymbol{K}$ is closed.

Theorem 2. *Let $\boldsymbol{K}$ be a class of algebraic systems. Then, in order that, for any cardinal number $\mathfrak{a}$, there exists a strongly free $\boldsymbol{K}$-algebraic system with a free generator system consisting of $\mathfrak{a}$ generators, it is necessary and sufficient that there exists a family A of composition-identities such that the closure $\overline{\boldsymbol{K}}$ is equal to the class $\boldsymbol{K}(A)$ of all the A-algebraic systems.*

Proof of necessity. Let A be the family consisting of all the defining composition-identities of the class $\boldsymbol{K}$. Hereafter we shall prove that the closure $\overline{\boldsymbol{K}}$ is equal to $\boldsymbol{K}(A)$. Now the class $\boldsymbol{K}$ is clearly contained in $\boldsymbol{K}(A)$, since every $\boldsymbol{K}$-algebraic system is an A-algebraic system. Hence the closure $\overline{\boldsymbol{K}}$ is contained in the class $\boldsymbol{K}(A)$, because $\boldsymbol{K}(A)$ is closed. On the other hand, let $\mathfrak{B}$ be an algebraic system in the class $\boldsymbol{K}(A)$, and let $\{b_\mu\mid\mu\in M\}$ be any generator system of $\mathfrak{B}$. Then, $\mathfrak{B}$ is homomorphic to a free A-algebraic system $F(\{a_\mu\mid\mu\in M\}, A)$ by the mapping $a_\mu\to b_\mu$. Hence it follows from Theorem 1 that $\mathfrak{B}$ is homomorphic to a strongly free $\boldsymbol{K}$-algebraic system $SF(\{f_\mu\mid\mu\in M\},\boldsymbol{K})$ by the mapping $f_\mu\to b_\mu$. Therefore it is easy to see from the definition of the closure that $\mathfrak{B}$ is contained in $\overline{\boldsymbol{K}}$. Hence the class $\boldsymbol{K}(A)$ is contained in $\overline{\boldsymbol{K}}$, and hence the closure $\overline{\boldsymbol{K}}$ is equal to the class $\boldsymbol{K}(A)$.

Proof of sufficiency. Suppose that there exists a family A of composition-identities such that the closure $\overline{\boldsymbol{K}}$ is equal to the class $\boldsymbol{K}(A)$. Then, for any cardinal number $\mathfrak{a}$, a free A-algebraic system $F(\{a_\mu \mid \mu \in M(\mathfrak{a})\}, A)$[5] is clearly a strongly free $\overline{\boldsymbol{K}}$-algebraic system with its free generator system $\{a_\mu \mid \mu \in M(\mathfrak{a})\}$. Hence a $\boldsymbol{K}$-algebraic system $\mathfrak{B}$ which is isomorphic to $F(\{a_\mu \mid \mu \in M(\mathfrak{a})\}, A)$ is a strongly free $\boldsymbol{K}$-algebraic system with a free system of $\mathfrak{a}$ generators. Hereafter we shall prove the existence of such a $\boldsymbol{K}$-algebraic system $\mathfrak{B}$. Since $F(\{a_\mu \mid \mu \in M(\mathfrak{a})\}, A)$ is contained in the closure $\overline{\boldsymbol{K}}$, there exists a $\boldsymbol{K}$-algebraic system $\mathfrak{B}$ with its generator system $\{b_\mu \mid \mu \in M(\mathfrak{a})\}$ such that the mapping $b_\mu \to a_\mu$ can be extended to a homomorphism φ of $\mathfrak{B}$ onto $F(\{a_\mu \mid \mu \in M(\mathfrak{a})\}, A)$. On the other hand, there exists a homomorphism ψ of $F(\{a_\mu \mid \mu \in M(\mathfrak{a})\}, A)$ onto $\mathfrak{B}$, which is an extension of the mapping $a_\mu \to b_\mu$, since $\mathfrak{B}$ is an A-algebraic system. Hence $\mathfrak{B}$ is clearly isomorphic to $F(\{a_\mu \mid \mu \in M(\mathfrak{a})\}, A)$, from the existence of the two homomorphisms φ and ψ. This completes the proof.

The following corollary can be easily obtained.

Corollary. *Let* $\boldsymbol{K}$ *be a class of algebraic systems. Then, the following three conditions are equivalent:*

(a) *There exists a family* A *of composition-identities such that the class* $\boldsymbol{K}$ *is equal to the class* $\boldsymbol{K}(A)$ *of all the* A*-algebraic systems.*

(b) *For any cardinal number* $\mathfrak{a}$, *there exists a strongly free* $\boldsymbol{K}$*-algebraic system with a free system of* $\mathfrak{a}$ *generators, and every homomorphic image of any* $\boldsymbol{K}$*-algebraic system is a* $\boldsymbol{K}$*-algebraic system.*

(c) *For any cardinal number* $\mathfrak{a}$, *there exists a strongly free* $\boldsymbol{K}$*-algebraic system with a free system of* $\mathfrak{a}$ *generators, and the class* $\boldsymbol{K}$ *is closed.*

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5) $M(\mathfrak{a})$ means a set with its cardinal number $\mathfrak{a}$.