44. *Curvature and Stability of Vector Bundles*[*)]

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The purpose of this note is to give a differential geometric condition for a holomorphic vector bundle to be stable or semi-stable.

1. Curvature and the Einstein condition. Let E be a holomorphic vector bundle of rank r over a compact complex manifold M of dimension n and let h be a hermitian structure in E. With respect to a system of linearly independent local holomorphic sections $s_1, \cdots, s_r$ of E, h is given by

(1.1) $$h_{i\bar{j}}=h(s_i, s_j), \quad i, j=1, \cdots, r.$$

Let $z^1, \cdots, z^n$ be a local coordinate system in M. Then the curvature of h is given by

(1.2) $$R_{i\bar{j}\alpha\bar{\beta}}=-\partial_\alpha\partial_{\bar{\beta}}h_{i\bar{j}}+h^{a\bar{b}}\partial_\alpha h_{i\bar{b}}\cdot\partial_{\bar{\beta}}h_{a\bar{j}}, \quad 1\leqq i, j\leqq r, \quad 1\leqq\alpha, \beta\leqq n,$$

where $\partial_\alpha=\partial/\partial z^\alpha$, $\partial_{\bar{\beta}}=\partial/\partial\bar{z}^\beta$, $(h^{a\bar{b}})$ is the inverse matrix of $(h_{i\bar{j}})$, and the summation sign with respect to $a, b=1, \cdots, r$ is omitted. The Ricci tensor of h is given by

(1.3) $$R_{\alpha\bar{\beta}}=h^{i\bar{j}}R_{i\bar{j}\alpha\bar{\beta}}.$$

Then the first Chern class $c_1(E)$ of E is represented by the closed form

(1.4) $$\frac{\sqrt{-1}}{2\pi}R_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta.$$

Now, in addition to a hermitian structure h in E, we fix a Kähler metric

(1.5) $$g=2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$$

on M. The associated Kähler form is given by

(1.6) $$\Phi=\sqrt{-1}\,g_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta.$$

The inverse matrix of $(g_{\alpha\bar{\beta}})$ is denoted by $(g^{\alpha\bar{\beta}})$. We set

(1.7) $$K_{i\bar{j}}=g^{\alpha\bar{\beta}}R_{i\bar{j}\alpha\bar{\beta}}.$$

By definition, $h=(h_{i\bar{j}})$ defines a hermitian bilinear form in each fibre of E. Similarly, $K=(K_{i\bar{j}})$ defines also a hermitian bilinear form in each fibre of E. We say that (E, h, M, g) satisfies the *Einstein condition* with factor φ if

(1.8) $$K_{i\bar{j}}=\varphi h_{i\bar{j}},$$

where φ is a real differentiable function on M.

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2. Stability. Let $\mathcal{F}$ be a coherent analytic sheaf over a compact Kähler manifold (M, g) of dimension n. Let

$$\deg \mathcal{F} = \int_M c_1(\mathcal{F}) \cdot \Phi^{n-1}, \tag{2.1}$$

where Φ is the Kähler form of g. Set

$$\mu(\mathcal{F}) = \deg \mathcal{F} / \operatorname{rank} \mathcal{F}. \tag{2.2}$$

Both $\deg \mathcal{F}$ and $\mu(\mathcal{F})$ depend on the choice of g. Following Mumford and Takemoto [4], we say that a holomorphic vector bundle E over a compact Kähler manifold (M, g) is *Φ-stable* (resp. *Φ-semi-stable*) if, for every subsheaf $\mathcal{F}$ of $\mathcal{O}(E)$ with $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} E$, we have

$$\mu(\mathcal{F}) < \mu(\mathcal{O}(E)), \qquad (\text{resp. } \mu(\mathcal{F}) \leqq \mu(\mathcal{O}(E))). \tag{2.3}$$

Remark. If H is an ample line bundle over M and if Φ is a positive (1, 1)-form representing the Chern class of H, then E is said to be H-stable (resp. H-semi-stable) when it is Φ-stable (resp. Φ-semi-stable).

