8. *Gauss-Manin System and Mixed Hodge Structure*

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Let $f: \boldsymbol{C}^{n+1}, 0 \to \boldsymbol{C}, 0$ be a holomorphic function with an isolated singularity. There are two ways of describing the degeneracy of a one-parameter family $\{X_t = f^{-1}(t)\}$. One is the theory of Gauss-Manin connection. Here, the Brieskorn lattice $\mathcal{H}^{(0)}_{X,0} = \Omega^{n+1}_{X,0}/df \wedge d\Omega^{n-1}_{X,0}$ plays an important role. The other is the theory of mixed Hodge structure of Steenbrink on the vanishing cohomology $H^n(X_\infty, \boldsymbol{C})$ (cf. (1.2)).

In [7], Scherk and Steenbrink constructed a filtration on $H^n(X_\infty, \boldsymbol{C})$ using $\mathcal{H}^{(0)}_{X,0}$, and asserted that the filtration coincides with Hodge filtration $\{F^{\cdot}_{St}\}$ of the mixed Hodge structure. But there is an example (e.g. $f = x^5 + y^5 + x^3y^3$), in which their filtration is not compatible with the monodromy decomposition $H^n(X_\infty, \boldsymbol{C}) = \oplus_\lambda H^n(X_\infty, \boldsymbol{C})_\lambda$, whereas $\{F^{\cdot}_{St}\}$ is compatible with it. Here $H^n(X_\infty, \boldsymbol{C})_\lambda := \{u \in H^n(X_\infty, \boldsymbol{C}) : (M_X - \lambda)^{n+1}u = 0\}$ and M_X is the local monodromy of f.

This contradiction comes from the following. In [9], Steenbrink proved that the Hodge subbundle of the flat vector bundle $H_Y = \coprod_{t \in S^*} H^n(Y_t, \boldsymbol{C})$ can be extended to the origin as a subbundle of Deligne's extension $\mathcal{L}'_Y$ of H_Y (cf. (1.3)). Here, $\bar{f}: Y \to S$ is a one-parameter projective family. But this limit filtration is not compatible with the monodromy decomposition $H^n(Y_\infty, \boldsymbol{C}) = \oplus H^n(Y_\infty, \boldsymbol{C})_\lambda$. Following the construction of Schmid, we have to take a base change such that the pullback of H_Y has a unipotent monodromy.

In this note, we give a correct formulation of their assertion and an outline of the proof.

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§ 1. Limit mixed Hodge structure. (1.1) Let $f: \boldsymbol{C}^{n+1}, 0 \to \boldsymbol{C}, 0$ be a holomorphic function with an isolated singularity. We may assume that f is a polynomial of degree d, where we can take d as large as we like, and that $\overline{f^{-1}(0)} \subset \boldsymbol{P}^{n+1}$ is smooth away from the origin. We define $Y := \overline{\{(x, t) \in \boldsymbol{C}^{n+1} \times S : f(x) = t\}} \subset \boldsymbol{P}^{n+1} \times S$ and $X := (B \times S) \cap Y$, where $S := \{t \in \boldsymbol{C} : |t| < \eta\}$ and $B := \{x \in \boldsymbol{C}^{n+1} : \|x\| < \varepsilon\}$. We put $Z := \boldsymbol{P}^{n+1} \times S$, $p: Z \to S$ the natural projection and $\bar{f} := p|_Y$. For $1 \gg \varepsilon \gg \eta > 0$, $\bar{f}: Y \to S$ is smooth away from 0 and $f: X \to S$ is a Milnor fibration, i.e., $f': X^* = X - f^{-1}(0) \to S^* = S - \{0\}$ is a C^∞-fibration. Hence, $H_X := R^n f_* \boldsymbol{C}_X|_{S^*}$ (resp. $H_Y := R^n \bar{f}_* \boldsymbol{C}_Y|_{S^*}$) is a local system with a monodromy

M_X (resp. M_Y).

