1. The Algebraic Derivative and Laplace's Differential Equation

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0. The purpose of the present note is to show that the differential equation with linear coefficients (so-called Laplace's differential equation)

$$a_2ty''(t)+(a_1t+b_1)y'(t)+(a_0t+b_0)y(t)=0 \tag{1}$$

is convertible into

$$\frac{Dy}{y}=\frac{q(s)}{p(s)}=\frac{(-2a_2+b_1)s-a_1+b_0}{a_2s^2+a_1s+a_0}. \tag{2}$$

Here D is "the operator of algebraic derivative" and s is "the operator of differentiation" in the operational calculus of J. Mikusiński (Pergamon Press (1959)), and fractions Dy/y and $q(s)/p(s)$ are "convolution quotients".

We shall show that, if the algebraic equation

$$p(z)=a_2z^2+a_1z+a_0 \tag{3}$$

has two distinct roots z_1 and z_2 so that

$$\frac{q(z)}{p(z)}=\frac{\gamma_1}{z-z_1}+\frac{\gamma_2}{z-z_2} \quad (\gamma_1 \text{ and } \gamma_2 \text{ are complex numbers}),$$

then the convolution quotient

$$y=C(s-z_1I)^{\gamma_1}(s-z_2I)^{\gamma_2} \quad (C \text{ is a non-zero constant}) \tag{4}$$

satisfies equation (2). In this way, we can solve Bessel differential equation, Laguerre differential equation and the like algebraically, by simply making use of the general binomial expansion

$$(1-\alpha z)^{\gamma}=\sum_{k=0}^{\infty}\binom{\gamma}{k}(-\alpha)^k z^k \quad (\text{convergent for } |\alpha z|<1),$$

without appeal to other analytical tools like the Laplace transform nor to the Fuchs theory of differential equations.

1. The definition of D and of $(s-z_1I)^{\gamma_1}$. Let $\mathcal{C}=C[0,\infty)$ be the totality of complex-valued continuous functions $f=\{f(t)\}$, $g=\{g(t)\}$, $\cdots$. $\mathcal{C}$ is a commutative ring by the sum $f+g=\{f(t)+g(t)\}$ and the (convolution) product $fg=\left\{\int_0^t f(t-u)g(u)du\right\}$. By virtue of Titchmarsh's convolution theorem, we have $fg=0$ ($(fg)(t)\equiv 0$) if and only if either $f=0$ or $g=0$. Hence the totality $\mathcal{C}/\mathcal{C}$ of fractions (convolution quotients) f/g ($f, g \in C$ and $g\neq 0$), $f_1/g_1\cdots$ constitutes a commutative ring by

$$\frac{f}{g}+\frac{f_1}{g_1}=\frac{fg_1+f_1g}{gg_1}, \qquad \frac{f}{g}\frac{f_1}{g_1}=\frac{ff_1}{gg_1}.$$

$\mathcal{C}$ is a subring of $\mathcal{C}/\mathcal{C}$ by identifying $f\in\mathcal{C}$ with $fg/g\in\mathcal{C}/\mathcal{C}$.

We denote $h=\{1\}$ (the operator of integration), $I=h/h=g/g$ (the operator of the product unit) and $s=I/h=g/hg$ (the operator of differentiation). We have

(5) $$h^n=\Gamma(n)^{-1}t^{n-1} \quad (n=1,2,\cdots;\ h^0=I),$$

(6) $$\begin{cases}\text{If } \{f^{(n)}(t)\}\in\mathcal{C}, \text{ then } f^{(n)}=s^nf-s^{n-1}[f(0)]-\cdots-[f^{(n-1)}(0)],\\ \text{where } [\alpha]=s\{\alpha\} \text{ for complex number } \alpha.\end{cases}$$

Definition of D. D is a mapping of $\mathcal{C}/\mathcal{C}$ into $\mathcal{C}/\mathcal{C}$ such that

