29. On Schlicht Functions. I

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It is known that $w=\frac{1+z}{1-z}$ is a schlicht (convex) function with a positive real part for $|z|<1$, and, when $|z|=1$, w corresponds to the imaginary axis. Hence, for $|z|<1$, $\left(\frac{1+z}{1-z}\right)^2$ is a schlicht function with the cut on the negative real axis. For any real number λ, a function

$$\left[\frac{1+z}{1-z}+i\lambda\right]^2; \quad |z|<1$$

is univalent, and for any positive number $\mu\geqq 0$,

$$\left\{\left[\frac{1+z}{1-z}+i\lambda\right]^2+\mu\right\}^{\frac{1}{2}}; \quad |z|<1$$

is a schlicht function with positive real part. For any two real numbers λ_1 and λ_2,

$$\left[\left\{\left[\frac{1+z}{1-z}+i\lambda_1\right]^2+\mu\right\}^{\frac{1}{2}}+i\lambda_2\right]^2$$

is univalent for $|z|<1$.

In such a way, we can form a class of schlicht functions which have a certain type of slits. That is, for any set of real numbers $\lambda_1, \lambda_2,\cdots,\lambda_k$, and for any positive numbers $\mu_1,\mu_2,\cdots,\mu_{k-1}$, a function defined by

$$(1)\qquad \left[\cdots\left\{\left[\left\{\left(\frac{1+z}{1-z}+i\lambda_1\right)^2+\mu_1\right\}^{\frac{1}{2}}+i\lambda_2\right]^2+\cdots+\mu_{k-1}\right\}^{\frac{1}{2}}+i\lambda_k\right]^2$$

is analytic and univalent for $|z|<1$, and the values form a region with a tree-shaped slit.

The chief object of this paper is to give properties of coefficients obtained by Taylor expansion of such a function.

Let $F_k(z)$ be denoted by the function (1), we have

$$(2)\qquad \begin{cases} F_0(z)=\dfrac{1}{(1-z)^2}(1+z)^2\equiv\dfrac{1}{(1-z)^2}[\varphi_0(z)]^2 \\ F_1(z)=\dfrac{1}{(1-z)^2}[1+z+i\lambda_1(1-z)]^2\equiv\dfrac{1}{(1-z)^2}[\varphi_1(z)]^2 \\ \cdots\cdots\cdots\cdots\cdots \\ F_k(z)=\dfrac{1}{(1-z)^2}[\cdots\{[1+z+i\lambda_1(1-z)]^2+\mu_1(1-z)^2\}^{\frac{1}{2}}+\cdots \\ \qquad +i\lambda_k(1-z)]^2\equiv\dfrac{1}{(1-z)^2}[\varphi_k(z)]^2 \\ \cdots\cdots\cdots\cdots\cdots . \end{cases}$$

If we put

$$(3)\qquad \begin{cases} \varphi_k(z)\equiv\alpha_{k0}+\alpha_{k1}z+\alpha_{k2}z^2+\cdots, \\ F_k(z)\equiv A_0^{(k)}+A_1^{(k)}z+A_2^{(k)}z^2+\cdots, \end{cases}$$

we have

$$(4)\qquad \begin{cases} A_0^{(k)}=\alpha_{k0}^2;\quad \alpha_{00}=1,\quad \alpha_{10}=1+i\lambda, \\ A_1^{(k)}=2\alpha_{k0}(\alpha_{k0}+\alpha_{k1});\quad \alpha_{01}=1 \\ A_2^{(k)}-A_1^{(k)}=\alpha_{k0}^2+2\alpha_{k0}\alpha_{k1}+2\alpha_{k0}\alpha_{k2}+\alpha_{k1}^2 \\ A_3^{(k)}-A_2^{(k)}=\alpha_{k0}^2+2\alpha_{k0}\alpha_{k1}+2\alpha_{k0}\alpha_{k2}+\alpha_{k1}^2+2\alpha_{k0}\alpha_{k3}+2\alpha_{k1}\alpha_{k2} \\ \cdots\cdots\cdots\cdots \\ A_n^{(k)}-A_{n-1}^{(k)}=\sum_{m=0}^{n}\sum_{\substack{i+j=m \\ i,j=0,1,2,\cdots}}\alpha_{ki}\alpha_{kj} \\ \cdots\cdots\cdots\cdots\ . \end{cases}$$

At first we show the next

Theorem 1. *For any set of real numbers $\lambda_1, \lambda_2, \cdots$, any set of positive numbers $\mu_1, \mu_2, \cdots$, and for any $k=0, 1, 2, \cdots$, we have*

$$|A_1^{(k)}|\geqq 4,$$

where $A_1^{(k)}$ is defined by (2) and (3). The equality is valid only when $k=0$.

This theorem can be proved by mathematical inductions. $A_1^{(0)}=4$ is clear by $\alpha_{00}=\alpha_{01}=1$. And $A_1^{(1)}=4(1+i\lambda_1)$ satisfies the theorem, and $\alpha_{10}=1+i\lambda_1$ has the positive real part 1.

