143. *An Extension Theorem*

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In an earlier paper [3], we obtained an extension theorem ([3, Theorem 2.2]; [4, § 79 Theorem 2]) about linear functionals on modulared linear spaces. From the proof of this theorem we conclude immediately:

Extension Theorem. *Let R be a linear space and m a functional on R subject to*

1) $\qquad 0\leqq m(x)\leqq +\infty \quad$ *for every* $x\in R$,

2) $\qquad m(\lambda x+\mu y)\leqq\lambda m(x)+\mu m(y) \quad$ *for* $\lambda+\mu=1;\ \lambda,\mu\geqq 0$.

For a linear manifold A of R, a linear functional φ on A and a real number γ, if

$$\varphi(x)\leqq\gamma+m(x) \quad \textit{for every } x\in A,$$

then we can find a linear functional ψ on R such that

$$\psi(x)=\varphi(x) \quad \textit{for every } x\in A,$$

$$\psi(x)\leqq\gamma+m(x) \quad \textit{for every } x\in R.$$

As an application of this extension theorem, we will prove here Ascoli's theorem [1, 2]: every closed convex set in a Banach space also is weakly closed. Using the terminologies in the book [4], we state this theorem in more general form:

Theorem. *Let R be a convex linear topological space, and A a closed convex set. For any $a\,\overline{\in}\,A$, we can find a continuous linear functional φ on R such that*

$$\varphi(a)>\sup_{x\in A}\varphi(x).$$

Proof. We can suppose $0\in A$ without loss of generality. Since A is closed, for any $a\,\overline{\in}\,A$ we can find a convex pseudo-norm for ichwh

$$\inf_{x\in A}\|x-a\|>0.$$

For such a convex pseudo-norm, putting

$$m(x)=\inf_{y\in A}\|x-y\|,$$

we see easily that $0\leqq m(x)<+\infty$; $m(x)=0$ for $x\in A$, and

$$m(\lambda x+\mu y)\leqq\lambda m(x)+\mu m(y)$$

for $\lambda+\mu=1;\ \lambda,\mu\geqq 0$, because A is convex. Furthermore, putting

$$\varphi_0(\xi a)=\xi m(a),$$

$$\gamma=\sup_{0\leqq\xi\leqq 1}\{\xi m(a)-m(\xi a)\},$$

we obtain a linear functional φ_0 on the linear manifold

$$\{\xi a;\ -\infty<\xi<+\infty\},$$

and we have $\gamma < m(a)$,

$$\varphi_0(\xi a) \leqq \gamma + m(\xi a) \quad \text{for } -\infty < \xi < +\infty,$$

because $m(\xi a)$ is a continuous convex function of ξ. By virtue of Extension Theorem, we can find then a linear functional φ on R for which $\varphi(\xi a) = \varphi_0(\xi a)$ and

$$\varphi(x) \leqq \gamma + m(x) \quad \text{for every } x \in R.$$

Since $m(x) \leqq \| x \|$, we have obviously then,

$$\varphi(\xi x) \leqq \gamma + \| \xi x \|,$$

and hence

$$\varphi(x) \leqq \frac{\gamma}{\xi} + \| x \| \quad \text{for } \xi > 0.$$

Making ξ tend to $+\infty$, we obtain $\varphi(x) \leqq \| x \|$ for every $x \in R$. From this fact, we conclude that φ is continuous. Furthermore, we have $\varphi(x) \leqq \gamma$ for every $x \in A$
and hence

$$\sup_{x \in A} \varphi(x) < \varphi(a).$$

References

[1] G. Ascoli: Sugli spazi lineari metrici e le loro varietà lineari, Annali di Math., **10**, 33 (1932).

[2] S. Mazur: Über konvexe Mengen in linearen normierten Räumen, Stud. Math., **4**, 70 (1933).

[3] H. Nakano: Modulared linear spaces, Jour. Fac. Sci. Univ. Tôkyô, I, **6**, 85 (1951).

[4] H. Nakano: Topology and Linear Topological Spaces, Tôkyô (1951).