84. *Continuity in Mixed Norms*

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1. Consider a normed vector lattice X over the real number field R with $X \supset R$. The norm $\|\circ\|_X$ on X is supposed to satisfy

(1) $$\min(a, \|x\|_X) \leq \|(a \vee x) \wedge \beta\|_X \leq \min(\beta, \|x\|_X) \ (\forall x \in X, \forall a, \forall \beta \in R, 0 \leq a \leq \beta).$$

Take a seminorm p_X on X satisfying the following:

(2) $$p_X(a) = 0 \ (\forall a \in R);$$

(3) $$p_X^2(x) = p_X^2(x \wedge a) + p_X^2(x \vee a) \ (\forall x \in X, \forall a \in R);$$

(4) $$\lim_{a' \uparrow a, \beta' \downarrow \beta} p_X((a' \vee x) \wedge \beta') = p_X((a \vee x) \wedge \beta) \ (\forall x \in X, \forall a, \forall \beta \in R).$$

With the aid of p_X we can define a new norm in X:

(5) $$|||x|||_X = \|x\|_X + p_X(x).$$

Let Y, $\|\circ\|_Y$, p_Y, and $|||\circ|||_Y$ be as above. Then we can show the following

Theorem. *Suppose that T is an isomorphism of $(X, \|\circ\|_X)$ onto $(Y, \|\circ\|_Y)$ as normed vector lattices with $T(a) = a(\forall a \in R)$. Then*

(6) $$\exists K : K^{-1} p_X(x) \leq p_Y(T(x)) \leq K p_X(x) \ (\forall x \in X)$$

if and only if

(7) $$\exists K : K^{-1} |||x|||_X \leq |||T(x)|||_Y \leq K |||x|||_X \ (\forall x \in X).$$

Proof. Since $\|T(x)\|_Y = \|x\|_X$, (6) clearly implies (7). To show the reversed implication let $A = \{x \in X \ x \geq 0, \|x\|_X \leq 1\}$. Then from (7) it follows that

(8) $$\exists K : p_Y(T(x)) \leq K(1 + p_X(x)) \ (\forall x \in A).$$

Fix an arbitrary $x \in A$ and define

$$x_i = n\left(\left(\frac{i-1}{n} \vee x\right) \wedge \frac{i}{n} - \frac{i-1}{n}\right) \ (i = 1, 2, \cdots, n).$$

By (1), $x_i \in A$ $(i = 1, 2, \cdots, n)$. Since T is an isomorphism of vector lattices with $T(a) = a(a \in R)$,

$$T(x_i) = n\left(\left(\frac{i-1}{n} \vee T(x)\right) \wedge \frac{i}{n} - \frac{i-1}{n}\right) (i = 1, 2, \cdots, n).$$

In view of (2), we see that

$$p_X(x_i) = n p_X\left(\left(\frac{i-1}{n} \vee x\right) \wedge \frac{i}{n}\right), \quad p_Y(T(x_i)) = n p_Y\left(\left(\frac{i-1}{n} \vee T(x)\right) \wedge \frac{i}{n}\right).$$

Repeated use of (3) yields

$$p_X^2(x) = \sum_{i=1}^n p_X^2 \left(\left(\frac{i-1}{n} \vee x\right) \wedge \frac{i}{n}\right), \quad p_Y^2(T(x)) = \sum_{i=1}^n p_Y^2 \left(\left(\frac{i-1}{n} \vee T(x)\right) \wedge \frac{i}{n}\right)$$

and therefore

$$(9)\qquad n^2p_X^2(x)=\sum_{i=1}^n p_X^2(x_i),\ n^2p_Y^2(T(x))=\sum_{i=1}^n p_Y^2(T(x_i)).$$

Since $x_i\in A$, (8) implies that

$$p_Y^2(T(x_i))\le K^2(1+2p_X(x_i)+p_X^2(x_i))\ (i=1,2,\cdots,n).$$

On summing up these n inequalities and using (9), we obtain

$$n^2p_Y^2(T(x))\le K^2\Big(n+2\sum_{i=1}^n p_X(x_i)+n^2p_X^2(x)\Big).$$

Setting $z_i^n=\left(\frac{i-1}{n}\vee x\right)\wedge\frac{i}{n}$ $(i=1,2,\cdots,n)$, we deduce

$$(10)\qquad p_Y^2(T(x))\le K^2p_X^2(x)+\frac{K^2}{n}+2K^2\cdot\frac{1}{n}\sum_{i=1}^n p_X(z_i^n).$$

Let $z_{i_n}^n$ be such that

$$p_X(z_{i_n}^n)=\max_{1\le i\le n}p_X(z_i^n).$$

we claim that

$$(11)\qquad \lim_{n\to\infty}p_X(z_{i_n}^n)=0.$$

Contrary to the assertion assume the existence of a subsequence $\{n(k)\}_{k=1}^\infty$ of positive integers such that $\varepsilon=\lim_{k\to\infty}p_X(z_{i_{n(k)}}^{n(k)})>0$. By choosing a suitable subsequence of $\{n(k)\}$, if necessary, we may assume that $\{i_{n(k)}/n(k)\}$ converges to a number c. By (4),

$$\lim_{m\to\infty}p_X\left(\left(\left(c-\frac{1}{m}\right)\vee x\right)\wedge\left(c+\frac{1}{m}\right)\right)=p_X((c\vee x)\wedge c)=p_X(c)=0.$$

