145. On the Singularity of the Spectral Measures of a Semi-Infinite Random System

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1. Introduction. H. Matsuda and K. Ishii [1] showed an exponential growth character of polynomials related to a second order difference operator with random coefficients by invoking a limit theorem of H. Furstenberg [4]. A. Casher and J. L. Lebowitz [3] then used this character to derive the singularity of the related spectral measure. We refer the reader to K. Ishii [2] for an improvement of the proof of [3] and for the related physical problems.

The purpose of this note is to simplify the proof of the Matsuda-Ishii theorem and to give an extension of Ishii's results. Let $(\Omega, \mathcal{B}, P)$ be a probability space on which are defined independent real random variables $\{\nu_n(\omega)\}_{n=0}^{\infty}$ with common distribution ν. We consider the following random system on the semi-infinite lattice $Z^+=\{0,1,2,3,\cdots\}$

$$\text{(a)}\qquad \begin{cases} i\dfrac{du_n(t)}{dt}=u_{n-1}(t)-(2+\nu_n)u_n(t)+u_{n+1}(t), \\ u_{-1}(t)=0,\ n\in Z^+,\ t\in[0,\infty). \end{cases}$$

Putting $u_n(t)=y_n e^{-i\lambda t}$, we are led to the following difference equation

$$\text{(b)}\qquad \lambda y_n=y_{n-1}-(2+\nu_n)y_n+y_{n+1},\ n\in Z^+,\ y_{-1}=0.$$

Let $\{p_n^{\omega}(\lambda)\}_{n=0}^{\infty}$ be the solution of (b) under the conditions $y_0=1$ and $y_{-1}=0$. Denote by l_0 the space of all functions on Z^+ with finite supports. We introduce an infinite Jacobi matrix $H^{\omega}=(h_{ij})$, $i,j\in Z^+$, with $h_{ij}=1$, $|i-j|=1$, $h_{ii}=-(2+\nu_i)$, $i\in Z^+$, and $h_{ij}=0$, $|i-j|>1$. $\{H^{\omega}\}$ are regarded as linear operators with domain l_0. Then H^{ω} is an essentially self-adjoint operator on $l^2(Z^+)$ for each $\omega\in\Omega$ and we denote its smallest closed extension by H^{ω} again [5]. We further introduce the resolvent $G^{\omega}(\lambda)=(\lambda-H^{\omega})^{-1}$. Then we have the following expression of $G_{mm}^{\omega}(\lambda)=(G^{\omega}(\lambda)e_m, e_m)$, $m\in Z^+$, [6].

$$G_{mm}^{\omega}(\lambda)=\{p_{mm}^{\omega}(\lambda)\}^2\sum_{i=m}^{\infty}\frac{1}{p_i^{\omega}(\lambda)p_{i+1}^{\omega}(\lambda)},\qquad \operatorname{Im}\lambda\neq 0.$$

Now let $E^{\omega}(\lambda)$ be the resolution of the identity of H^{ω}. K. Ishii [2] showed that, for almost every fixed $\omega\in\Omega$, $\rho_n^{\omega}(\lambda)=(E^{\omega}(\lambda)e_n, e_n)$, $n\in Z^+$, are singular with respect to the Lebesgue measure $d\lambda$ under the assumption that the support of ν is finite and is not a single point. We will show that this is still true under the weaker assumptions that $\int_{-\infty}^{\infty}|c|\,d\nu(c)<\infty$ and that the support of ν is not a single point

(Theorem 2).

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2. Furstenberg's theorem and its applications to the random system. Let μ be a probability measure on unimodular matrices $SL(m, R^1)$ and G be the smallest closed subgroup of $SL(m, R^1)$ containing the support of μ. Let $\{X_n\}_{n=0}^{\infty}$ be the G-valued independent random variables with common distribution μ. For each $g \in SL(m, R^1)$, $|||g|||$ denotes $\sup_{\|x\|=1} \|gx\|$. G is called irreducible if the subspace of R^m invariant under G is either R^m or $\{0\}$. Otherwise it is called reducible.

