34. A Characteristic Property of L_p-Spaces ($p>1$). II

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(Comm. by K. KUNUGI, M.J.A., March 12, 1960)

In the previous paper,[1] we gave a characteristic property of L_p-spaces ($p>1$). The purpose of this paper is to give another characterization.

In the case of L_p ($p>1$), the transformation

$$Tx(t)=|x(t)|^{p-1}\cdot \operatorname{sgn} x(t) \tag{1}$$

is a one-to-one correspondence between L_p and L_q ($q=p/p-1$), and the functional (called a modular)

$$m(x)=\int_0^1 (T\xi x, x)\, d\xi=\frac{1}{p}\int_0^1 |x(t)|^p\, dt \tag{2}$$

is well defined. Putting

$$\|x\|=\inf_{m(\xi x)\leqq 1}\frac{1}{|\xi|}, \tag{3}$$

we get a norm of L_p and

$$\|x\|=\left(\frac{1}{p}\int_0^1 |x(t)|^p\, dt\right)^{\frac{1}{p}} \qquad (x\in L_p).$$

The conjugate norm of it is

$$\|\bar{x}\|=\sup_{\|x\|\leqq 1}|(\bar{x}, x)|=p^{\frac{1}{p}}\left(\int_0^1 |\bar{x}(t)|^q\, dt\right)^{\frac{1}{q}} \qquad (\bar{x}\in L_q). \tag{4}$$

Then, it is easily seen that the transformation (1) is *norm-preserving*:

$$\|x\|=\|y\| \text{ in } L_p \text{ implies } \|Tx\|=\|Ty\| \text{ in } L_q.$$

In this paper, we will prove that this property of T is characteristic for L_p ($p>1$) among such Banach spaces that have some transformations like (1), namely, conjugately similar spaces.

Definition. A universally continuous semi-ordered linear space R is said to be *conjugately similar*,[2] if R is reflexive and there exists a one-to-one transformation T from R onto its conjugate space $\bar{R}$ with the following properties:

(i) $T(-a)=-Ta$ $(a\in R)$;
(ii) $Ta\leqq Tb$ if and only if $a\leqq b$ $(a, b\in R)$;
(iii) $(Ta, a)=0$ implies $a=0$.

The above transformation T is called a *conjugately similar correspondence*.

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1) K. Honda and S. Yamamuro [1].

2) Throughout this paper, notations and terminologies are according to H. Nakano [2].

In the conjugately similar space, the modular $m(x)$ and the norm $\|x\|$ are defined by the formulas (2) and (3). It is known that, for the conjugate norm (4), we have

$$\|\bar{x}\|=\inf_{\xi>0}\frac{1+\bar{m}(\xi\bar{x})}{\xi} \qquad (\bar{x}\in\bar{R}),$$

where

$$(5) \qquad \bar{m}(\bar{x})=\sup_{x\in R}\{(\bar{x},x)-m(x)\}.^{3)}$$

The correspondence T has the following properties:

$$(6) \qquad m(x)+\bar{m}(Tx)=(Tx,x);$$

$$(7) \quad x\frown y=0 \quad \text{implies} \quad Tx\frown Ty=0;$$

$$(8) \quad x\frown y=0 \quad \text{implies} \quad m(x+y)=m(x)+m(y)$$

$$\text{and} \quad \bar{m}(T(x+y))=\bar{m}(Tx)+\bar{m}(Ty).$$

The fact we are going to prove in this paper is the following

Theorem. *Let R be a universally continuous semi-ordered linear space which has at least two linearly independent elements. If there exists a conjugately similar correspondence T which is norm-preserving for the norms defined by* (3) *and* (4), *then we can find a number $p>1$ such that*

$$T\xi x=\xi^{p-1}Tx$$

for any number $\xi>0$ and $x\in R$.

Before proceeding to the proof, we state the following

Lemma. *Let R be a conjugately similar space with its conjugately similar correspondence T. For the modulars defined by* (2) *and* (5), *if there exists a positive number γ such that, for any $x\in R$,*

$$(*) \qquad \bar{m}(Tx)=\gamma\cdot m(x),$$

then $T\xi x=\xi^{\gamma}Tx$ for every $x\in R$ and $\xi>0$.