(2.4) **Theorem.** *Let (E, h) be a hermitian vector bundle over a compact Kähler manifold (M, g). If (E, h, M, g) satisfies the Einstein condition, then E is Φ-semi-stable and (E, h) is isomorphic to a direct sum $(E_1, h_1) \oplus \cdots \oplus (E_k, h_k)$ of Φ-stable hermitian vector bundles (E_1, h_1), $\cdots$, (E_k, h_k).*

Remark. I do not know if every Φ-stable bundle E over (M, g) admits a hermitian structure h such that (E, h, M, g) satisfies the Einstein condition. My guess is that the Einstein condition is slightly stronger than but very close to the Φ-semi-stability. In my earlier paper [2] I showed that the Einstein condition implies the semi-stability in the sense of Bogomolov.

3. Conformal invariance and deformations. Let (E, h) be a hermitian vector bundle over a compact Kähler manifold (M, g). Let a be a real positive function on M, and consider a new hermitian structure $h' = ah$. We denote various curvatures of h' by $R'_{i\bar{j}\alpha\bar{\beta}}$, $R'_{\alpha\bar{\beta}}$, $K'_{i\bar{j}}$, etc. Then

$$K'_{i\bar{j}} = aK_{i\bar{j}} - (\Delta \log a) a h_{i\bar{j}}, \quad \text{where} \quad \Delta = g^{\alpha\bar{\beta}} \partial_\alpha \partial_{\bar{\beta}}. \tag{3.1}$$

It follows that if (E, h, M, g) satisfies the Einstein condition with factor φ, then (E, h', M, g) satisfies also the Einstein condition with factor

$$\varphi' = \varphi - \Delta \log a. \tag{3.2}$$

With a suitable choice of a, the factor φ' becomes a constant function. This constant is given by

$$\varphi' = \left(2n\pi \int_M c_1(E) \Phi^{n-1}\right) \Big/ r \cdot \operatorname{vol}(M), \quad \text{where} \quad \operatorname{vol}(M) = \int_M \Phi^n. \tag{3.3}$$

In particular, a special case of the question raised at the end of § 2 is whether a given Φ-semi-stable bundle E of degree 0 admits a hermitian structure h satisfying

$$K_{i\bar{j}} = 0. \tag{3.4}$$

This differential equation is for anti-self-dual solutions in the Yang-Mills theory [1].

By simple calculation we see that every hermitian structure h satisfies the following inequality:

$$(3.5)\quad \int_M \|K\|^2\Phi^n \geqq \frac{(2n\pi\cdot\deg E)^2}{r\cdot\mathrm{vol}(M)}, \qquad \deg E=\int c_1(E)\Phi^{n-1},$$

where $\|K\|^2=h^{i\bar{j}}h^{k\bar{m}}K_{i\bar{m}}\bar{K}_{j\bar{k}}$. Moreover, the equality holds if and only if h satisfies the Einstein condition with a constant factor φ.

Fix a hermitian structure h satisfying the Einstein condition with a constant factor φ. Let V be the space of all $v=(v_{i\bar{j}})$ which define a hermitian bilinear form on each fibre of E and are parallel with respect to the hermitian connection defined by h. It is a real vector space of finite dimension. Let V_0 denote the 1-dimensional subspace spanned by h. Consider the family of all hermitian structures satisfying the Einstein condition with a constant factor φ. Then V may be regarded as the space of infinitesimal deformations of h. If we do not distinguish two conformally related hermitian structures, then V/V_0 should be considered as the space of infinitesimal deformations of h. It follows that if the holonomy of h is irreducible, then there is no deformation of h except conformal changes.

Let $v\in V$ such that $h'=h+v$ is still positive definite and so defines a hermitian structure. Then h' satisfies also the Einstein condition with the same factor φ.

4. $c_1(E)$ and $c_2(E)$. Let (E, h) be a hermitian vector bundle over a compact Kähler manifold (M, g) satisfying the Einstein condition. Lübke [3] has established the following inequality:

$$(4.1)\qquad \int_M (2rc_2(E)-(r-1)c_1(E)^2)\Phi^{n-2}\geqq 0,$$

where r is the rank of E and n is the dimension of M.