Therefore we can find an m such that

$$p_X((a\vee x)\wedge\beta)<\varepsilon,\ a=c-1/m,\ \beta=c+1/m.$$

For all sufficiently large k, $a<(i_{n(k)}-1)/n(k)<i_{n(k)}/n(k)<\beta$. Again by (3) we see that

$$p_X(z_{i_{n(k)}}^{n(k)})\le p_X((a\vee x)\wedge\beta)<\varepsilon,$$

which is a contradiction, and therefore (11) is valid. Hence

$$0\le\frac{1}{n}\sum_{i=1}^n p_X(z_i^n)\le p_X(z_{i_n}^n)\to 0$$

as $n\to\infty$. On making $n\to\infty$ in (10), we now conclude that $p_Y(T(x))\le Kp_X(x)$. The same argument applied for T^{-1} gives $K^{-1}p_X(x)\le p_Y(T(x))$.

Thus (6) is valid at least for $x\in A$. For general x, write x as

$$x=\|x\|(y\vee 0+y\wedge 0),\ y=\frac{1}{\|x\|}x.$$

Observe that (3) implies

$$p_X^2(x)=\|x\|^2(p_X^2(y\vee 0)+p_X^2(y\wedge 0)),$$
$$p_Y^2(T(x)=\|x\|^2(p_Y^2(T(y)\vee 0)+p_Y^2(T(y)\wedge 0)).$$

Since $y\vee 0$ and $-(y\wedge 0)$ are in A, (6) for $y\vee 0$ and $-(y\wedge 0)$ with the above conclude (6) for x.

2. We state an application of Theorem 1. Let Ω and Ω' be open sets in the m-dimensional Euclidean space E^m. We denote by $W(\Omega)$ the class of functions f in the local Sobolev space $W^{1;2}_{\mathrm{loc}}(\Omega)$ with finite Dirichlet integrals over Ω:

$$D_\Omega(f)=\int_\Omega \sum_{i=1}^m \left(\frac{\partial f}{\partial x^i}\right)^2 dx^1\cdots dx^m<\infty.$$

Theorem. *Suppose that $y=t(x)$ is a homeomorphism of Ω onto Ω' with the property*

(12) $$g\in W(\Omega')\cap L^\infty(\Omega')\rightleftarrows g\circ t\in W(\Omega)\cap L^\infty(\Omega).$$

Then $g\in W(\Omega')$ if and only if $g\circ t\in W(\Omega)$ and

(13) $$\exists K<\infty: K^{-2}D_\Omega(f)\leq D_{\Omega'}(f\circ t^{-1})\leq K^2 D_\Omega(f)\ (f\in W(\Omega)).$$

Proof. Let $X=W(\Omega)\cap L^\infty(\Omega)$, $Y=W(\Omega')\cap L^\infty(\Omega')$, $\|\circ\|_X=\|\circ\|_{L^\infty(\Omega)}$, $\|\circ\|_Y=\|\circ\|_{L^\infty(\Omega')}$, $p_X(f)=(D_\Omega(f))^{1/2}$, and $p_Y(g)=(D_{\Omega'}(g))^{1/2}$. Then conditions (1)–(4) are met by these. Define $T(f)=f\circ t^{-1}$. Then T satisfies the condition of Theorem 1. Moreover $(X, |||\circ|||_X)$ and $(Y, |||\circ|||_Y)$ are Banach spaces and T is a closed map. Hence by the closed graph theorem, T is bicontinuous, i.e. (7) is valid. Therefore we obtain (13) for $f\in X=W(\Omega)\cap L^\infty(\Omega)$, and then for $f\in W(\Omega)$.

Remark. It can be seen that *a homeomorphism t with* (13) *is a quasiconformal mapping if $m=2$ and a quasiisometry if $m\geq 3$, and vice versa.* Thus (12) may be considered as a new definition of quasiconformal mappings for $m=2$ and quasiisometry for $m\geq 3$. A homeomorphism t with (12) induces a ring isomorphism of $M(\Omega)=W(\Omega)\cap L^\infty(\Omega)\cap C(\Omega)$ onto $M(\Omega')=W(\Omega')\cap L^\infty(\Omega')\cap C(\Omega)$. The converse is also true: *Any ring isomorphism of $M(\Omega)$ onto $M(\Omega')$ is induced by a homeomorphism t with* (12). The ring $M(\Omega)$ is called the Royden algebra. These results are also valid if Ω and Ω' are replaced by Riemannian manifolds. Details of results mentioned in this remark will be published elsewhere.