83. *The Solution Spaces of Non-Linear Partial Differential Equations of Elliptic Type on Compact Manifolds*

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§ 1. In this paper, we shall deal with some geometric properties concerning with the *solution space* of non-linear partial differential equations of elliptic type defined on compact manifolds, by using a unified method, namely that of the "linearization" of the non-linear operators.

Throughout the present paper, let M denote a compact C^∞-manifold of dimension n, and $C^\infty(M)$ the linear space of C^∞-functions on M with the C^∞-topology. Further, let m be an arbitrary non-negative integer. Then we define a (non-linear) differential operator L of order m on M as a mapping:

$$L\colon C^\infty(M)\to C^\infty(M),$$

which can be expressed, locally, in terms of coordinates, as a C^∞-function in the partial derivatives of order $\leqq m$. To state more precisely, let $x_1,\cdots,x_n$ denote the local coordinates of M with the coordinate domain U, then the operator: $C^\infty(U)\to C^\infty(U)$ induced by L has the form $L(u)=F(x,D_x^\alpha u)$, where $F(x,y^{(\alpha)})$ is an element of $C^\infty(U\times \boldsymbol{R}^N)$ (N denotes the number of multi-indices $\alpha=(\alpha_1,\cdots,\alpha_n)$ with $|\alpha|=\sum\alpha_i\leqq m$), and D_x^α denotes the partial derivative $\partial^{\alpha_1}/\partial x_1^{\alpha_1}\cdots\partial^{\alpha_n}/\partial x_n^{\alpha_n}$.

In our case, the *linearization* (the Gateaux derivative) of L at $f\in C^\infty(M)$ is given by

$$d_fL(u)=\lim_{h\to 0}\frac{L(f+hu)-L(f)}{h}.$$

If L has the local expression as above, it can be expressed by

$$d_fL(u)=\sum_{|\alpha|\leqq m}\frac{\partial F(x,y^{(\alpha)})}{\partial y^{(\alpha)}}\bigg|_{y^{(\alpha)}=D_x^\alpha f}D^\alpha u.$$

Hence, d_fL is a *linear* differential operator with C^∞-coefficients of order m.

The operator L will be called an *elliptic operator* (of order m), if, for each $f\in C^\infty(M)$ and for each local parameter, the highest order term of d_fL

$$\sum_{|\alpha|\leqq m}\frac{\partial F(x,y^{(\alpha)})}{\partial y^{(\alpha)}}\bigg|_{y^{(\alpha)}=D_x^\alpha f}\xi^\alpha$$

is non-zero for any $x \in U$ and $\xi \in \boldsymbol{R}^n-(0)$, namely, if $d_f L$ is a linear elliptic operator for every f.

Since the *linearization* means, in a sense, to replace locally the non-linear mapping L by an approximating linear mapping, and further we know that the dimension of the solution space of a linear elliptic equation is finite, we may therefore expect that the *dimension* of the solution space of any non-linear elliptic equation ought to be finite. This note is devoted to a verification of this conjecture which is originary due to M. Ise.

Now we introduce some notations:

$$\mathfrak{S}=\{u \in C^\infty(M)\,;\, L(u)=0\},$$
$$T_u(\mathfrak{S})=\{v \in C^\infty(M)\,;\, d_uL(v)=0\}\,;$$

whereby we consider $\mathfrak{S}$ as a topological subspace of $C^\infty(M)$ with the induced topology.

Theorem 1. *Let L be a non-linear elliptic differential operator. Then the solution space $\mathfrak{S}$ is locally a finite dimensional subset in $C^\infty(M)$.* More precisely, *for any $u \in \mathfrak{S}$ there is a neighborhood $\mathfrak{U}$ of u in $C^\infty(M)$ with respect to the C^∞-topology such that $\mathfrak{U} \cap \mathfrak{S}$ is diffeomorphic to a locally closed set of the finite dimensional vector space $T_u(\mathfrak{S})$.*

This theorem is in a sence a generalization of a result of J. Moser [1], who proved the zero-dimensionality of the solution space when d_uL is injective.

The precise statements together with complete proofs, and with some applications to differential and analytic geometry will be published elsewhere.

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§ 2. The statement of Theorem 1 being local in its nature, we take $u \in \mathfrak{S}$ as a fixed element and consider the operator d_uL. We choose moreover a Riemannian structure (g_{ij}) on M, and let $dM(g) =\sqrt{g}\,dx_1\wedge \cdots \wedge dx_n$ denote the volume elements on M, $(f, h)_0=\int_M f\cdot h\, dM(g)$ the L^2-inner product on $C^\infty(M)$, and $(d_uL)^*$ the formal adjoint operator of d_uL; namely for every $f, h \in C^\infty(M)$

$$(d_uL(f), h)_0=(f, (d_uL)^*h)_0.$$

Then, using the theorem of Hodge-Kodaira for the linear elliptic operator d_uL, we have the direct decompositions:

$$C^\infty(M)=T_u(\mathfrak{S}) \oplus \mathrm{Im}\,(d_uL)^*,$$
$$C^\infty(M)=\mathrm{Ker}\,(d_uL)^* \oplus \mathrm{Im}\, d_uL,$$

which are orthogonal with respect to the L^2-inner product.

