2. On the Relative Pseudo-Rigidity

By Kazuhiko MAKIO
Department of Mathematics, University of Tokyo

(Comm. by Kunihiko KODAIRA, M. J. A., Jan. 12, 1973)

In this paper we establish a generalization of the results in [1].

In what follows, by a *pair* (W, S) we mean the pair of a complex manifold W and a compact submanifold S of W. By a *deformation* of a pair (W, S) we mean the quintuple $(\mathcal{W}, \mathcal{S}, B, o, \pi)$ of connected complex manifolds $\mathcal{W}, B$, a closed submanifold $\mathcal{S}$ of $\mathcal{W}$, a point o of B and a smooth holomorphic map π of $\mathcal{W}$ onto B such that $\pi^{-1}(o)=W$, $\pi^{-1}(o)\cap\mathcal{S}=S$ and the restriction of π to $\mathcal{S}$ is a proper smooth holomorphic map.

For convenience sake we list here some notations whose meanings are the same wherever they occur. Let $(\mathcal{W}, \mathcal{S}, B, o, \pi)$ be a deformation of a pair (W, S).

$m=\dim B$

$(t_1, \cdots, t_m)=$ a local coordinate of B with center o

$B(\varepsilon)=\{(t_1, \cdots, t_m)\in B\,;\,|t_i|<\varepsilon,\ i=1, \cdots, m\}$

$\mathcal{W}(\varepsilon)=\pi^{-1}(B(\varepsilon))$

$\mathcal{W}|_U=\pi^{-1}(U)$, $U\subset B$

$\Xi=$ the sheaf over W of germs of holomorphic vector fields which are tangential to S at each point of S.

$\tilde{\Xi}=$ the sheaf over $\mathcal{W}$ of germs of holomorphic vector fields along fibres which are tangential to $\mathcal{S}$ at each point of $\mathcal{S}$.

We say that a deformation $(\mathcal{W}, \mathcal{S}, B, o, \pi)$ of a pair (W, S) is *relatively trivial* if there exists a biholomorphic map g of $\mathcal{W}$ onto $W\times B$ which induces a biholomorphic map of $\mathcal{S}$ onto $S\times B$ such that $g|W$ is the identity map and $pr_B\circ g=\pi$ where pr_B is the canonical projection of $W\times B$ onto B.

Definition 1. A deformation $(\mathcal{W}, \mathcal{S}, B, o, \pi)$ of a pair (W, S) is said to be *relatively pseudo-trivial at* o if, for any relative compact subset N of W, there exist a positive number ε and a submanifold $\mathcal{N}$ of $\mathcal{W}(\varepsilon)$ such that $(\mathcal{N}, \mathcal{N}\cap\mathcal{S}, B(\varepsilon), o, \pi|\mathcal{N})$ is a relative trivial deformation of the pair $(N, N\cap S)$.

Definition 2. A pair (W, S) is said to be *relatively pseudo-rigid* if any deformation of (W, S) is relatively pseudo-trivial at o.

Lemma. *Let* $(\mathcal{W}, \mathcal{S}, B, o, \pi)$ *be a deformation of a pair* (W, S). *If the stalk* $(R^1\pi_*\tilde{\Xi})_o=0$, *then* $(\mathcal{W}, \mathcal{S}, B, o, \pi)$ *is relatively pseudo-trivial at* o.

The proof of this lemma goes parallel with that of "Theorem 5.1" in [3] or "Proposition 1" in [1], so we omit it.

The following theorem generalizes the "Theorem" in [1].

Theorem. *Let V be a complex manifold, W be an open relative compact submanifold of V and S be a compact submanifold of W.*

If W is the strongly pseudo-convex manifold in the sense of [2] *and $H^1(W, \Xi)=0$, then (W, S) is relatively pseudo-rigid.*

Proof. We begin with quoting a result obtained in [4].

Definition ([4], [5]). Let $\mathcal{W}'$ and B' be complex manifolds. A holomorphic map π of $\mathcal{W}'$ onto B' is called 1-*convex map* if, for $\forall_t \in B'$, there exist an open neighborhood U_t of t in B', a function φ of $\mathcal{W}'|_{U_t}$ into $\boldsymbol{R}$ and a constant $c_0 \in \boldsymbol{R}$ such that:

(i) the restriction of φ to $\{p \in \mathcal{W}'|_{U_t}; \varphi(p) > c_0\}$ is a strongly plurisubharmonic function (cf. [2] §1, Definition 1).

(ii) the restriction of φ to $\{p \in \mathcal{W}'|_{U_t}; \varphi(p) \leq c\}$ is a proper map for all $c \in \boldsymbol{R}$,

where φ is called the "Exhaustions function" of $\pi|_{\pi^{-1}(U_t)}$ and c_0 is called the "Ausnahmekonstante".

Now let $\mathcal{W}'_c = \{p \in \mathcal{W}'; \varphi(p) < c\}$ and $\pi_c = \pi | \mathcal{W}'_c$.