(7) $$\begin{cases}Df=\{-tf(t)\} \quad \text{for } f\in\mathcal{C},\\ D\dfrac{f}{g}=\dfrac{(Df)g-f(Dg)}{g^2} \quad \text{for } \dfrac{f}{g}\in\mathcal{C}/\mathcal{C}\end{cases}$$

and it is not difficult to prove that

(8) $$\begin{cases}D\left(\dfrac{f}{g}+\dfrac{f_1}{g_1}\right)=D\dfrac{f}{g}+D\dfrac{f_1}{g_1}, \qquad D\left([\alpha]\dfrac{f}{g}\right)=[\alpha]\left(D\dfrac{f}{g}\right),\\ D\left(\dfrac{f}{g}\dfrac{f_1}{g_1}\right)=\left(D\dfrac{f}{g}\right)\dfrac{f_1}{g_1}+\dfrac{f}{g}\left(D\dfrac{f_1}{g_1}\right).\end{cases}$$

Moreover, it is not difficult to show that

(8)′ $$\begin{cases}\text{If } a=\dfrac{m}{n}\in\mathcal{C}/\mathcal{C} \text{ and } b=\dfrac{p}{q}\in\mathcal{C}/\mathcal{C}, \text{ then}\\ D\dfrac{a}{b}=\dfrac{(Da)b-a(Db)}{b^2}=D\dfrac{mq}{np}.\end{cases}$$

We have thus

(9) $$\begin{cases}Dh^n=-nh^{n+1} \text{ and } Ds^n=ns^{n-1} \ (n=1,2,\cdots;\ s^0=I), \text{ in}\\ \text{particular } Dh^0=DI=0,\ Ds^0=DI=0 \text{ and } Ds=I.\end{cases}$$

Proof. $Dh^n=\{-t\Gamma(n)^{-1}t^{n-1}\}=-\Gamma(n)^{-1}\Gamma(n+1)h^{n+1}$.

The hitherto formulas are proved by J. Mikusiński in his book mentioned above. We now define and prove the following.

For any complex number γ,

(10) $$\begin{cases}Dh^\gamma=-\gamma h^{\gamma+1}, \text{ where } h^\gamma=\dfrac{h^{\gamma+n}}{h^n}=\dfrac{\{\Gamma(\gamma+n)^{-1}t^{\gamma+n-1}\}}{\{\Gamma(n)^{-1}t^{n-1}\}},\\ n \text{ being any integer} \geqq 1 \text{ such that } \mathrm{Re}(\gamma+n)>1.\end{cases}$$

Proof. Easy from (5), (8)′ and (9).

We have thus, by (8)′ and (10),

(10)′ $$Ds^\gamma=D\frac{I}{h^\gamma}=\frac{-Dh^\gamma}{h^{2\gamma}}=\gamma h^{\gamma+1-2\gamma}=\gamma s^{\gamma-1}.$$

The next formula is very important:

(11) $$\begin{cases}D(I-\alpha h)^\gamma=\gamma(I-\alpha h)^{\gamma-1}\alpha h^2, \text{ where}\\ (I-\alpha h)^\gamma=\displaystyle\sum_{k=0}^{\infty}\binom{\gamma}{k}(-\alpha)^kh^k.\end{cases}$$

Proof. $\displaystyle\sum_{k=0}^{\infty}\binom{\gamma}{k}(-\alpha)^kh^k=I+\left\{\sum_{k=1}^{\infty}\binom{\gamma}{k}(-\alpha)^k\Gamma(k)^{-1}t^{k-1}\right\}\in\mathcal{C}/\mathcal{C}$ and the infinite series $\sum_{k=1}^{\infty}\binom{\gamma}{k}(-\alpha)^k\Gamma(k)^{-1}t^{k-1}$ is, thanks to the convergent factors $\Gamma(k)^{-1}$, convergent at every t and it can be differentiated with respect to t by term-wise differentiation.

