164. An Integral of the Denjoy Type. II

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1. Introduction. The author [3] introduced the approximately continuous Denjoy integral (AD-integral) which is based on the descriptive definition of the general Denjoy integral.

The AD-integral is an extension of Burkill's approximately continuous Perron integral (AP-integral) [3] and of Denjoy's general integral (D-integral). In section 2 we shall state some fundamental properties of the AD-integral and it will be proved that our integral and the GM-integral defined by H. W. Ellis [1] are not compatible. An integral of the Perron type equivalent to the AD-integral is given in section 3.

2. The approximately continuous Denjoy integral. A real valued function $f(x)$ is said to be $\underline{AC}$ on a linear set E if, to each positive number ε, there exists a number $\delta>0$ such that

$$\sum\{f(b_k)-f(a_k)\}>-\varepsilon$$

for all finite non-overlapping sequences of intervals $\{(a_k, b_k)\}$ with end points on E and such that $\sum(b_k-a_k)<\delta$. There is a corresponding definition $\overline{AC}$ on E. If the set E is the sum of a countable number of sets E_k on each of which $f(x)$ is $\underline{AC}$ then $f(x)$ is said to be $\underline{ACG}$ on E. If the set E_k are assumed to be closed, then $f(x)$ is said to be $(\underline{ACG})$ on E. Similarly we can define $\overline{ACG}$ and $(\overline{ACG})$ on E. A function is (ACG) on E if it is both $(\underline{ACG})$ and $(\overline{ACG})$ on E.

Let $f(x)$ be a function defined on $[a, b]$ and suppose there exists a function $F(x)$ such that

(i) $F(x)$ is approximately continuous on [a, b],

(ii) $F(x)$ is (ACG) on [a, b],

(iii) $AD\ F(x)=f(x)$ a.e.,

then $f(x)$ is said to be integrable on $[a, b]$ in the approximately continuous Denjoy sense or AD-integrable. We than say that the function $F(x)$ is an indefinite AD-integral of $f(x)$ which is uniquely determined except an additive constant (Lemma 3.1 below).

The author [3] proved that the AD-integral is more general than Burkill's approximately continuous Perron integral (AP-integral) [6].

The condition in (ACG) that the set E_k be closed gives no restriction when $F(x)$ is continuous since the continuity of $F(x)$ is sufficient to ensure that if $F(x)$ is AC on a set, it is AC on the closure of this set. Hence it follows from the descriptive definition of the general Denjoy integral (D-integral) ([5], p. 241) that our integral includes the D-integral.

We now state some fundamental properties of the AD-integral.

Theorem 2.1. (i) *If $f(x)$ and $g(x)$ are AD-integrable on $[a, b]$, then $\alpha f(x)+\beta g(x)$ is AD-integrable and*

$$(AD)\int_a^b(\alpha f+\beta g)dt=\alpha(AD)\int_a^b fdt+\beta(AD)\int_a^b gdt.$$

(ii) *If $f(x)$ is AD-integrable on $[a, b]$ and $f(x)=g(x)$ a.e., then $g(x)$ is also AD-integrable and*

$$(AD)\int_a^b f(t)\,dt=(AD)\int_a^b g(t)dt.$$

(iii) *If $f(x)$ is AD-integrable on $[a, b]$ then $f(x)$ is also so in every subinterval.*

Proof. The proof follows directly from the definition of the integral.

Theorem 2.2. *A non-negative function $f(x)$ which is AD-integrable on* [a, b] *is necessarily L-integrable on* [a, b] *and both integrals coincide each other.*

Proof. Since $f(x)$ is AD-integrable, there exists a function $F(x)$ which is approximately continuous, (ACG) and $AD\ F(x)=f(x)$ a.e. Hence $AD\ F(x)\geqq 0$ a.e. It follows from Theorem 1 in [3] (Lemma 3.1 below) that $F(x)$ is non-decreasing on [a, b] and therefore $F'(x)$ is summable on [a, b] and $AD\,F(x)=F'(x)=f(x)$ a.e. Hence $f(x)$ is L-integrable on [a, b] and

$$(AD)\int_a^b f(t)\,dt=(L)\int_a^b f(t)dt.$$

Theorem 2.3. *Given a non-decreasing sequence $\{f_n\}$ of functions which are AD-integrable on $[a, b]$ and whose AD-integral over $[a, b]$ constitute a sequence bounded above, the function $f(x)=\lim f_n(x)$ is itself AD-integrable on $[a, b]$ and*

$$\lim\,(AD)\int_a^b f_n(t)dt=(AD)\int_a^b f(t)dt.$$

Proof. Since f_n-f_1 is non-negative, it follows from Theorem 2.2 that f_n-f_1 is L-integrable. Hence, by Lebesgue's theorem,

$$\lim\,(L)\int_a^b(f_n-f_1)\,dt=(L)\int_a^b(f-f_1)dt.$$

Since the sequence of integrals $(AD)\int_a^b f_n(t)dt$ is bounded above, the sequence of integrals $(L)\int_a^b(f_n-f_1)dt$ is also so, and therefore

$$0\leqq(L)\int_a^b(f-f_1)dt<\infty$$

which implies L-integrability of the function $f-f_1$. Hence $f-f_1$ is

AD-integrable, and f is also so. The equality follows directly.