Proof. Since R is conjugately similar, for any $a\in R$, $(T\xi a, a)$ is a continuous function of ξ.[4]
Setting $\Phi(\xi)=m(\xi a)$, by (2) and (6), we have

$$\frac{d}{d\xi}\Phi(\xi)=\frac{\gamma+1}{\xi}\Phi(\xi).$$

As the solution of the above differential equation, we have $m(\xi a)=\xi^{\gamma+1}m(a)$ and hence $T\xi x=\xi^{\gamma}\cdot Tx$. q.e.d.

Proof of Theorem. It is enough only to prove the existence of a positive number γ which satisfies the condition $(*)$ in the above lemma. Since the modulars m and $\bar{m}$ are simple and finite,[5] we have that

$\|x\|=1$ is equivalent to $m(x)=1$ and hence

3) This is called the *conjugate modular* of $m(x)$.

4) See H. Nakano [2, §60, Th. 60.2].

5) m is said to be *simple*, if $m(x)=0$ implies $x=0$ and is said to be *finite*, if $m(\xi x)<+\infty$ for every ξ and $x\in R$. Also, see H. Nakano [2, §60, Th. 60.10].

$$\|x\|=1 \text{ implies } \|Tx\|=1+\overline{m}(Tx).^{6)}$$

Therefore, we may set $\overline{m}(Tx)=\gamma$ for every $\|x\|=1$.

We may prove the theorem as for two cases that R has not discrete elements and has complete system of discrete elements,[7] because R is a direct sum of the former and the latter.

I. Let R be non-atomic.

If $m(a)=n$ (positive integer), there exist mutually orthogonal positive elements such that

$$|a|=a_1+a_2+\cdots+a_n \quad \text{and} \quad m(a_i)=1 \quad (i=1,2,\cdots,n).$$

Hence we have $\overline{m}(Ta)=\sum_{i=1}^{n}\overline{m}(Ta_i)=n\cdot\gamma$.

Next, let a be a non-complete element and be $m(a)=1/n$ (n is a positive integer). Then, we can find $a_1, a_2,\cdots, a_{n-1}$ and c of R with the following properties:

1) $a\frown a_i=0$ $(i=1,2,\cdots,n-1)$, $a_i\frown a_j=0$ if $i\neq j$ and $c\frown(a+a_1+a_2+\cdots+a_{n-1})=0$;

2) $m(a_i)=1/n$ $(i=1,2,\cdots,n-1)$ and $m(c)=(n-1)/n$.

Then, we have

$$\overline{m}(Ta)=\overline{m}(T(a+c))-\overline{m}(Tc)=\gamma-\overline{m}(Tc)$$

and

$$\overline{m}(Ta_i)=\overline{m}(T(a_i+c))-\overline{m}(Tc)=\gamma-\overline{m}(Tc),$$

because $m(a+c)=1$ and $m(a_i+c)=1$.

Therefore

$$\overline{m}(Ta)=\frac{1}{n}\left\{\overline{m}(Ta)+\sum_{i=1}^{n-1}\overline{m}(Ta_i)\right\}$$

$$=\frac{1}{n}\{\overline{m}(T(a+a_1+a_2+\cdots+a_{n-1}))\}=\gamma/n,$$

because $m(a+a_1+a_2+\cdots+a_{n-1})=1$. Thus, we have that $m(a)=1/n$ implies $\overline{m}(Ta)=\gamma/n$, if a is not a complete element.

If a is a complete element and $m(a)=1/n$, we have such a partition that $|a|=a_1+a_2$, $a_i>0$, $m(a_i)=1/2n$ and $a_1\frown a_2=0$ and hence $\overline{m}(Ta)=\overline{m}(Ta_1)+\overline{m}(Ta_2)=\gamma/n$, because a_i is not complete elements.

When $m(a)$ is a rational number k, we have $\overline{m}(Ta)=k\cdot\gamma$ by the same methods as above. Therefore, since Ta is continuous with respect to order-topology,[8] we have $\overline{m}(Ta)=\gamma\cdot m(a)$ for any element $a\in R$.

II. Let R be a discrete space with its discrete base $\{e\}_{\lambda\in\Lambda}$:

$$m(e_\lambda)=1 \ (\lambda\in\Lambda) \text{ and } e_\lambda\frown e_\mu=0 \ (\lambda\neq\mu;\ \lambda,\mu\in\Lambda)$$

where Λ is a set of indices.

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6) See S. Yamamuro [4, Th. 3.2.1].