Theorem ([4]). *Let $\pi: \mathcal{W}' \to B'$ be a 1-convex map, $\mathcal{F}$ be a coherent sheaf over $\mathcal{W}'$ and other symbols be as above.*

If cohomology classes $\xi_1, \xi_2, \cdots, \xi_l$ of $H^1(\mathcal{W}'_c(\varepsilon), \mathcal{F})$ generate the image of the canonical map of $R^1(\pi_c)_*\mathcal{F}$ to $R^1(\pi_c)_*(\mathcal{F}/\sum_{i=1}^m t_i^{\nu_i}\mathcal{F})$ for all sufficiently large $(\nu_1, \nu_2, \cdots, \nu_m) \in N^m$, then they already generate the stalk $(R^1(\pi_c)_*\mathcal{F})_o$.

Now let $(\mathcal{W}, \mathcal{S}, B, o, \pi)$ be any deformation of (W, S) and N be any relative compact subset of W. By the hypothesis that W is a strongly pseudo-convex submanifold of V, there exist a neighborhood U of W in V, a pluri-subharmonic function $\varphi_0: U \to \boldsymbol{R}$ and a positive constant δ_0 such that $W = \{p \in U; \varphi_0(p) < 0\}$ and the restriction of φ_0 to $\{p \in U; -\delta_0 < \varphi_0(p)\}$ is strongly pluri-subharmonic.

Let ε and δ be positive numbers which satisfy the following conditions:

i) $\delta < \delta_0$

ii) $N \cup S \Subset U_\delta = \{p \in U; \varphi_0(p) < -\delta\}$

iii) there exists a complex submanifold $\mathcal{U}_\delta$ of $\mathcal{W}$ such that $(\mathcal{U}_\delta, \mathcal{S}(\varepsilon), B(\varepsilon), o, \pi)$ is a deformations of (U_δ, S) where $U_\delta = \pi^{-1}(o) \cap \mathcal{U}_\delta$ and is relatively differentiably trivial, that is, there exists a diffeomorphism f of $\mathcal{U}_\delta$ onto $U_\delta \times B(\varepsilon)$ which induces a diffeomorphism of $\mathcal{S}(\varepsilon)$ onto $S \times B(\varepsilon)$ such that $f|_{U_\delta}$ is the identity map and $pr_{B(\varepsilon)} \circ f = \pi$ where $pr_{B(\varepsilon)}$ is the canonical projection of $U_\delta \times B(\varepsilon)$ onto $B(\varepsilon)$.

Let a and b be negative numbers such that

i) $-\delta_0 < b < a < -\delta$

ii) $K = \{p \in U\,;\, b \leq \varphi_0(p) \leq a\} \subset U_\delta - S$

iii) $S \cup N \subset \{p \in U\,;\, \varphi_0(p) < a\}$

Let $\hat{\varphi}(p) = \varphi_0(pr_{U_\delta} \circ f(p))$ for all $p \in \mathcal{U}_\delta$ where pr_{U_δ} is the canonical projection of $U_\delta \times B(\varepsilon)$ onto U_δ, and let $\hat{\hat{\varphi}} = \hat{\varphi} + M \circ \pi^*(\sum_{\lambda=1}^m t_\lambda \bar{t}_\lambda)$ on $\mathcal{U}_\delta$. As K is compact, for sufficiently large number M, $\hat{\hat{\varphi}}$ is strongly pluri-subharmonic on K. Then there exists an open neighborhood $\mathcal{U}$ of K in $\mathcal{U}_\delta$ where $\hat{\hat{\varphi}}$ is strongly pluri-subharmonic.

Let ε' be a sufficiently small number such that $\{p \in \mathcal{U}_\delta\,;\, b \leq \hat{\hat{\varphi}}(p) \leq a\} \cap \mathcal{W}(\varepsilon') \subset \mathcal{U} \cap \mathcal{W}(\varepsilon')$ and $\mathcal{S}(\varepsilon') \subset \{p \in \mathcal{U}_\delta\,;\, \hat{\hat{\varphi}}(p) < a\}$.

Let $\mathcal{W}' = \{p \in \mathcal{U}_\delta\,;\, \hat{\hat{\varphi}}(p) < a\} \cap \mathcal{W}(\varepsilon')$, then $\pi : \mathcal{W}' \to B' = B(\varepsilon')$ is a 1-convex map whose "Exhaustions function" is $-\log(a - \hat{\hat{\varphi}})$ and whose "Ausnahmekonstante" is $-\log(a-b)$.

Now, taking a negative constant c' such that $b < c' < a$ and $N \subset \{p \in U_\delta\,;\, \varphi_0(p) < c'\}$, let $c = -\log(a - c')$. To complete the proof, it is sufficient to prove $R(\pi_c)_*(\tilde{\Xi}/\sum_{i=1}^m t_i^{\nu_i}\tilde{\Xi}) = 0$ for all $(\nu_1, \cdots, \nu_m) \in N^m$. Because, if so, $(R(\pi_c)_*\tilde{\Xi})_o = 0$ by the theorem quoted above and then, by the lemma, $(\mathcal{W}'_c, \mathcal{S}(\varepsilon'), B(\varepsilon'), o, \pi)$ is relatively pseudo-trivial at o.