(1.2) Let $\pi:\tilde{S}\ni\tilde{t}\mapsto t=\tilde{t}^e\in S$ be an e-fold covering of S, such that $\tilde{H}_Y:=\pi^*H_Y$ has a unipotent monodromy. We set $X_\infty:=X\times_S U$ and $Y_\infty:=Y\times_S U$, where $U\to S^*$ is a universal covering.

(1.3) As was proved by Manin and Deligne (see [3]), the local system H_X can be uniquely extended to the origin as a locally free $\mathcal{O}_S$-Module $\mathcal{L}_X$ (resp. $\mathcal{L}'_X$) with a regular singular connection ∇, such that ∇ has a simple pole and the residue of ∇ has its eigenvalues in $(-1,0]$ (resp. $[0,1)$). We have an isomorphism $H^n(X_\infty,\boldsymbol{C})\xrightarrow{\sim}\mathcal{L}_X(0):=\mathcal{L}_{X,0}/t\mathcal{L}_{X,0}$ defined by $u\mapsto\exp(-\log M_X\ \log t/2\pi\sqrt{-1})u$, where we regard u as a multivalued section of H_X.

We can obtain similar extensions for H_Y, $\tilde{H}_Y$ and $\tilde{H}_X:=\pi^*H_X$, and denote them by $\mathcal{L}_Y$, $\tilde{\mathcal{L}}_Y$, etc.

(1.4) **Proposition** (Schmid [8]). *Let $\{\mathcal{F}^\cdot\}$ be the Hodge subbundles of H_Y (i.e., $\{\mathcal{F}^\cdot(t)\}$ is the Hodge filtration of $H^n(Y_t,\boldsymbol{C})$ for $\forall t\in S^*$). Then $\tilde{\mathcal{F}}^\cdot:=\pi^*\mathcal{F}^\cdot$ can be extended to the origin as holomorphic vector subbundles of $\tilde{\mathcal{L}}_Y$, and the limit Hodge filtration $F^\cdot_S\subset\tilde{\mathcal{L}}_Y(0)\xleftarrow{\sim}H^n(Y_\infty,\boldsymbol{C})$ forms a mixed Hodge structure with the monodromy weight filtration.*

(1.5) **Proposition** (Steenbrink [9]). *$H^n(X_\infty,\boldsymbol{C})$ has a mixed Hodge structure such that $i^*:H^n(Y_\infty,\boldsymbol{C})\to H^n(X_\infty,\boldsymbol{C})$ is a morphism of mixed Hodge structures.*

§ 2. **Gauss-Manin systems.** (2.1) **Definition.** The Gauss-Manin systems are the integration of systems defined by

$$\mathcal{H}_X:=\mathcal{H}^0(\boldsymbol{R}p'_*DR_{Z'/S}(\mathcal{B}_{X|Z'})[n+1]),$$
$$\mathcal{H}_Y:=\mathcal{H}^0(\boldsymbol{R}p_*DR_{Z/S}(\mathcal{B}_{Y|Z})[n+1]),$$

where $Z':=B\times S$, $p':=p|Z'$, $\mathcal{B}_{X|Z}:=\mathcal{H}^1_{[X]}(\mathcal{O}_{Z'})$ and $DR_{Z'/S}(\mathcal{B}_{X|Z'}):=\Omega^\cdot_{Z'/S}\otimes\mathcal{B}_{X|Z'}$ (cf. [4]).

$\mathcal{H}_X$ (resp. $\mathcal{H}_Y$) is a holonomic system with a regular singularity such that $DR_S(\mathcal{H}_X)=R^nf_*\boldsymbol{C}_X$ (resp. $\mathcal{H}^0(DR_S(\mathcal{H}_Y))=R^n\bar{f}_*\boldsymbol{C}_Y$). We remark that $\mathcal{H}_X$ (resp. $\mathcal{H}_Y$) contains $\mathcal{L}_X$ (resp. $\mathcal{L}_Y$) and the action of $\nabla_{d/dt}$ coincides with the action of $\partial_t\in\mathcal{D}_S$.

