195. On a Set Theory Suggested by Dedecker and Ehresmann. II

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The preceding theorem shows how the general notions, referring to classes, are related to the corresponding ones, *restricted* to sets. If we wish to extend the above relations to the other concepts of set theory like, for instance, the notions of ordered pair, function, ordinal number, and cardinal number, it is clear that several modifications must be made in the general definitions (borrowed from [13]) or in the definitions of the *restricted* notions (adapted from [9], appendix). However as we have defined V-union ($\cup_V$), V-intersection ($\cap_V$) and V-complement ($\sim_V$), we may define V-ordered pair, V-function, V-choice function, etc., following [9].

The behaviour of sets is regulated by the following postulates:

Postulate of subsets.

(P5) $$x \in V \supset \exists y(y \in V \ \& \ \forall z(z \subset x \supset z \in y)).$$

Postulate of union.

(P6) $$x \in V \ \& \ y \in V \supset x \cup_V y \in V.$$

Postulate of substitution.

(P7) $$x \text{ is a } V\text{-function } \& \text{ dom}_V x \in V \supset \text{range}_V x \in V.$$

Postulate of amalgamation.

(P8) $$x \in V \supset \cup_V x \in V.$$

Postulate of regularity.

(P9) $$x \in V \ \& \ x \neq \emptyset \supset \exists z(z \in x \ \& \ x \cap_V z = \emptyset).$$

Postulate of infinity.

(P10) $$\exists y(y \in V \ \& \ \emptyset \in y \ \& \ \forall z(x \in y \supset x \cup_V \{x\}_V \in V)).$$

Theorem 9. $\vdash \exists x(x \in V)$.

Corollary. $\vdash V \neq \emptyset$.

Postulate of choice.

(P11) $$\exists x(x \text{ is a } V\text{-choice function } \& \text{ dom}_V x = \{y : y \neq \emptyset\}_V).$$

From the above postulates, we may deduce that all the theorems of the Kelley-Morse system, conveniently translated, are true for the elementary classes of $\boldsymbol{D}$. Thus, $\boldsymbol{D}$ is strictly stronger than the Kelley-Morse set theory.

In the next theorems, we use the set-theoretic terminology of [13], with evident adaptations.

Theorem 10. $\vdash \mathrm{Can}\,(V)$;
$\vdash \forall x(x \in V \supset \mathrm{Can}\,(x))$;
$\vdash \neg\, \mathrm{Can}\,(U)$.

Proof. Employing (P3), it is possible to construct a function such that $\vdash V \text{ sm USC } (V)$; the proofs of the other parts are standard.

Theorem 11. $\vdash x$ is a proper class $\supset \{x\}$ is a proper class.

Theorem 12. $\vdash V$ is a proper class;
$\vdash U$ is a proper class.

Theorem 13. $\vdash \mathrm{Nc}\,(U) \neq \mathrm{Nc}\,(\{U\})$.

Theorem 14. $\vdash \mathrm{Nc}\,(V) \neq \mathrm{NcSC}\,((V))$.

Theorem 15. $\vdash \mathrm{Nc}\,(U) = \mathrm{Nc}\,(\mathrm{SC}\,(U))$.

Theorems 11-15 answer (in D) the questions (a) and (b) raised by Dedecker at the end of his paper [5].

Rosser's axiom of counting is provable in D.

Theorem 16. $\vdash \forall x(x \in \mathrm{Nn} \supset \{y : y \in \mathrm{Nn} \;\&\; 0 < y \leqslant x\} \in x)$.

Proof. In V we may identify the set ω_V of finite V-integers. Now, clearly

$$\vdash \forall t \forall x(t \in \mathrm{Nn} \;\&\; x \in t \;\&\; x \in \omega_V \supset x +_V l_V \in t + l).$$

So, we easily infer

$$\vdash \forall t(t \in \mathrm{Nn} \supset \exists x(x \in \omega_V \;\&\; x \in t)).$$

However, as $\vdash x \in V \;\&\; x \text{ sm}_V \text{ USC}_V(x) \supset x \text{ sm USC } (x)$, we have $\vdash \forall x(x \in \omega_V \supset \mathrm{Can}\,(x))$.

So, by Theorem XI. 2.61 of Rosser [13],

$$\vdash \forall x(x \in \mathrm{Nn} \supset \forall t(t \in x \supset \mathrm{Can}\,(x)).$$

Hence, the axiom of counting follows by Theorem XIII.1.3 of [13].

Remark. It is shown in Orey [11] that the axiom of counting cannot be proved in NF, if this system is consistent. Hence, there is no hope to prove in NF our axioms by a suitable definition of V. Consequently, D is strictly stronger than NF.

Theorem 17. $\vdash \forall x(x \in V \supset \mathrm{st\ Can}\,(x))$.

Definition 9. Ord_V is the elementary class of V-ordinals. (Restricting the corresponding notion of [9].)

Definition 10. $Q = \mathrm{Ord}_V \rceil \in \lceil \mathrm{Ord}_V$. (The meaning of this definition is clear.)

Theorem 18. $\vdash \forall x(x \in \mathrm{Ord}_V \supset \mathrm{No}\,(x\rceil \in \lceil x) \in \mathrm{No})$.

Proof. By transfinite induction in V.

Theorem 19. $\vdash \forall x(x \in \mathrm{Ord}_V \supset x\rceil \in \lceil x \in \mathrm{Word})$.

Theorem 20. $\vdash \mathrm{Ord}_V\rceil \in \lceil \mathrm{Ord}_V \in \mathrm{Word}$.

Theorem 21. $\vdash \mathrm{st\ Can}\,(AV(Q))$.

Definition 11. Under the usual conventions referring to variables, the ordinal α is inaccessible if:

1) $\alpha=\omega_\beta$, where β is a limit ordinal;

2) $\forall x(\forall y((y \in x \supset y<\alpha) \ \&\ \mathrm{Nc}\,(x)<\mathrm{Nc}\,(\{\theta : \theta<\alpha\})$
$\supset \exists z(z<\alpha \ \&\ \forall y(y \in x \supset y<z)))$.

Theorem 22. $\vdash \mathrm{Nc}\,(Q)$ is inaccessible.[1]

Theorem 23. If $\boldsymbol{D}$ is consistent, then the systems of Quine-Rosser and Kelley-Morse are also consistent.

3. Concluding remarks. Though $\boldsymbol{D}$ seems apt to serve as a foundation for the (elementary) category theory, a real interesting solution would be to employ instead of the Kelley-Morse theory, one of our systems T or T^* (cf. [2] and [3]): in this way, we obtain two of the most powerful and beautiful systems of set theory.[2]

Evidently, it is also possible to combine other systems of set theory, as we have combined the theories of Quine-Rosser and Kelley-Morse, to construct systems similar to $\boldsymbol{D}$.

References

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1) This (among other things) shows that D is a strong system and in consequence a dangerous one. In a note, *Un nouveau système formel suggéré par Dedecker*, C.R. Acad. Sc. Paris, 265, Série A (1967), pp. 85–88, we have presented a new system D^* very similar to D. (Professor W. S. Hatcher has proven that Rosser's axiom of counting is a theorem of $\boldsymbol{D}^*$.) Apparently, $\boldsymbol{D}$ and $\boldsymbol{D}^*$ are consistent.

2) A solution of the problem of founding category theory, embedded in the general structure of NF is the following: in this system we define a convenient notion of *universe* and assume the existence of a suitable "number" of such sets. This type of solution will be treated in forthcoming papers.

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