Now we have, by the above,

$$\begin{aligned} D(I-\alpha h)^{\gamma} &= DI+\left\{-t\sum_{k=1}^{\infty}\binom{\gamma}{k}(-\alpha)^k\Gamma(k)^{-1}t^{k-1}\right\} \\ &= \sum_{k=1}^{\infty}\binom{\gamma}{k}(-\alpha)^{k-1}\alpha k h^{k+1} \\ &= \gamma\sum_{k=1}^{\infty}\binom{\gamma-1}{k-1}(-\alpha)^{k-1}h^{k-1}\alpha h^2=\gamma(I-\alpha h)^{\gamma-1}\alpha h^2. \end{aligned}$$

As a corollary of (11), we have

(12) $D(s-\alpha I)^{\gamma}=\gamma(s-\alpha I)^{\gamma-1}$, where $(s-\alpha I)^{\gamma}=\dfrac{(I-\alpha h)^{\gamma}}{h^{\gamma}}$.

Proof.
$$\begin{aligned} D\frac{(I-\alpha h)^{\gamma}}{h^{\gamma}} &= \frac{(D(I-\alpha h)^{\gamma})h^{\gamma}-(I-\alpha h)^{\gamma}(Dh^{\gamma})}{h^{2\gamma}} \\ &= \frac{\gamma(I-\alpha h)^{\gamma-1}(\alpha h^2h^{\gamma}+(I-\alpha h)h^{\gamma+1})}{h^{2\gamma}} \\ &= \frac{\gamma(I-\alpha h)^{\gamma-1}h^{\gamma+1}}{h^{2\gamma}}=\gamma(s-\alpha I)^{\gamma-1}. \end{aligned}$$

2. Proof of (4) and examples. Assuming that $y(t)\not\equiv 0$ is twice continuously differentiable, we can rewrite (1) by (6) and (7) as follows:

$$\begin{aligned} -a_2D(s^2y-s[y(0)]-[y'(0)])+(-a_1D+b_1)(sy-[y(0)]) \\ +(-a_0D+b_0)y=0. \end{aligned}$$

Hence, by $Ds=I$, we obtain (2) assuming the initial condition of $y(t)$:

(13) $$y(0)=0 \text{ if } a_2\neq b_1.$$

This proves (4) by (12).

Example 1 (Bessel differential equation). For the equation

(1)′ $$ty''(t)-(2\alpha-1)y'(t)+ty(t)=0,$$

we have $a_2-b_1=2\alpha$ and

(2)′ $$\frac{Dy}{y}=\frac{-\alpha-1/2}{s+iI}+\frac{-\alpha-1/2}{s-iI}.$$

Hence

(4)′ $$\begin{aligned} y &= C(s+iI)^{-\alpha-1/2}(s-iI)^{-\alpha-1/2}=C(s^2+I)^{-\alpha-1/2} \\ &= C(h^2(I+h^2)^{-1})^{\alpha+1/2}=C\left(\sum_{k=0}^{\infty}\binom{-\alpha-1/2}{k}h^{2k}\right)h^{2\alpha+1}. \end{aligned}$$

satisfies (2)′. If $\mathrm{Re}\,\alpha\geqq 0$, we obtain

$$\binom{-\alpha-1/2}{k}=\frac{(-1)^k\Gamma(2\alpha+2k+1)\Gamma(\alpha+1)}{2^{2k}\Gamma(k+1)\Gamma(2\alpha+1)\Gamma(\alpha+k+1)}.$$

Thus if $\mathrm{Re}\,\alpha>1$, then

$$y=C\frac{\Gamma(\alpha+1)2^{2\alpha}}{\Gamma(2\alpha+1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(\alpha+k+1)}\left(\frac{t}{2}\right)^{2k+2\alpha}$$

is twice continuously differentiable in t for $t\geqq 0$ including $t=0$. Thus the solution

(14) $$y_\alpha(t)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(\alpha+k+1)}\left(\frac{t}{2}\right)^{2k+2\alpha}$$

of (2)′ satisfying (13) is a solution of (1)′ for $t\geqq 0$. This means that, when $t\geqq 0$ and $\mathrm{Re}\,\alpha>1$, the coefficient of $t^{2k+2\alpha-1}$ in the infinite series given by

(15) $$ty''_\alpha(t)-(2\alpha-1)y'_\alpha(t)+ty_\alpha(t)$$

must vanish as an analytic function of α ($k=0, 1, 2, \cdots$).