(2.2) **Definition.**

$$DR_{Z'/S}(\mathcal{B}_{X|Z'})(k):=\{\Omega^\cdot_{Z'/S}\otimes\mathcal{B}_{X|Z'}(\cdot+k-n-1)\}$$

for $k\in\boldsymbol{Z}_+:=\{\mathrm{m}\in\boldsymbol{Z}:m\geq 0\}$, where $\mathcal{B}_{X|Z'}(m):=\mathcal{D}_{Z'}(m)\delta(f(x)-t)$ and $\mathcal{D}_{Z'}(m):=\{\sum_{|\nu|\leq m}a_\nu\partial^\nu\}\subset\mathcal{D}_{Z'}$ for $m\in\boldsymbol{Z}$. We define $DR_{Z/S}(\mathcal{B}_{Y|Z})(k)$ and $\mathcal{B}_{Y|Z}(m)$ similarly.

(2.3) **Definition.**

$$\mathcal{H}_X^{(k)}:=\mathrm{Im}\,(\mathcal{H}^0(\boldsymbol{R}p'_*(DR_{Z'/S}(\mathcal{B}_{X|Z'})(k))[n+1])\longrightarrow\mathcal{H}_X)\qquad k\in\boldsymbol{Z}_+,$$
$$\mathcal{H}_Y^{(k)}:=\mathrm{Im}\,(\mathcal{H}^0(\boldsymbol{R}p_*(DR_{Z/S}(\mathcal{B}_{Y|Z})(k))[n+1])\longrightarrow\mathcal{H}_Y)\qquad k\in\boldsymbol{Z}_+.$$

We remark that the natural inclusion $i:X\to Y$ induces a morphism $i^*_{\mathcal{D}}:\mathcal{H}_Y\to\mathcal{H}_X$ such that $i^*_{\mathcal{D}}(\mathcal{H}_Y^{(k)})\subset\mathcal{H}_X^{(k)}$, and we have $\mathcal{H}_X^{(k)}=\partial_t^k\mathcal{H}_X^{(0)}$ for $\forall k\in\boldsymbol{Z}_+$.

(2.4) **Proposition** (cf. [2]). *$\mathcal{H}_Y^{(k)}|_{S^*}$ is a holomorphic subbundle of H_Y. Furthermore, it coincides with the Hodge bundle $\mathcal{F}^{n-k}$* (cf. (1.4)).

(2.5) **Theorem.** *If the degree d of f is sufficiently large, then we have $i_{\mathcal{D}}^*(\mathcal{H}_Y)=\mathcal{H}_X$, $i_{\mathcal{D}}^*(\mathcal{H}_Y^{(k)})=\mathcal{H}_X^{(k)}$ for $\forall k\in \boldsymbol{Z}_+$ and $\mathcal{K}:=\operatorname{Ker} i_{\mathcal{D}}^*$ is a free $\mathcal{O}_S$-Module of finite rank,* i.e.

(2.5.1) $$0\longrightarrow \mathcal{K}=\oplus\mathcal{O}_S\longrightarrow \mathcal{H}_Y\longrightarrow \mathcal{H}_X\longrightarrow 0$$
is an exact sequence of $\mathcal{D}_S$-Modules.

We remark that $DR_S(\mathcal{K})\subset R^n\bar{f}_*\boldsymbol{C}_Y$ is the sheaf of invariant cycles of $\bar{f}\colon Y\to S$ and the exact sequence (2.5.1) does not split if there is an invariant cycle in H_X.

This theorem follows from the theory of microlocalization (cf. [4]) and the next lemma.

(2.6) **Lemma** (Scherk [7]). *If d is sufficiently large, there is a $\boldsymbol{C}\{t\}$ basis $\{w_i\}_{i=1,\dots,\mu}$ of $\mathcal{H}_{X,0}^{(0)}$ such that* res $(w_i/(f-t))$ *can be extended to holomorphic relative n-forms on $Y-\{0\}$.*

Remark. The exact sequence (2.5.1) was found independently by F. Pham.