For a function

$$\varphi(z)=\alpha_0+\alpha_1z+\alpha_2z^2+\cdots$$

and for any real number λ, we have

$$[\varphi(z)+i\lambda(1-z)]^2=(\alpha_0+i\lambda)^2+2(\alpha_0+i\lambda)(\alpha_1-i\lambda)z$$
$$+\{2(\alpha_0+i\lambda)\alpha_2+(\alpha_1-i\lambda)^2\}z^2+[2(\alpha_0+i\lambda)\alpha_3+2(\alpha_1-i\lambda)\alpha_2]z^3+\cdots\ .$$

If we put

$$F(z)=\frac{1}{(1-z)^2}[\varphi(z)+i\lambda(1-z)]^2$$
$$\equiv A_0+A_1z+A_2z^2+\cdots,$$

we have

$$A_0=(\alpha_0+i\lambda)^2$$
$$A_1=2(\alpha_0+i\lambda)(\alpha_0+\alpha_1).$$

Now we assume that, for any real number λ, the real part of α_0 is positive and the absolute value of A_1 is not less than 4. Let, for any real number λ' and any positive number μ, $G(z)$ be a function defined by

$$G(z)=\frac{1}{(1-z)^2}[\{[\varphi(z)]^2+\mu(1-z)^2\}^{\frac{1}{2}}+i\lambda'(1-z)]^2$$
$$\equiv\frac{1}{(1-z)^2}[(\beta_0+\beta_1z+\beta_2z^2+\cdots)+i\lambda'(1-z)]^2$$
$$\equiv B_0+B_1z+B_2z^2+\cdots\ .$$

From the relation

$$(\alpha_0+\alpha_1z+\alpha_2z^2+\cdots)^2+\mu(1-z)^2=(\beta_0+\beta_1z+\beta_2z^2+\cdots)^2,$$

we have

$$\begin{cases} \beta_0^2=\alpha_0^2+\mu \\ \beta_0\beta_1=\alpha_0\alpha_1-\mu \\ 2\beta_0\beta_2+\beta_1^2=2\alpha_0\alpha_2+\alpha_1^2+\mu \\ 2\beta_0\beta_3+2\beta_1\beta_2=2\alpha_0\alpha_3+2\alpha_1\alpha_2 \\ \cdots\cdots\cdots \\ \sum\limits_{\substack{i+j=n\geq 3 \\ i,j=0,1,2,\cdots}} \beta_i\beta_j = \sum\limits_{i+j=n\geq 3} \alpha_i\alpha_j, \end{cases}$$

and hence

$$B_1=2(\beta_0+i\lambda)(\beta_0+\beta_1)=2\alpha_0\left(1+\frac{i\lambda'}{\sqrt{\alpha_0^2+\mu}}\right)(\alpha_0+\alpha_1),$$

where the absolute value of $2(\alpha_0+i\lambda)(\alpha_0+\alpha_1)$ is not less than 4, and α_0 has a positive real part.

The minimum absolute value of $\alpha_0+i\lambda$ is reached by the positive real part of α_0, and when λ' varies on the real axis, the minimum absolute value of $1+\frac{i\lambda'}{\sqrt{\alpha_0^2+\mu}}$ is not less than absolute value of $1+\frac{i\lambda}{\alpha_0}$. Accordingly, the absolute value of $\alpha_0\left(1+\frac{i\lambda'}{\sqrt{\alpha_0^2+\mu}}\right)$ is greater than the real part of α_0 when $\mu>0$, and then, the absolute value of the coefficient B_1 is greater than 4. It is easily verified that the real part of β_0 is positive from $\beta_0=\sqrt{\alpha_0^2+\mu}$. Thus, the theorem has been established.

Next, we consider an asymptotic relation of a coefficient $A_n^{(k)}$ when n approaches to infinity.

From the relation $\varphi_k(1)=2$ for every positive integer k, the following equation

$$(5) \qquad \lim_{n\to\infty}(A_n^{(k)}-A_{n-1}^{(k)})=[\varphi_k(1)]^2=4$$

follows at once from (4). For an arbitrary small positive number ε and a sufficiently large fixed integer n_0, we can verify

$$|A_n^{(k)}-A_{n_0}^{(k)}-4(n-n_0)|<(n-n_0)\varepsilon$$

from (4), and hence we have

$$\lim_{n\to\infty}\frac{A_n^{(k)}}{n}=4.$$

Now the following theorem follows at once.

Theorem 2. *For any positive integer $k\neq 0$, we have a following asymptotic relation*

$$\lim_{n\to\infty}\frac{A_n^{(k)}}{nA_1^{(k)}}=\kappa; \qquad |\kappa|<1.$$

Remark. This theorem means that, for n sufficiently large, we have

$$\begin{cases} \left|\frac{A_n^{(k)}}{A_1^{(k)}}\right|<n; \qquad k\neq 0 \\ \frac{A_n^{(0)}}{A_1^{(0)}}=n. \end{cases}$$