Theorem (H. Furstenberg [4]). *Let G be a non compact subgroup of $SL(m, R^1)$ such that no subgroup of finite index is reducible and $\int |||g|||\, d\mu(g) < \infty$. Then there exists a positive constant α such that $P\{\omega\,;\, \lim_{n\to\infty} (n+1)^{-1} \log \|X_n \cdots X_0 x\| = \alpha\} = 1$ for each $x \in R^m - \{0\}$.*

Now let $\{x_n\}_{n=0}^{\infty}$ be independent real random variables with common distribution φ. Set $X_n = \begin{pmatrix} x_n & -1 \\ 1 & 0 \end{pmatrix}$, then $\{X_n\}_{n=0}^{\infty}$ are independent $SL(2, R^1)$-valued random variables with common distribution $\tilde{\varphi}$ induced by φ. Applying Furstenberg's theorem, we have the following.

Lemma 1. *Suppose that $\int_{-\infty}^{\infty} |c|\, d\varphi(c) < \infty$ and the support of φ contains more than one point. Then there exists a positive constant γ such that $P\{\omega\,;\, \lim_{n\to\infty} (n+1)^{-1} \log \|X_n \cdots X_0 x\| = \gamma\} = 1$ for each $x \in R^2 - \{0\}$.*

Proof. First we note that $\int |||g|||\, d\tilde{\varphi}(g) \leqq \int_{-\infty}^{\infty} (|c|+1) d\varphi(c) < \infty$. Let G be the smallest closed subgroup of $SL(2, R^1)$ containing the support of $\tilde{\varphi}$. Since G contains at least two matrices of the type $\begin{pmatrix} e & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} e' & -1 \\ 1 & 0 \end{pmatrix}$, $e \neq e'$, we see that $\begin{pmatrix} e & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & e \end{pmatrix} \in G$, $\begin{pmatrix} 0 & 1 \\ -1 & e \end{pmatrix} \begin{pmatrix} e' & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e-e' & 1 \end{pmatrix} \in G$ and $\begin{pmatrix} 1 & 0 \\ e-e' & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ n(e-e') & 1 \end{pmatrix} \in G$. Therefore G is non compact. Note that $\begin{pmatrix} e' & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & e \end{pmatrix} = \begin{pmatrix} 1 & e'-e \\ 0 & 1 \end{pmatrix} \in \mathrm{G}$. Let G_0 be an arbitrary subgroup of G of finite index. Then there exist positive integers n, m, such that $\begin{pmatrix} 1 & 0 \\ e-e' & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ n(e-e') & 1 \end{pmatrix} \in G_0$, $\begin{pmatrix} 1 & e'-e \\ 0 & 1 \end{pmatrix}^m = \begin{pmatrix} 1 & m(e'-e) \\ 0 & 1 \end{pmatrix} \in G_0$. Therefore, G_0 contains at least two distinct matrices of the type $\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ f' & 1 \end{pmatrix}$, $ff' \neq 0$. Let A be a subspace of R^2 such that $G_0 A = A$. Put $R = \{g \in M_2(R^1)\,;\, gA \subset A\}$, $M_2(R^1)$ being the space of all 2×2 real matrices. Let us show that $R = M_2(R^1)$. Clearly R is an algebra containing G_0. Since $\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & f\\ 0 & 0\end{pmatrix}\in R$, $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\in R$. Similarly $\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\in R$. Hence $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\in R$ and $\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\in R$. Therefore $R=M_2(R^1)$, which in turn implies that A is either R^2 or $\{0\}$, proving the irreducibility of G_0. Q.E.D.

Let us return to the random system described in 1. From the definition of $\{p_n^\omega(\lambda)\}_{n=0}^\infty$,

$$\begin{pmatrix}p_{n+1}^\omega(\lambda)\\ p_n^\omega(\lambda)\end{pmatrix}=T_n\cdots T_0\begin{pmatrix}1\\ 0\end{pmatrix},\qquad T_n=\begin{pmatrix}2+\lambda+\nu_n & -1\\ 1 & 0\end{pmatrix}.$$

Hence we have the next theorem by Lemma 1.

Theorem 1. *If* $\int_{-\infty}^{\infty}|c|\,d\nu(c)<\infty$ *and the support of* ν *is not a single point, then there exists a positive constant* $\beta(\lambda)$ *such that*

$$P\Big\{\omega\,;\lim_{n\to\infty} n^{-1}\log\{(p_{n+1}^\omega(\lambda))^2+(p_n^\omega(\lambda))^2\}=2\beta(\lambda)\Big\}=1$$

for each $\lambda\in R^1$.

Lemma 2. *Under the assumptions of Theorem* 1,

$$P\{\omega\,;\operatorname{Im} G_{mm}^\omega(\lambda-i0)=0\}=1$$

for each $\lambda\in R^1$ *and* $m\in Z^+$.