§ 3. The Gauss Manin system determines the Hodge filtration.

(3.1) **Definition.** We define a decreasing filtration $\{F^{\boldsymbol{\cdot}}_{\mathcal{H}}\}$ on $H^n(X_\infty, \boldsymbol{C})$ by
$$F^k_{\mathcal{H}}:=\operatorname{Im}(\pi^*(\mathcal{H}_X^{(n-k)}\cap \mathcal{L}_X)\cap\tilde{\mathcal{L}}_X\longrightarrow \tilde{\mathcal{L}}_X(0)\simeq H^n(X_\infty,\boldsymbol{C})),$$
where $\pi^*(\mathcal{H}_X^{(n-k)})\cap\mathcal{L}_X)$ is an $\mathcal{O}_{\tilde{S}}$-submodule of $\tilde{\mathcal{L}}_X\otimes_{\mathcal{O}_{\tilde{S}}}\mathcal{O}_{\tilde{S}}[\tilde{t}^{-1}]$ generated by $\pi^*\,w$ with $w\in\mathcal{H}_X^{(n-k)}\cap\mathcal{L}_X$.

(3.2) **Theorem.** *We have $\{F^{\boldsymbol{\cdot}}_{\mathcal{H}}\}=\{F^{\boldsymbol{\cdot}}_{St}\}$, where $\{F^{\boldsymbol{\cdot}}_{St}\}$ is the Hodge filtration of Steenbrink on $H^n(X_\infty,\boldsymbol{C})$* (cf. (1.5)).

(3.3) Outline of the proof. The inclusion $F^k_{\mathcal{H}}\subset F^k_{St}$ is obvious from (1.4), (1.5), (2.4) and (2.5). To prove the reverse inclusion, we need two results: the duality of exponents due to Steenbrink (cf. [5] [9]) and the following lemma due to Kyoji Saito (cf. [11]).

(3.4) **Lemma.** *Let $\{w_i\}$ be a $\boldsymbol{C}\{t\}$ basis of $\mathcal{H}_{X,0}^{(0)}$ and let $\{\gamma_i(t)\}$ be a basis of multivalued horizontal sections of $\coprod_{t\in S^*}H_n(X_t,\boldsymbol{C})$. Then we have $\left(\det\left(\int_{\gamma_i(t)}w_j\right)\right)^2=t^{\mu(n-1)}g(t)$, where $g(t)$ is a holomorphic function such that $g(0)\neq 0$.*

The rest of the proof is almost the same as Varchenko [10, Lemma 2].

(3.5) **Remarks.** 1. If we define $\{'F^{\boldsymbol{\cdot}}\}$ by $'F^k:=\operatorname{Im}(\pi^*\mathcal{H}_X^{(n-k)}\cap\tilde{\mathcal{L}}_X\to\tilde{\mathcal{L}}_X(0))$, we have $'F^k\supset F^k_{\mathcal{H}}$. But the equality does not hold in general (e.g. $f=x^5+y^5+z^5+x^3y^3$).

2. Varchenko defined a similar filtration $\{F^{\boldsymbol{\cdot}}_a\}$ (cf. [10]). But it is different from $\{F^{\boldsymbol{\cdot}}_{\mathcal{H}}\}$, if there exsists $k\leq n-1$ such that $h_\lambda^{k,n+1-k}\neq 0$ $(\lambda\neq 1)$ or $h_1^{k,n+2-k}\neq 0$ (cf. [6]).

(3.6) **Corollary.** *Hodge number $h_\lambda^{p,q}:=\dim_{\boldsymbol{C}} Gr_F^p Gr_{p+q}^W H^n(X_\infty,\boldsymbol{C})_\lambda$*

and exponents (*cf.* [5]) *are constant under μ-constant deformations* (cf. [7]).

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