Note that $\pi^{-1}(o) \cap \mathcal{W}'_c = \{p \in W\,;\, \varphi_0(p) < c'\}$ and $H^1(\pi^{-1}(o) \cap \mathcal{W}'_c, \Xi) \cong H^1(W, \Xi)$. But $R^1(\pi_c)_*(\tilde{\Xi}/\sum_{i=1}^m t_i\tilde{\Xi}) = H^1(\pi^{-1}(o) \cap \mathcal{W}'_c, \Xi)$, therefore $R^1(\pi_c)_*(\tilde{\Xi}/\sum_{i=1}^m t_i\tilde{\Xi}) = 0$.

We shall prove $R^1(\pi_c)_*(\tilde{\Xi}/\sum_{i=1}^m t_i^{\nu_i}\tilde{\Xi}) = 0$ under the condition $R^1(\pi_c)_*(\tilde{\Xi}/\sum_{i=1}^m t_i\tilde{\Xi}) = 0$ by the induction with respect to $m = \dim B$ and $(\nu_1, \cdots, \nu_m) \in N^m$ which is ordered in such a way that $(\nu_1, \cdots, \nu_m) < (\nu'_1, \cdots \nu'_m) \Leftrightarrow \nu_1 = \nu'_1, \cdots, \nu_k = \nu'_k, \nu_{k+1} < \nu'_{k+1}$.

Let $m=1$ and suppose $R^1(\pi_c)_*(\tilde{\Xi}/t^\nu\tilde{\Xi}) = 0$. We have the isomorphism

$$t^\nu\tilde{\Xi}/t^{\nu+1}\tilde{\Xi} \cong \tilde{\Xi}/t\tilde{\Xi}$$

and the exact sequence

$$0 \to t^\nu\tilde{\Xi}/t^{\nu+1}\tilde{\Xi} \to \tilde{\Xi}/t^{\nu+1}\tilde{\Xi} \to \tilde{\Xi}/t^\nu\tilde{\Xi} \to 0.$$

Consequently we obtain $R^1(\pi_c)_*(\tilde{\Xi}/t^{\nu+1}\tilde{\Xi}) = 0$.

Now let $m \geqslant 2$ and suppose our assertion is true if $\dim \leqslant m-1$ and moreover

$$R^1(\pi_c)_*\left(\tilde{\Xi}\Big/\sum_{i=1}^{m-1} t_i^{\nu_i}\tilde{\Xi} + t_m^{\nu_m - 1}\tilde{\Xi}\right) = 0.$$

Let $\tilde{\Xi}_1 =$ the restriction of $\tilde{\Xi}$ to $\pi^{-1}\{(t_1, \cdots, t_m)\,;\, t_m = 0\}$, then we have the isomorphism

$$\sum_{i=1}^{m-1} t_i^{\nu_i}\tilde{\Xi} + t_m^{\nu_m - 1}\tilde{\Xi}\Big/\sum_{i=1}^{m} t_i^{\nu_i}\tilde{\Xi} \cong \tilde{\Xi}_1\Big/\sum_{i=1}^{m-1} t_i^{\nu_i}\tilde{\Xi}_1.$$

By the hypothesis of the induction we obtain

$$R^1(\pi_c)_*\left(\sum_{i=1}^{m-1} t_i^{\nu_i}\tilde{\Xi} + t_m^{\nu_m - 1}\tilde{\Xi}\Big/\sum_{i=1}^{m} t_i^{\nu_i}\tilde{\Xi}\right) = 0,$$

then the exact sequence

$$0 \to \sum_{i=1}^{m-1} t_i^{\nu_i} \tilde{\Xi} + t_m^{\nu_m - 1} \tilde{\Xi} \Big/ \sum_{i=1}^{m} t_i^{\nu_i} \tilde{\Xi} \to \tilde{\Xi} \Big/ \sum_{i=1}^{m} t_i^{\nu_i} \tilde{\Xi} \to \tilde{\Xi} \Big/ \sum_{i=1}^{m-1} t_i^{\nu_i} \tilde{\Xi} + t_m^{\nu_m - 1} \tilde{\Xi} \to 0$$

infers

$$R^1(\pi_c)_* \left(\tilde{\Xi} \Big/ \sum_{i=1}^{m} t_i^{\nu_i} \tilde{\Xi} \right) = 0.$$

Q.E.D.

References

[1] A. Andreotti and E. Vesentini: On the pseudo-rigidity of Stein manifolds. Ann. Scuola Norm. Sup. Pisa, **16** (3), 213–223 (1962).
[2] H. Grauert: Über Modifikationen und exzeptionelle analytische Mengen. Math. Ann., **146**, 331–368 (1962).
[3] K. Kodaira and D. C. Spencer: On deformations of complex analytic structures. I. Ann. of Math., **67** (2), 328–401 (1958).
[4] K. Knorr: Noch ein Theorem der analytischen Garbentheorie (to appear).
[5] K. Knorr und Michael Schneider: Relativexzeptionelle analytische Mengen (to appear).