Next we shall consider the relationship between the AD-integral and the GM-integral defined by H. W. Ellis [1].

The function $f(x)$ is GM-integrable on $[a, b]$ if there exists a mean continuous function $F(x)$ that is (ACG) on $[a, b]$ and is such that $AD\ F(x)$ exists and equal to $f(x)$ almost everywhere on $[a, b]$. A function is mean [Cesàro] continuous at x if $M(F, x, x+h) \to F(x)$ as $h \to 0$, where

$$M(F, x, x+h)=\frac{1}{h}\int_x^{x+h} F(t)dt$$

the integral being taken in the general [special] Denjoy sense. The function $F(x)$ is called an indefinite GM-integral of $f(x)$ on $[a, b]$. The definite GM-integral of f over $[a, b]$ is designated by

$$(GM)\int_a^b f(t)dt=F(b)-F(a).$$

We call two definitions of integration compatible if every function which is integrable in both senses is integrable to the same value in both senses.

Theorem 2.4. *The AD-integral and the GM-integral are not compatible.*

Proof. We use the example given by H. W. Ellis [2] to show the theorem.

For $n=1, 2, \cdots$, $F_n(x)$ is defined on $[-1, 1]$ to be zero everywhere except at the points of an interval I'_n of length $1/2^{n+1}$ strictly contained in $I_n=[1/(n+1), 1/n]$. On I_n, $F_n(x)$ is defined in such a way as to be non-negative, absolutely continuous on I_n, with finite derivative on I_n and with

$$\int_{1/(n+1)}^{1/n} F_n(t)dt=1/n(n+1).$$

The function $F(x)$, $G(x)$, and $f(x)$ on $[-1, 1]$ are defined as follows:

$$\begin{aligned} F(x)&=\sum F_n(x) && (x>0) & G(x)&=\sum F_n(x) && (x>0)\\ &=0 && (x\leqq 0), & &=-1 && (x\leqq 0), \end{aligned}$$

$$\begin{aligned} f(x)&=DF(x)=DG(x) && (x\neq 0)\\ &=0 && (x=0). \end{aligned}$$

Then $F(x)$ is approximately continuous and $G(x)$ is Cesàro-continuous (*a priori* mean continuous) on $[-1, 1]$ (cf. [2]). $F(x)$ and $G(x)$ are (ACG) on $[-1, 1]$, for $[-1, 1]=[-1, 0]\cup\bigcup_{n=1}^{\infty}[1/(n+1), 1/n]$ and on each closed interval they are absolutely continuous. Since $DF(x)=DG(x)=f(x)$ a.e., $f(x)$ is both AD-integrable and GM-integrable on $[-1, 1]$. But

$$(AD)\int_{-1}^{1} f(t)dt=F(1)-F(-1)=0,$$

$$(GM)\int_{-1}^{1} f(t)dt=G(1)-G(-1)=-1.$$

This completes the proof.

3. An integral of Perron's type equivalent to the AD-integral. Let $f(x)$ be a function defined on $[a, b]$. The function $U(x)$ is called upper function of $f(x)$ in $[a, b]$ if

(i) $U(a)=0$,

(ii) $U(x)$ is approximately continuous on $[a, b]$,

(iii) $U(x)$ is $(\underline{ACG})$ on $[a, b]$,

(iv) $AD\,U(x)\geqq f(x)$ a.e.

The lower function $L(x)$ is defined similarly. If $f(x)$ has upper and lower functions in $[a, b]$ and $\inf_U U(b)=\sup_L L(b)$, then $f(x)$ is termed integrable AP^*-sense or AP^*-integrable on $[a, b]$. The common value of the two bounds is called the definite AP^*-integral and is denoted by $(AP^*)\int_a^b f(t)dt$.

Lemma 3.1. ([3], p. 715.) *If $f(x)$ is approximately continuous and $(\underline{ACG})$ and if $AD\ F(x)\geqq 0$ almost everywhere on $[a, b]$, then $f(x)$ is non-decreasing on $[a, b]$.*

The direct consequence of this theorem is the following theorem.