Therefore, since $y_\alpha(t)$ with $\mathrm{Re}\,\alpha\geqq 0$ is also twice continuously differentiable in $t>0$, we see, as in the case of $\mathrm{Re}\,\alpha>1$, that the formula

$$ty''_\alpha(t)-(2\alpha-1)y'_\alpha(t)+ty_\alpha(t)$$

must vanish, because the coefficients of $t^{2k+2\alpha-1}$ all vanish.

Thus we have proved that, when $\mathrm{Re}\,\alpha>0$ or $\alpha=0$, $y_\alpha(t)$ given in (14) is a solution of (1)′ at every $t>0$ satisfying (13). Hence we have obtained Bessel function of the first kind and of order α ($\mathrm{Re}\,\alpha>0$ or $\alpha=0$):

(16) $$J_\alpha(t)=t^{-\alpha}y_\alpha(t)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(\alpha+k+1)}\left(\frac{t}{2}\right)^{2k+\alpha}$$

which satisfies the original Bessel equation

(17) $$t^2J''_\alpha(t)+tJ'_\alpha(t)+(t^2-\alpha^2)J_\alpha(t)=0 \qquad \text{for } t>0.$$

Example 2 (Laguerre differential equation). For the equation

(1)″ $$ty''(t)-(t+\alpha-1)y'(t)+(\alpha+\lambda)y(t)=0$$

we have $a_2-b_1=\alpha$ and

(2)″ $$\frac{Dy}{y}=\frac{(-2+1-\alpha)s+1+\alpha+\lambda}{s^2-s}=\frac{-1-\alpha-\lambda}{s}+\frac{\lambda}{s-I}.$$

Hence, for $\mathrm{Re}\,\alpha>0$ or for $\alpha=0$,

(4)″ $$\begin{aligned}y_{\alpha,\lambda}&=Cs^{-1-\alpha-\lambda}(s-I)^\lambda=Ch^{1+\alpha}(I-h)^\lambda\\&=C\sum_{k=0}^{\infty}\binom{\lambda}{k}(-1)^k\Gamma(k+\alpha+1)^{-1}t^{k+\alpha}\end{aligned}$$

is a solution of (2)″ and $t^{-\alpha}y_{\alpha,\lambda}$ reduces to a polynomial in t if and only if $\lambda=n$ ($=0, 1, 2, \cdots$). So we have, by taking C as $(n!)^{-1}\Gamma(n+\alpha+1)$,

$$\begin{aligned}t^{-\alpha}y_{\alpha,n}&=\frac{\Gamma(n+\alpha+1)}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(-t)^k}{\Gamma(k+\alpha+1)}\\&=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-t)^k}{k!}=L_n^{(\alpha)}(t).\end{aligned}$$

We have thus obtained the n-th Laguerre polynomial of order α: $L_n^{(\alpha)}(t)$. When $\mathrm{Re}\,\alpha>0$ or $\alpha=0$, $t^\alpha L_n^{(\alpha)}(t)$ is surely a solution of (1)″ with $\lambda=n$ for $t>0$ and satisfies (13).

Remark. The equation of the form

$$\frac{Dy}{y}=\frac{\gamma}{(s-\alpha)^2}$$

is satisfied by $y=Ce^{\alpha t}e^{-\gamma h}$ which belongs to C/C. We omit the proof.