The equality occurs in (4.1) if and only if the pull-back p^*E of E to the universal covering space $p:\tilde{M}\to M$ splits into a direct sum of hermitian line bundles with the same curvature:

$$(4.2)\qquad (p^*E, p^*h)=(L_1, h_1)\oplus\cdots\oplus(L_r, h_r),$$

and the curvature 2-forms $\Omega_1, \cdots, \Omega_r$ of $h_1, \cdots, h_r$ are all equal. In particular, if M is simply connected, the equality in (4.1) holds if and only if E is a direct sum of line bundles $L_1, \cdots, L_r$ such that $c_1(L_1)=\cdots=c_1(L_r)$.

With the method establishing (4.1) it can be shown that if a hermitian vector bundle (E, h, M, g) with $c_1(E)=0$ and $\int c_2(E)\Phi^{n-2}=0$ satisfies the Einstein condition, then the bundle is flat and hence comes from a unitary representation of the fundamental group of M. It is therefore natural to ask whether every Φ-semi-stable bundle E with

$c_1(E)=0$ and $\int c_2(E)\Phi^{n-2}=0$ comes from a unitary representation of the fundamental group of M. The answer is affirmative when M is an Abelian surface or a hyperelliptic surface and Φ comes from an ample line bundle H over M (Umemura [5]).

5. **Examples.** We shall give examples of hermitian vector bundles satisfying the Einstein condition. We fix a compact Kähler manifold (M, g).

(5.1) Every hermitian line bundle over M satisfies the Einstein condition, (trivial).

Given vector bundles with Einstein condition we can generate more by using the following facts:

(5.2) If a hermitian vector bundle (E, h) over M satisfies the Einstein condition with factor φ, then its dual bundle (E^*, h^*) satisfies the Einstein condition with factor $-\varphi$.

(5.3) If hermitian vector bundles (E_1, h_1) and (E_2, h_2) satisfy the Einstein condition with factors φ_1 and φ_2, respectively, then $(E_1\otimes E_2, h_1\otimes h_2)$ satisfies the Einstein condition with factor $\varphi=\varphi_1+\varphi_2$.

(5.4) Let (E_1, h_1) and (E_2, h_2) be hermitian vector bundles over M. Then $(E_1\oplus E_2, h_1\oplus h_2)$ satisfies the Einstein condition with factor φ if and only if (E_1, h_1) and (E_2, h_2) satisfy the Einstein condition with the same factor φ.

As a consequence,

(5.5) If (E, h) satisfies the Einstein condition with factor φ, then its exterior powers $(\Lambda^k E, \Lambda^k h)$ and symmetric powers $(S^k E, S^k h)$ satisfy the Einstein condition with factor $k\varphi$.

(5.6) Let $p: \tilde{M}\to M$ be a finite unramified covering. If (E, h) over M satisfies the Einstein condition with factor φ, the pull-back bundle (p^*E, p^*h) satisfies the Einstein condition with factor $p^*\varphi$, (trivial).

(5.7) Let $p: \tilde{M}\to M$ be a finite unramified covering. If a hermitian vector bundle $(\tilde{E}, \tilde{h})$ over $\tilde{M}$ satisfies the Einstein condition with constant factor φ, then its direct image $(p_*(\tilde{E}), p_*(\tilde{h}))$ satisfies the Einstein condition with the same factor φ.

(5.8) Let $\rho: \pi_1(M)\to U(r)$ be a representation of the fundamental group into the unitary group $U(r)$. Then the natural hermitian structure in $E=\tilde{M}\times_\rho \boldsymbol{C}^r$, (where $\tilde{M}$ denotes the universal covering space of M), is flat and hence satisfies the Einstein condition with factor 0. By (2.4), E is Φ-stable if ρ is irreducible.

(5.9) Let $M=K/V$ be a Kähler C-space (where K is compact and semi-simple). If E is a homogeneous vector bundle associated with an irreducible representation of V, then its (essentially unique) invariant hermitian structure satisfies the Einstein condition with constant factor. If K is simple, then (2.4) and differential geometric arguments give Umemura's result [6] that E is Φ-stable.

In particular,

(5.10) Every irreducible homogeneous vector bundle over a compact irreducible hermitian symmetric space satisfies the Einstein condition and is Φ-stable. For example, all irreducible tensor bundles such as symmetric powers S^kT of the tangent bundle $T=T(P_n)$ of a projective space P_n is Φ-stable.

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