Proof. By the assumption, $\{|\nu_n|\}_{n=0}^\infty$ are independent identically distributed random variables with finite expectation. We put $A(\lambda)=\{\omega\,;\lim_{n\to\infty} n^{-1}\log\{(p_{n+1}^\omega(\lambda))^2+(p_n^\omega(\lambda))^2\}=2\beta(\lambda)\}$, $\lambda\in R^1$, and $B=\{\omega\,;|\nu_n|=O(n)\}$. Theorem 1 and the strong law of large numbers then imply $P(A(\lambda)\cap B)=1$. Using now the expression (c) of $G_{mm}^\omega(\lambda)$, and noticing $\sum_{n=0}^\infty(n+1)e^{-n}<\infty$ and the identity

$$\frac{1}{p_{n-1}^\omega(\lambda)p_n^\omega(\lambda)}+\frac{1}{p_n^\omega(\lambda)p_{n+1}^\omega(\lambda)}=\frac{\lambda+2+\nu_n}{p_{n-1}^\omega(\lambda)p_{n+1}^\omega(\lambda)},\qquad \operatorname{Im}\lambda\neq 0$$

we can combine the method of [2] with the Lebesgue dominated convergence theorem to obtain that $\operatorname{Im} G_{mm}^\omega(\lambda-i0)=0$ for every $\omega\in A(\lambda)\cap B$. Q.E.D.

Consider the product space $(R^1\times\Omega, \mathscr{B}(R^1)\times\mathscr{B}, d\lambda\times dP)$, where $(R^1, \mathscr{B}(R^1), d\lambda)$ is the real line with the Lebesgue measure $d\lambda$.

Lemma 3. $\{(\lambda,\omega)\,;\operatorname{Im} G_{mm}^\omega(\lambda-i0)=0\}\in\mathscr{B}(R^1)\times\mathscr{B}$

Proof. Since the function $f_n(\lambda,\omega)=\operatorname{Im} G_{mm}^\omega(\lambda-i(1/n))$ is continuous in $\lambda\in R^1$ for each $\omega\in\Omega$, $f_n(\lambda,\omega)$ is $\mathscr{B}(R^1)\times\mathscr{B}$ measurable and so is $\operatorname{Im} G_{mm}^\omega(\lambda-i0)$. Q.E.D.

Fubini's theorem together with Lemmas 2 and 3 implies the following.

Lemma 4. *Under the assumptions of Theorem* 1, *for almost every fixed* $\omega\in\Omega$, $\operatorname{Im} G_{mm}^\omega(\lambda-i0)=0$ *a.e.* $\lambda\in R^1$.

Theorem 2. *Under the assumptions of Theorem* 1, $P\{\omega\,;d\rho_m^\omega(\lambda)$ *is singular with respect to the Lebesgue measure for all* $m\in Z^+\}=1$

Proof. We see that

$$\operatorname{Im} G^{\omega}_{mm}(\lambda) = -\int_{-\infty}^{\infty} \frac{\lambda''}{(\lambda'-\mu)^2+\lambda''^2} d\rho^{\omega}_{m}(\mu), \qquad \lambda = \lambda' + i\lambda''.$$

When $\lambda'' \to 0-$, the left-hand side converges to $\operatorname{Im} G^{\omega}_{mm}(\lambda' - i0)$ and the right-hand side converges to $d\rho^{\omega}_{m}(\lambda')/d\lambda'$ a.e. λ' by Fatou's theorem [7]. Theorem 2 now follows from Lemma 4. Q.E.D

Finally we consider the solution $\{u_n(t)\}_{n=0}^{\infty}$ of the evolution equation (a) under the initial condition $u_n(0) = \delta_{nN}$, with $N \in Z^+$ being arbitrarily fixed. We say that the weak absence of diffusion takes place if $\int_0^{\infty} |u_N(t)|^2 \, dt$ diverges for almost all $\omega \in \Omega$.

Theorem 3. *Under the assumptions of Theorem* 1, *the weak absence of diffusion takes place.*

This theorem was obtained by K. Ishii [2] when the support of ν is finite and is not a single point. By Lemma 4 and the Stieltjes inversion formula, almost every operator H^{ω} has the property (A_N) of [2] even in the present case. By the standard argument involving the uniform integrability, we can then prove Theorem 3.

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