7) An element $a\in R$ is said to be *discrete*, if for every element $x\in R$ such that $|x|\leqq|a|$ there exists a real number α for which $x=\alpha a$. A subset N of R is said to be *complete* in R, if $|x|\frown|y|=0$ for all $x\in N$ implies $y=0$. We say that R is *discrete*, if R has a complete system of discrete elements, and is *non-atomic*, if R has no discrete element.

8) See 4).

For any number ξ and element e_λ, $T\xi e_\lambda$ is a discrete element in $\overline{R}$, because ξe_λ is a discrete element in the conjugate similar space R. Therefore, there exists an increasing continuous function $\varphi_{e_\lambda}(\xi)$ of ξ depending on e_λ such that $T\xi e_\lambda=\varphi_{e_\lambda}(\xi)Te_\lambda$. But then by the assumption, i.e. $\|e_\lambda\|=\|e_\mu\|$ implies $\|T\xi e_\lambda\|=\|T\xi e_\mu\|$ for every $\xi>0$, $\lambda,\mu\in\Lambda$, we know $\varphi_{e_\lambda}(\xi)$ is independent of $\lambda\in\Lambda$, and hence it follows that $(T\xi e_\lambda, e_\lambda)$ is independent of $\lambda\in\Lambda$. Therefore, by (2) and (6) we have

$$m(\xi e_\lambda)=m(\xi e_\mu) \quad \text{and} \quad \overline{m}(T\xi e_\lambda)=\overline{m}(T\xi e_\mu)$$

for every positive number ξ and $\lambda,\mu\in\Lambda$.

Now, we will prove that

$$\overline{m}(T\xi e_\lambda)=\gamma\cdot m(\xi e_\lambda) \quad (\xi>0). \tag{9}$$

For this purpose, define a non-decreasing continuous function $f(\rho)$ as $m(\xi e_\lambda)=\rho$ implies $\overline{m}(T\xi e_\lambda)=f(\rho)\cdot\gamma$.

Then, we have

$$f(\rho)+f(1-\rho)=1 \quad \text{if} \quad 0\leqq\rho\leqq 1 \tag{10}$$

$$f(2\rho)=2f(\rho) \quad \text{if} \quad \rho\geqq 0. \tag{11}$$

To prove (10), take e_λ, e_μ $(\lambda\neq\mu)$ and ρ such that $0<\rho<1$. Then, if $m(\alpha e_\lambda)=\rho$, we can find $\beta>0$ such that $m(\beta e_\mu)=1-\rho$. By (8), we have

$$m(\alpha e_\lambda+\beta e_\mu)=m(\alpha e_\lambda)+m(\beta e_\mu)=1$$

and

$$\overline{m}(T(\alpha e_\lambda+\beta e_\mu))=\overline{m}(T\alpha e_\lambda)+\overline{m}(T\beta e_\mu) = [f(\rho)+f(1-\rho)]\cdot\gamma.$$

To prove (11), take an arbitrary $\rho>0$, and fix e_λ and e_μ $(\lambda\neq\mu)$. Then, there exists $\alpha>0$ such that $m(\alpha(e_\lambda+e_\mu))=2\rho$, which implies that $m(\alpha e_\lambda)=m(\alpha e_\mu)=\rho$ and $\overline{m}(T\alpha e_\lambda)=\overline{m}(T\alpha e_\mu)=f(\rho)\cdot\gamma$. On the other hand, $\overline{m}(T\alpha(e_\lambda+e_\mu))=f(2\rho)\cdot\gamma$.

Hence it follows that $f(2\rho)=2f(\rho)$.

Since $f(\rho)$ is continuous, by (10) and (11), we have $f(\xi)=\xi$, which implies (9).

From the fact that every element x of R is represented as $x=\sum_{\lambda\in\Lambda}\alpha_\lambda e_\lambda$, it follows that $\overline{m}(Tx)=\gamma\cdot m(x)$ for any $x\in R$.

Thus the proof is completed.

Remark 1. If R has at least three linearly independent elements, we can generalize the lemma's assumption (*) as "*there exists a real function $g(\xi)$ such that $\overline{m}(Tx)=g(m(x))$ for every $x\in R$*". Because, the above assumption brings the fact that R has a unique indicatrix.[9]

Remark 2. It is easily seen that the case of one-dimensional is exceptional.

In conclusion, I wish to express my sincere thanks to Prof. S. Yamamuro for his encouragement and kind advice.

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9) See H. Nakano [3, Anhang II, Satz II.6].

References

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