# 44. The Riemann-Roch Theorem and Bernoulli Polynomials

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**0. Introduction.** Let $X$ be a non-singular algebraic variety with $\dim X=N$ over an algebraically closed field. In this paper we shall prove the following formula

$$\chi(tK_X)=\sum_{r=0}^{[N/2]}\frac{\phi_{N-2r}(t)}{(N-2r)!}K_X^{N-2r}R_r.$$

Here the $\phi_n(t)$ denote the Bernoulli polynomials, defined by

$$\frac{xe^{tx}}{e^x-1}=\sum_n\frac{\phi_n(t)}{n!}x^n,$$

$R_n=R_n(c_1,\cdots,c_{2n})$ is a polynomial of Chern classes, defined by

$$T_{2n+1}(c_1,\cdots,c_{2n})=(1/2)c_1R_n(c_1,\cdots,c_{2n})$$

where $T_r$ is the $r$-th todd class of $X$.

**1. Preliminaries.** We start by recalling the following elementary facts.

**Lemma 1.**

(1-1) $$\phi_0(t)=1,\qquad \phi_1(t)=t-(1/2).$$

(1-2) $$(d/dt)\phi_n(t)=n\cdot\phi_{n-1}(t).$$

(1-3) $$\phi_{2n+1}(0)=\phi_{2n+1}(1/2)=0\qquad \textit{for } n\geqq 1.$$

(1-4) $$\phi_n(t+1)-\phi_n(t)=nt^{n-1}.$$

(1-5) $$\phi_n(t)=\sum_{r=0}^{n}\binom{n}{r}\phi_r(0)t^{n-r},\qquad \phi_{2n}(t)=\sum_{r=0}^{m}\binom{2m}{2r}\phi_{2r}(0)t^{2m-2r}-mt^{2m-1}.$$

(1-6) $$\sum_{r=0}^{m}\binom{2m}{2r}\frac{2^{2r}\phi_{2r}(0)}{2m-2r+1}=1.$$

*Proof.* We only prove (1-6). From (1-5) we have

$$\frac{\phi_{2m+1}(t)}{2m+1}=\sum_{r=0}^{m}\binom{2m}{2r}\frac{\phi_{2r}(0)}{2m-2r+1}t^{2m-2r+1}-\frac{1}{2}t^{2m}.$$

Put $t=1/2$. Then

$$0=\sum_{r=0}^{m}\binom{2m}{2r}\frac{\phi_{2r}(0)}{2m-2r+1}\cdot\frac{1}{2^{2m-2r+1}}-\frac{1}{2^{2m+1}}.$$

From this (1-6) follows. Q.E.D.

We define the symbols $c_1,\cdots,c_N$; $p_1,\cdots,p_N$; $z_1,\cdots,z_N$; $x_1,\cdots,x_N$; and polynomials $A_i(p_1,\cdots,p_i)$, $T_i(c_1,\cdots,c_i)$ $(0\leqq i\leqq N)$ and $R_j(c_1,\cdots,c_{2j})$ $(0\leqq j\leqq[N/2])$ as follows:

(1) $$z_i=x_i^2\qquad\text{for } 1\leqq i\leqq N.$$

(2) $p_i$ is the $i$-th elementary symmetric function of $x_1,\cdots,x_N$.

(3) $c_i$ is the $i$-th elementary symmetric function of $z_1,\cdots,z_N$.

$$(4)\qquad \sum_{i=1}^{N}\frac{2x_iT}{\sinh 2x_iT}=\sum_{i=0}^{N}A_i(p_1,\cdots,p_i)\cdot T^i \qquad (\mathrm{mod}\ T^{N+1}).$$

$$(5)\qquad \sum_{i=1}^{N}\frac{z_iT}{1-\exp(-z_iT)}=\sum_{i=0}^{N}T_i(c_1,\cdots,c_i)\cdot T^i \qquad (\mathrm{mod}\ T^{N+1}).$$

$$(6)\qquad T_{2j+1}(c_1,\cdots,c_{2j})=(1/2)c_1R_j(c_1,\cdots,c_{2j}).$$

From these

$$A_1=-(2/3)p_1,\qquad A_2=(2/45)(-4p_2+7p_1^2),$$
$$A_3=(-4/945)(16p_3-44p_2p_1+31p_1^3),\quad \cdots;$$
$$T_1=(1/2)c_1,\quad T_2=(1/12)(c_2+c_1^2),\quad T_3=(1/24)c_2c_1.$$
$$R_0=1,\quad R_1=(1/12)c_2,\quad R_2=(1/720)(-c_1^2c_2+c_1c_3-c_4+3c_2^2),$$
$$R_3=(1/60480)(2c_1^4c_2+2c_1^2c_4-2c_1^3c_3-10c_1^2c_2^2+11c_1c_2c_3-c_3^2-9c_2c_4-2c_1c_5+10c_2^3+2c_6),$$
$$R_4=(1/3628800)(-3c_1^6c_2+3c_1^5c_2+21c_1^4c_2^2-3c_1^4c_4-29c_1^3c_2c_3+3c_1^3c_5-42c_1^2c_2^3+8c_1^2c_3^2+26c_1^2c_2c_4-3c_1^2c_6+50c_1c_2^2c_3-16c_1c_2c_5-13c_1c_3c_4+3c_1c_7+21c_2^4-34c_2^2c_4-8c_2c_3^2+13c_2c_6+3c_3c_5+5c_4^2-3c_8).$$

**Remark.** If we regard $c_i$ as the $i$-th chern class of $X$, then $T_r$ represents the $r$-th Todd class of $X$.

**Lemma 2.**

$$(7)\qquad T_r(c_1,\cdots,c_r)=\sum_{s=0}^{[r/2]}\frac{1}{2^{4s}(r-2s)!}\left(\frac{1}{2}c_1\right)^{r-2s}A_s(p_1,\cdots,p_s).$$

*Especially $T_{2r+1}$ is a polynomial in $c_1,\cdots,c_{2r}$ which can be devided by $c_1$.*

*Proof.* See Todd [2].