Thus, we can define the operator:

$$\Phi : C^\infty(M)\to T_u(\mathfrak{S}) \oplus \mathrm{Im}\, d_uL,$$

by $\Phi(f)=\pi(f)\oplus\omega(Lf)$, where the mappings π and ω are the L^2-orthogonal projections of $T_u(\mathfrak{S})$ and $\operatorname{Im}(d_uL)$ respectively.

Theorem 2. *Φ maps a neighborhood $\mathfrak{U}$ of u in $C^\infty(M)$ diffeomorphically onto a neighborhood of $\Phi(u)=(\pi(u),0)$.*

In order to prove this theorem, we shall utilize a slight variation of the so-called *implicit function theorem*, which will be discussed in the later sections.

By using a standard method of non-linear functional analysis, the proof of Theorem 1 can be derived from this theorem: In fact, let $\mathfrak{U}$, $\mathfrak{V}_1$, an $\mathfrak{V}_2$ be neighborhood respectively, in $C^\infty(M)$, $T_u(\mathfrak{S})$, and $\operatorname{Im} d_uL$ such that $u\in\mathfrak{U}$, $\pi(u)\in\mathfrak{V}_1$, $0\in\mathfrak{V}_2$ and the mapping $\Phi:\mathfrak{U}\to\mathfrak{V}_1\oplus\mathfrak{V}_2$ is diffeomorphic. Then a C^∞-mapping $Q:\mathfrak{V}_1\oplus\mathfrak{V}_2\to\operatorname{Ker}(d_uL)^*\oplus\operatorname{Im} d_uL$ is defined such that in the diagram:

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{\Phi} & \mathfrak{V}_1\oplus\mathfrak{V}_2 \\ \big\downarrow{\scriptstyle L} & & \big\downarrow{\scriptstyle Q} \\ C^\infty(M) & = & \operatorname{Ker}(d_uL)^*\oplus\operatorname{Im} d_uL, \end{array}$$

the commutative velation $Q\Phi=L$ is valid. If we define the mapping $R:\mathfrak{V}_1\oplus\mathfrak{V}_2\to\operatorname{Ker}(d_uL)^*$ by $R=\pi'L\Phi^{-1}$, we have

$$Q(f_1\oplus f_2)=(R(f_1,f_2),f_2),$$

where $f_i\in\mathfrak{V}_i(i=1,2)$. Hence it follows that under Φ, $\mathfrak{S}_1=\{(f,0)\in\mathfrak{V}_1\oplus\mathfrak{V}_2\,;\,R(f,0)=0\}$ is diffeomorphic to $\mathfrak{S}\cap\mathfrak{U}$. This proves that $\mathfrak{S}\cap\mathfrak{U}$ is diffeomorphic to the set of zeros of C^∞-functions defined on an open set $\mathfrak{V}_1$ in the finite dimensional vector space $T_u(\mathfrak{S})\,;\,R(\cdot,0):\mathfrak{V}_1\to\operatorname{Ker}(d_uL)^*$.

§ 3. We sketch here an outline of the proof of Theorem 2: Let us now consider the two inner-products in $C^\infty(M)$ related to d_uL, which are defined by

$$[f,h]_1^k=\int_M (d_uL^*d_uL+1)^k f\cdot h\,dM(g),$$

$$[f,h]_2^k=\int_M (d_uLd_uL^*+1)^k f\cdot h\,dM(g)\,;$$

where k is a non-negative integer and $f,h\in C^\infty(M)$. Denote further by $H_i^k(M)$ the completions of $C^\infty(M)$ with respect to $[\ ,\ ]_i^k$ $(i=1,2)$. Then, if we use the Gårding inequality and *a priori* estimate of an elliptic operator, it is easily verified that $H_i^k(M)$ $(i=1,2)$ are isomorphic to the Sobolev space, W^{km}, as Banach spaces.

In this section, we shall consider the case where

$$k\geqq k_0=\frac{1}{m}\left(m+\left[\frac{n}{2}\right]+1\right).$$

Proposition 1. i) *$L:C^\infty(M)\to C^\infty(M)$ can be extended to the C^∞-mapping $L^{(k)}$ of $H_1^k(M)$ to $H_2^{k-1}(M)$ for any $k\geqq k_0$.* ii) *For any $f\in C^\infty(M)$, the Gateaux derivative of $L^{(k)}$ at f coincides with the extension of d_fL: that is $(d_fL)^{(k)}:H_1^k(M)\to H_2^{k-1}(M)$.*

The proof can be found in a monograph of R. S. Palais [2].