Theorem 3.1. *For any upper function $U(x)$ and any lower function $L(x)$, the function $U(x)-L(x)$ is non-decreasing on $[a, b]$.*

Then we can develope the theory of Perron scale of integration as usual and have the following theorems (cf. [4]).

Theorem 3.2. *If $f(x)$ is AP^*-integrable on $[a, b]$ then $f(x)$ is also so in $[a, x]$ for $a<x<b$.*

Let $f(x)$ be an AP^*-integrable function on $[a, b]$. Then we define the indefinite AP^*-integral of $f(x)$ as

$$F(x)=(AP^*)\int_a^x f(t)dt.$$

Theorem 3.3. *For any upper function $U(x)$ and any lower function $L(x)$, the function $U(x)-F(x)[F(x)-L(x)]$ is non-decreasing on $[a, b]$.*

Theorem 3.4. *The indefinite integral $F(x)$ is approximately continuous on $[a, b]$.*

Theorem 3.5. *The indefinite AP^*-integral $F(x)$ is approximately differentiable almost everywhere on $[a, b]$ and $AD\ F(x)=f(x)$ a.e.*

Theorem 3.6. *The AD-integral is equivalent to the AP^*-integral.*

Proof. Suppose that $f(x)$ is AD-integrable on $[a, b]$. Then there exists a function $F(x)$ which is approximately continuous, (ACG) and $AD\,F(x)=f(x)$ a.e. Hence the function $F(x)-F(a)$ is an upper function and at the same time a lower function of $f(x)$ in $[a, b]$. Thus $f(x)$ is AP^*-integrable on $[a, b]$ and

$$(AP^*)\int_a^b f(t)\,dt = F(b)-F(a) = (GM)\int_a^b f(t)\,dt.$$

Next we shall show that the AD-integral includes the AP^*-integral. Suppose that $f(x)$ is AP^*-integrable on $[a, b]$ and that

$$F(x) = (AP^*)\int_a^x f(t)dt.$$

Then $F(x)$ is approximately continuous on $[a, b]$ and $AD\,F(x) = f(x)$ a.e. by Theorems 3.4 and 3.5. We must show that $F(x)$ is (ACG) on $[a, b]$. Since $f(x)$ is AP^*-integrable, there exists a sequence of upper functions $\{U_k(x)\}$ and a sequence of lower functions $\{L_k(x)\}$ such that

$$(1)\qquad \lim U_k(b) = F(b) = \lim L_k(b).$$

Since $U(x)-F(x)$ and $F(x)-L(x)$ are non-decreasing by Theorem 3.3, it holds that

$$(2)\qquad \lim U_k(x) = F(x) = \lim L_k(x) \quad \text{for} \quad a \leqq x \leqq b.$$

The interval $[a, b]$ is expressible as the sum of a countable number of closed sets E_k such that any U_k is AC on any E_k and at the same time any L_k is AC on any E_k. It is sufficient to prove that $F(x)$ is AC on E_k. For this purpose we shall show that $F(x)$ is both $\underline{AC}$ and $\overline{AC}$ on E_k.

Suppose that $F(x)$ is not $\underline{AC}$ on E_k. Then there exists an $\varepsilon > 0$ and a finite sequence of non-overlapping intervals $\{(a_\nu, b_\nu)\}$ with end points on E_k such that for any small δ

$$\sum (b_\nu - a_\nu) < \delta$$

but

$$(3)\qquad \sum \{F(b_\nu) - F(a_\nu)\} \leqq -\varepsilon.$$

Since we can find a natural number p such that

$$U_p(b) - F(b) \leqq 1/2\cdot\varepsilon,$$

and since $U_p(x) - F(x)$ is non-decreasing on $[a, b]$, we have

$$(4)\qquad \begin{aligned} &\sum\{U_p(b_\nu) - U_p(a_\nu)\} - \sum\{F(b_\nu) - F(a_\nu)\} \\ &= \sum [\{U_p(b_\nu) - F(b_\nu)\} - \{U_p(a_\nu) - F(a_\nu)\}] \\ &\leqq \quad U_p(b) - F(b) \leqq 1/2\cdot\varepsilon. \end{aligned}$$

It follows from (3) and (4) that

$$\begin{aligned} \sum \{U_p(b_\nu) - U(a_\nu)\} &\leqq \sum \{F(b_\nu) - F(a_\nu)\} + 1/2\cdot\varepsilon \\ &\leqq -1/2\cdot\varepsilon. \end{aligned}$$

This contradicts the fact that $U_p(x)$ is $\underline{AC}$ on E_k. Hence $F(x)$ is $\underline{AC}$ on E_k.

Similarly we can prove that $F(x)$ is $\overline{AC}$ on E_k. Thus $F(x)$ is (ACG) on $[a, b]$. This completes the proof.

References

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