Hence, from the definition of $R_r$,

$$(8)\qquad R_r(c_1,\cdots,c_{2r})=\sum_{s=0}^{r}\frac{1}{2^{4s}(2r-2s+1)!}\left(\frac{1}{2}c_1\right)^{2r-2s}A_s(p_1,\cdots,p_s).$$

## 2. Proof of the formula.

**Lemma 3.** $\displaystyle T_M=\sum_{r=0}^{[M/2]}\frac{\phi_{M-2r}(0)}{(M-2r)!}K_X^{M-2r}R_r.$

*Proof.* If $M$ is odd, then $\phi_{M-2r}(0)=0$ for $r<[M/2]$. Thus

$$\sum_{r=0}^{[M/2]}\frac{\phi_{M-2r}(0)}{(M-2r)!}c_1^{M-2r}R_r=\phi_1(0)K_XR_{[M/2]}=\frac{1}{2}c_1R_{[M/2]}=T_M.$$

We assume that $M$ is even, say $M=2n$. Then we shall show

$$T_{2n}=\sum_{r=0}^{n}\frac{\phi_{2n-2r}(0)}{(2n-2r)!}c_1^{2n-2r}R_r.$$

Actually, by (8), the right hand side is written as

$$\begin{aligned}
&\sum_{r=0}^{n}\frac{\phi_{2n-2r}(0)}{(2n-2r)!}c_1^{2n-2r}R_r\\
&=\sum_{r=0}^{n}\frac{\phi_{2n-2r}(0)}{(2n-2r)!}\left(\frac{1}{2}c_1\right)^{2n-2r}\left\{\sum_{s=0}^{r}\frac{2^{2n-2r}}{2^{4s}(2r-2s+1)!}\left(\frac{1}{2}c_1\right)^{2r-2s}A_s\right\}\\
&=\sum_{s=0}^{n}\left\{\sum_{r=s}^{n}\frac{2^{2n-2r}\phi_{2n-2r}(0)}{(2n-2r)!\cdot(2r-2s+1)!}\right\}\frac{1}{2^{4s}}\left(\frac{1}{2}c_1\right)^{2n-2s}A_s\\
&=\sum_{s=0}^{n}\left\{\sum_{q=0}^{n-s}\frac{2^{2q}\phi_{2q}(0)}{(2q)!\cdot(2n-2s-2q+1)!}\right\}\frac{1}{2^{4s}}\left(\frac{1}{2}c_1\right)^{2n-2s}A_s.
\end{aligned}$$

Putting $m=n-s$, (1-6) yields

$$\sum_{q=0}^{n-s} \frac{2^{2q}\phi_{2q}(0)}{(2q)!\,(2n-2s-2q+1)!}=\frac{1}{(2n-2s)!}.$$

Hence the above sum is $T_{2n}$ by (7). Q.E.D.

Let

$$(*) \qquad P(t):=\sum_{r=0}^{[N/2]} \frac{\phi_{N-2r}(t)}{(N-2r)!} K_X^{N-2r} R_r.$$

If we substitute $D/K_X$ for $t$ in $(*)$, then $(*)$ can be regarded as a polynomial in $D, K_X$ and $c_1, \cdots, c_N$.

**Theorem 4.** *Let $X$ be a non-singular complete variety of dimension $N$, $D$ a line bundle on $X$, $c_1, \cdots, c_N$ chern classes of $X$, and let $K_X$ be a canonical line bundle of $X$. Then*

$$\chi(\mathcal{O}_X(D))=P(D/K_X).$$

*Proof.* By the Hirzebruch Riemann-Roch formula

$$\chi(\mathcal{O}_X(D))=\sum_{s=0}^{N} \frac{1}{s!} D^s T_{N-s}.$$

On the other hand, the term of $D^s$ of $P(D/K_X)$ is equal to a multiple of $(D/K_X)^s$ and of the coefficient of $t^s$ in $P(t)$. Noting that

$$\frac{\phi_{N-2r}(t)}{(N-2r)!}=\sum_{s=0}^{N-2r} \frac{\phi_{N-2r-s}(0)}{r!\,(N-2r-s)!} t^s,$$

the term of $D^s$ of $P(D/K_X)$ is

$$(**) \qquad \sum_{r=0}^{N} \frac{\phi_{N-2r-s}(0)}{r!\,(N-2r-s)!} K_X^{N-2r-s} R_r D^s.$$

By Lemma 3, $(**)$ is equal to

$$\sum_{s=0}^{N} \frac{1}{s!} D^s T_{N-s}.$$

This completes the proof.

Putting $D=tK_X$ we obtain the following formula stated in the Introduction:

$$\chi(tK_X)=\sum_{r=0}^{[N/2]} \frac{\phi_{N-2r}(t)}{(N-2r)!} K_X^{N-2r} R_r.$$

## References

[1] F. Hirzebruch and K. H. Mayer: Topological methods in algebraic geometry. Grundlehren 131, 3rd ed., Springer-Verlag, Heidelberg, ix+232 pp. (1966).

[2] J. A. Todd: The arithmetical invariants of algebraic loci. Proc. London Math. Soc., **43**, 190–225 (1937).