41. *A Characterization of Homogeneous Self-Dual Cones*

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§1. It is known that every homogeneous self-dual cone is a Riemannian symmetric space of non-positive sectional curvature with respect to the canonical Riemannian metric (cf. [2], [4]). In 1965, Prof. Y. Matsushima raised the question whether every Riemannian symmetric homogeneous convex cone is self-dual or not. The purpose of the present note is to announce an affirmative answer to the above question. Furthermore, we will give an application. The detailed results with their complete proofs will be published elsewhere [6].

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§2. In the present note, we will employ the following notations and terminologies. Let V be a homogeneous convex cone in the n-dimensional real number space $\boldsymbol{R}^n$ with an inner product $\langle\ ,\ \rangle$. We denote by V^* the dual cone of V with respect to this inner product $\langle\ ,\ \rangle$. A cone V is called *self-dual* if the dual cone V^* with respect to a suitable inner product coincides with V. The *characteristic function* φ_V of V is defined by

$$\varphi_V(x)=\int_{V^*}\exp(-\langle x, y\rangle)dy$$

for every $x\in V$, where dy is a canonical Euclidean measure on $\boldsymbol{R}^n$. Let us take a system of linear coordinates $(x_1, x_2, \cdots, x_n)$ of $\boldsymbol{R}^n$. Then we can define a $G(V)$-invariant Riemannian metric g on V by

$$g=\sum_{i,j}\frac{\partial^2\log\varphi_V}{\partial x_i\partial x_j}dx_i dx_j,$$

where $G(V)=\{A\in GL(n, \boldsymbol{R})\,;\, AV=V\}$. This Riemannian metric g is called the *canonical metric* of V.

§3. It is known in [7] that there exists a natural bijection between the set of all linear equivalence classes of homogeneous convex cones and the set of all isomorphism classes of T-algebras. We recall briefly this bijection. (For the details, see [7].)

Let $\mathfrak{A}=\sum_{1\leq i,j\leq r}\mathfrak{A}_{ij}$ be a T-algebra of rank r with an involution $*$. We put

$$T(\mathfrak{A})=\{t=(t_{ij})\in\mathfrak{A}\,;\, t_{ii}>0 \text{ and } t_{ij}=0 \text{ for } i>j\}$$

and

$$V(\mathfrak{A})=\{tt^*\,;\, t\in T(\mathfrak{A})\}.$$

Then the set $V(\mathfrak{A})$ is a homogeneous convex cone in the real vector space $X(\mathfrak{A})=\{x\in\mathfrak{A}\,;\,x^*=x\}$. Conversely every homogeneous convex cone can be realized in this form up to a linear equivalence. The number r is called the *rank* of $V(\mathfrak{A})$. The Lie group $T(\mathfrak{A})$ acts simply transitively on V as linear transformations. We put $e=(e_{ij})$ by $e_{ii}=1$ and $e_{ij}=0$ for $i\neq j$. Then we can naturally identify the tangent space of $V(\mathfrak{A})$ at the point e with the Lie algebra $\mathfrak{t}(\mathfrak{A})$ of $T(\mathfrak{A})$. We define the *trace* of an element $a=(a_{ij})\in\mathfrak{A}$ by

$$Sp\, a=\sum_{1\leq i\leq r} n_i a_{ii},$$

where

$$n_i=1+\frac{1}{2}\sum_{1\leq k<i} n_{ki}+\frac{1}{2}\sum_{i<k\leq r} n_{ik},\quad n_{ij}=\dim\mathfrak{A}_{ij}.$$

Then the canonical metric g of $V(\mathfrak{A})$ at the point e is given by

$$g(a,\,b)=Sp((a+a^*)(b+b^*))$$

for every $a,\,b\in\mathfrak{t}(\mathfrak{A})$.

§4. Let $V(\mathfrak{A})$ be the homogeneous convex cone of rank r which corresponds to a T-algebra $\mathfrak{A}=\sum_{1\leq i,j\leq r}\mathfrak{A}_{ij}$. Then by using the theory of invariant connections in [3], we can calculate the Riemannian connection and the curvature tensor of the canonical metric on $V(\mathfrak{A})$ in terms of the Lie algebra $\mathfrak{t}(\mathfrak{A})$ and the trace Sp.

First of all, by calculating the covariant derivatives of the curvature tensor we have the following

Lemma 1. *If $V(\mathfrak{A})$ is Riemannian symmetric with respect to the canonical metric and* rank $V(\mathfrak{A})=r\geq 3$. *Then for each triple $(i,\,j,\,k)$ with $1\leq i<j<k\leq r$ which satisfies the conditions $n_{ij}n_{ik}\neq 0$ or $n_{jk}n_{ik}\neq 0$, the equalities $n_{ij}=n_{jk}=n_{ik}$ hold.*

We can easily show the following

Lemma 2. *Let $\mathfrak{A}=\sum_{1\leq i,j\leq r}\mathfrak{A}_{ij}$ be a T-algebra with* rank $\mathfrak{A}=r\geq 3$ *which satisfies the following two conditions:*

(1) *For each pair $(i,\,j)$ with $i<j$, there exists a series $i_0,\,i_1,\,\cdots,\,i_m$ such that $i_0=i$, $i_m=j$ and $n_{i_{\lambda-1}i_\lambda}\neq 0$ for $1\leq\lambda\leq m$.*

(2) *For each triple $(i,\,j,\,k)$ with $1\leq i<j<k\leq r$ satisfying the conditions $n_{ij}n_{ik}\neq 0$ or $n_{jk}n_{ik}\neq 0$, the equalities $n_{ij}=n_{jk}=n_{ik}$ hold.*
Then n_{ij} is constant for $1\leq i<j\leq r$.

A homogeneous convex cone is linearly equivalent to a direct product of irreducible homogeneous convex cones; the decomposition is unique up to an order. If a homogeneous convex cone $V(\mathfrak{A})$ of rank ≥ 3 is irreducible, then the condition (1) in Lemma 2 is satisfied (cf. [1]). On the other hand, $V(\mathfrak{A})$ is self-dual if and only if each irreducible factor of $V(\mathfrak{A})$ is self-dual. For an irreducible cone $V(\mathfrak{A})$, $V(\mathfrak{A})$ is self-dual if and only if n_{ij} is constant for every $i,\,j$ with $1\leq i<j\leq r$ (cf. [8]). It is easy to see that a homogeneous convex cone of rank

one is the cone of positive real numbers and an irreducible cone of rank two is the circular cone. And these cones are self-dual. Therefore, combining the above lemmas we have an affirmative answer to the Matsushima's problem stated in § 1 as follows:

Theorem. *Let V be a homogeneous convex cone. If V is Riemannian symmetric with respect to the canonical metric, then V is self-dual.*

§ 5. Finally we state an application of the above theorem. It is known in [5] that a homogeneous convex cone V in $\boldsymbol{R}^n$ is self-dual if and only if the tube domain $D(V)=\{z \in \boldsymbol{C}^n; \operatorname{Im} z \in V\}$ over V is Hermitian symmetric with respect to the Bergman metric of $D(V)$. As an application of the above theorem, we have the following

Corollary. *For a homogeneous convex cone V, the following three conditions are equivalent:*

(1) *V is Riemannian symmetric with respect to the canonical metric.*

(2) *V is self-dual.*

(3) *The tube domain $D(V)$ over V is Hermitian symmetric with respect to the Bergman metric.*

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