We consider the extension $(d_uL)^{(k)}: H_1^k(M) \to H_2^{k-1}(M)$. An argument similar to that of the L^2-orthogonal decomposition theorem shows that $H_i^k(M)$ can be decomposed as follows:

$$H_1^k(M) = T_u(\mathfrak{S}) \oplus \operatorname{Im}(d_uL^*)^{(k+1)},$$
$$H_2^k(M) = \operatorname{Ker}(d_uL)^* \oplus \operatorname{Im}(d_uL)^{(k+1)},$$

and their decompositions are orthogonal in the L^2-inner product, too. Then it is easy to see that the restriction $(d_uL)^{(k)}|\operatorname{Im}(d_uL^*)^{(k+1)}$: $\operatorname{Im}(d_uL^*)^{(k+1)} \to \operatorname{Im}(d_uL)^{(k)}$ is an isomorphism.

Next we define the operator

$$\Phi_k : H_1^k(M) \to T_u(\mathfrak{S}) \oplus \operatorname{Im}(d_uL)^{(k)},$$

by putting $\Phi_k(f) = \pi(f) \oplus \omega(L^{(k)}f)$ where $f \in H_1^k(M)$. From the above arguments, we have $\Phi_k | H_1^l(M) = \Phi_l$ for $l \geqq k$ and $\Phi_k | C^\infty(M) = \Phi$, and $d_u\Phi_k$ gives an isomorphism. If we apply the Implicit Function Theorem to the C^∞-mapping Φ_k, we have the following proposition.

Proposition 2. *Φ_k maps a neighborhood $\mathfrak{U}_k$ of u in $H_1^k(M)$ diffeomorphically onto a neighborhood $\mathfrak{V}_{k-1}$ of $\Phi(u)$ in $T_u(\mathfrak{S}) \oplus \operatorname{Im}(d_uL)^{(k)}$.*

This proposition, however, does not immediately lead to our desired result, because we cannot take in general a sequence $\{\mathfrak{U}_k\} k \geqq k_0$ such that $\mathfrak{U}_k \cap H_1^l(M) = \mathfrak{U}_l$ for $l \geqq k$.

But, if we make use of a result of A. Douglis and L. Nirenberg [3], we have the following weak form:

Proposition 3. *There exists a sequence of neighborhoods $\{\mathfrak{U}_k\}_{k \geq k_0}$ of u in $H_1^k(M)$ and $\{\mathfrak{V}_k\}_{k \geq k_0}$ of $\Phi(u)$ in $T_u(\mathfrak{S}) \oplus \operatorname{Im}(d_uL)^{(k+1)}$ such that*

i) *$\Phi_k : \mathfrak{U}_k \to \mathfrak{V}_{k-1}$ is a diffeomorphism for $k \geqq k_0$,*

ii) *$C^\infty(M) \cap \mathfrak{U}_k = C^\infty(M) \cap \mathfrak{U}_l$, $C^\infty(M) \cap \mathfrak{V}_k = C^\infty(M) \cap \mathfrak{V}_l$ for $l, k \geqq k_0$.*

It follows then immediately from Proposition 3 that Φ gives a local diffeomorphism.

§ 4. Under some additional assumptions, we shall state more precise results for the solution spaces. Namely, we introduce the notations:

$$\rho = \min_{f \in C^\infty(M)} (\dim \operatorname{Ker} d_fL),$$
$$\mathfrak{S}_0 = \{u \in C^\infty(M) ; L(u) = 0, \dim \operatorname{Ker} d_uL = \rho\},$$

Theorem 3. *$\mathfrak{S}_0$ is open in $\mathfrak{S}$ and further $\mathfrak{S}_0$ is a ρ-dimensional smooth manifold which is tangent to the affine subspace $u + T_u(\mathfrak{S})$ at $u \in \mathfrak{S}_0$.*

$L: C^\infty(M) \to C^\infty(M)$ will be called an *analytic operator along the fiber*, if for each local expression of $L, F(x, y^{(\alpha)})$ is an analytic function of $(y^{(\alpha)})$.

Theorem 4. *If L is analytic along the fiber, the solution space $\mathfrak{S}$ is an analytic space in the usual sense.*

The proof of this theorem is a slight modification of that of Theorem 1.

§ 5. Finally, we note that our results in the preceding sections can be extended to the case of operators acting on the space of sections of fiber bundles, following to the formulation of R. S. Palais [2].

References

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