86. *On Locally Cyclic Semigroups*

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A locally cyclic semigroup is defined as follows:

Definition 1. A semigroup S is called locally cyclic if for every $a, b \in S$ there is an element $c \in S$ and positive integers m and n such that $a=c^m$ and $b=c^n$.

A locally cyclic semigroup is a commutative archimedean semigroup [1].

In Dean and Oehmke's paper [2], they proved that the lattice of congruences of a locally cyclic semigroup is a distributive lattice. However the structure of those semigroups has not been studied. In this note we will report the results of the study of the structure of locally cyclic semigroups without detailed proof.

A locally cyclic group is defined to be a locally cyclic semigroup which is a group. This is different from that given by [4]:

(1) A group G is called locally cyclic in the sense of [4] if every finitely generated subgroup of G in the sense of groups is a cyclic group.

This condition is equivalent of the following:

(i) For every $a, b \in G$ there is an element $c \in G$ and integers m and n such that $a=c^m$ and $b=c^n$.

If G is a locally cyclic group in our definition, then G is locally cyclic in the sense of [4], but the converse need not be true. In the case of locally cyclic semigroups, the analogy to (1) is not effective, that is, a locally cyclic semigroup does not necessarily satisfy the following:

(2) Every finitely generated subsemigroup is a cyclic semigroup.

However, we have

Proposition. *S is a locally cyclic semigroup if and only if every finitely generated subsemigroup of S is a subsemigroup of a cyclic subsemigroup of S.*

Definition 1. A semigroup S is said to be "power-joined" if for any two elements $a, b \in S$, there are positive integers m and n such that $a^m=b^n$.

Definition 2. A semigroup S is said to be "power-cancellative" if for any two elements $a, b \in S$, $a^m=b^m$ implies that $a=b$ where m is an arbitrary positive integer.

Lemma 1. *Let S be a commutative semigroup. Then S is power-joined and power-cancellative if and only if S can be embedded into the semigroup R_a^+ of all positive rational numbers with addition.*

Proof. The part of "if" is obvious. The proof of the part of "only if": Let $a \in S$ be fixed. For $b \in S, b \neq a$ there exist positive integers m_1 and n_1 such that

$$a^{m_1} = b^{n_1}.$$

Define a map φ of S into R_a^+ as follows:

$$\varphi(b) = \frac{m_1}{n_1}, \qquad \varphi(a) = 1.$$

Then we can prove that φ is an isomorphism of S into R_a^+.

Theorem 1. *A nonpotent*[1] *locally cyclic semigroup S can be embedded into R_a^+.*

It is sufficient to show that S is power joined and power cancellative.

We will now study the case of a locally cyclic semigroup S which contains an idempotent. We will also assume that S is not a group.

Let $a, b, x \in S$; let m, n be positive integers. Suppose $a = x^n$ and $b = x^m$. We define the relation $\geq$: $a \geqq b$ iff $n \geqq m$. The relation $\geq$ is a compatible quasiordering. If we define the relation ρ by $a \rho b$ if and only if $a \geqq b$ and $b \geqq a$, then ρ is a congruence. Let G be a congruence class of S modulo ρ containing more than one element. Then G is a group. Moreover, G is the only congruence class containing more than one element.

Let $x \in S \backslash G$. Let $[x] = \{x, x^2, x^3, \cdots, x^n, \cdots, x^{n+m-1}\}$ be the cyclic subsemigroup of S generated by x; let r be the index of $[x]$. Then $x^i \in S \backslash G$ for all $i < r$; $S \backslash G$ is linearly ordered by $\geq$. Let $a \in S \backslash G$ be held fixed. Let $b \in S \backslash G$. Then there is $x \in S \backslash G$ such that

$$a = x^n, \quad b = x^m.$$

Set $\varphi(b) = \frac{m}{n}$, $\varphi(a) = 1$. Then the map φ is a partial isomorphism of $S \backslash G$ into R_a^+, and is an order preserving map. Let $R' = \varphi(S \backslash G)$. Then the least upper bound *lub* R' of R' is finite, and we see that for $c \in G$ there exists $x \in S \backslash G$ such that $c = x^n$ for some integer n. If $x \in S \backslash G$ and if $x^n \in G$ then

$$n\varphi(x) \geqq r = lub\ R'.$$

Suppose $\frac{p}{q} \in R'$ and suppose $x \in S \backslash G$ satisfies $\varphi(x) = \frac{p}{q}$. There is a positive integer m such that

$$x^m \in G, \ x^{m-1} \in S \backslash G.$$

Define

1) S is called "nonpotent" if S has no idempotent element.

$$T\left(\frac{p}{q}\right)=\left\{i\frac{p}{q};\quad i\geqq m\right\}$$

and let $R''=\bigcup_{\frac{p}{q}\in R'} T\left(\frac{p}{q}\right)$.

Note that for all $u\in R''$, $u\geq r=lub\, R'$.

Let $x, y\in S\backslash G$ satisfy $\varphi(x)=\frac{p_1}{q_1}$, $\varphi(y)=\frac{p_2}{q_2}$, respectively. Then $n_1\frac{p_1}{q_1}=n_2\frac{p_2}{q_2}$ implies $x^{n_1}=y^{n_2}$.

Now we define a relation τ on R'' as follows:

Let $A, B\in R''$ and let $A, B\in T\left(\frac{p}{q}\right)$: $A=n_1\frac{p}{q}$, $B=n_2\frac{p}{q}$. Let $x\in S\backslash G$ satisfy $\varphi(x)=\frac{p}{q}$. Then

$$A\,\tau\, B \text{ if and only if } x^{n_1}=x^{n_2}.$$

Let $S'=R'\cup R''\subseteq R_a^+$. Then S' is closed under addition of rational numbers, and we now extend τ to all of S'. Let $A, B\in S'$:

If $A, B\in R'$ then $A\,\tau\, B$ if and only if $A=B$.

If $A\in R', B\in R''$, then A is not τ-related to B.

If $A\in R'', B\in R''$, then τ is the same as in R''.

Then τ is a congruence on S', and S'/τ is isomorphic onto S. Thus we have

Theorem 2. *If S is a locally cyclic semigroup with an idempotent and if S is not a group, then S is the homomorphic image of a subsemigroup of R_a^+.*

If the idempotent e of S is a zero, then $G=Se=\{e\}$. In this case S is linearly ordered by $\geq$ and e is the largest element of S under this ordering. Also all congruence classes of S under ρ are singleton classes. Theorem 2 is certainly valid for this case.

The following theorem is valid for all types of locally cyclic semigroups: nonpotent, unipotent semigroups which are not groups and locally cyclic groups.

Theorem 3. *S is a locally cyclic semigroup if and only if*

$$S=\bigcup_{n=1}^{\infty} S_n,\ S_1\subseteq S_2\subseteq S_3\subseteq \cdots \subseteq S_n\subseteq \cdots$$

where S_n is a cyclic subsemigroup of S for all positive integers n.

Proof. If S is either nonpotent or a unipotent semigroup which is not a group, then S is countable. This follows from Theorems 1 and 2. If S is a unipotent locally cyclic semigroup which is also a group, then S is also countable. This was obtained by using the fact[2] that a locally cyclic (in the semigroup sense) p-primary group

2) This was proved by Dr. Doyle Cutler.

G is either cyclic or isomorphic to a quasicyclic group [3] $C(p^\infty)$. Let $a_1, a_2, \cdots, a_n, \cdots$ be an enumeration of S. Let $S_1=[a_1]$, the cyclic subsemigroup of S generated by a_1. If $S_1 \subsetneqq S$, then let a_{i_1} be the element of S, of least index, not in S_1. Then there is $b_2 \in S$ such that

$$a_1 = b_2^{n_2}, \quad a_{i_1} = b_2^{m_2}.$$

Set $S_2=[b_2]$. Then $S_1 \subseteq S_2$ and $a_1, a_2, \cdots, a_{i_1} \in S_2$. Continuing in this fashion, we arrive at a sequence of cyclic subsemigroups of S satisfying the desired condition. The proof of the converse is entirely obvious.

References

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