71. On a Class of Nonhyperbolic Microdifferential Equations with Involutory Double Characteristics

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§1. **Introduction.** In this note, we study a class of microdifferential equations with involutory double characteristics. Explicitly, let M be a real analytic manifold of dimension n (≥ 4) with a complexification X. We consider a microdifferential equation defined in a neighborhood of $\rho_0 \in T^*_M X \setminus M$

(1) $$Pu=\{(P_1+\sqrt{-1}P_2)P_3+Q\}u=0.$$

Here we set $p_j=\sigma(P_j)(1\leq j\leq 3)$ and assume the following conditions.

(2) $\operatorname{ord}(P_1)=\operatorname{ord}(P_2)=m_1$, $\operatorname{ord}(P_3)=m_2$ and $\operatorname{ord}(Q)=m_1+m_2-1$.

(3) p_1, p_2 and p_3 are real valued on $T^*_M X$.

(4) $p_j(\rho_0)=0 \quad (1\leq j\leq 3)$.

(5) dp_1, dp_2, dp_3 and the canonical 1-form ω of $T^*_M X$ are linearly independent at ρ_0.

(6) $\{p_i, p_j\}=0$ if $p_i=p_j=0$ $(1\leq i, j\leq 3)$ where $\{\cdot, \cdot\}$ denotes Poisson bracket on $T^*_M X$.

By Sato *et al.* [4], the structure of microdifferential equation (1) is completely studied outside the regular involutory submanifold

(7) $$\Sigma=\{\rho\in T^*_M X\,;\, p_1(\rho)=p_2(\rho)=p_3(\rho)=0\}.$$

Thus, we interest ourselves in studying the structure of solutions on Σ. By employing the theory of 2-microlocalization due to M. Kashiwara and Y. Laurent (see [1], [3]), we show a result about the propagation of 2-microlocal singularities as a byproduct of N. Tose [6]. More precisely, we see the equation (1) is 2-microlocally equivalent to $(D_1+\sqrt{-1}D_2)u=0$ or $D_3u=0$ or $u=0$.

§2. **Preliminary.** 2.1. *2-microdifferential operators.* Let X be an open subset in $\boldsymbol{C}^{n+d}$ and let T^*X be its cotangent bundle. We take a coordinate of X as (w, z) with $w\in\boldsymbol{C}^n$ and $z\in\boldsymbol{C}^d$. Then $\rho=(w, z\,;\,\theta dw+\zeta dz)$ denotes a point of T^*X with $\theta\in\boldsymbol{C}^n$ and $\zeta\in\boldsymbol{C}^d$. For microdifferential operators, see M. Sato *et al.* [4] and P. Schapira [5].

Hereafter in §2.1, Λ is the regular involutory submanifold in $T^*X\setminus X$: $\Lambda=\{(w, z\,;\,\theta, \zeta)\,;\,\zeta=0\}$. We identify Λ with a submanifold of $\Lambda\times\Lambda$ through the embedding $T^*X\simeq T^*_X(X\times X)\subsetneq T^*(X\times X)$. By definition, $\tilde{\Lambda}$ is the union of bicharacteristic leaves of $\Lambda\times\Lambda$ issued from Λ. We take a coordinate of $T^*_\Lambda\tilde{\Lambda}$ as $(w, z\,;\,\theta\,;\,z^*)$ with $(w, z\,;\,\theta)\in\Lambda$ and $z^*\in\boldsymbol{C}^d$.

$T^*_\Lambda\tilde{\Lambda}$ is endowed with the sheaf $\mathcal{E}^{2,\infty}_\Lambda$ of 2-microdifferential operators of infinite order constructed in Y. Laurent [3].

Definition 1. For an open subset U of $T^*_\Lambda\tilde{\Lambda}$, a formal sum

$\sum_{(i,j)\in Z^2} P_{ij}(w, z, \theta, z^*)$ belongs to $\mathcal{E}_A^{2,\infty}(U)$ if and only if the following conditions (8) and (9) are satisfied.

(8) P_{ij} is holomorphic on U and homogeneous of order j with respect to (θ, z^*) and of order i with respect to z^*.

(9) For any compact subset K of U, there exists a positive number C_K and for any positive ε and a compact subset K, we can take a positive $C_{\varepsilon,K}$ such that

$$\sup_k |P_{1,i+k}| \leq \begin{cases} C_{\varepsilon,K}\varepsilon^{i+k}/i!\,k! & (i, k \geq 0) \\ C_{\varepsilon,K}^{-k}\varepsilon^i(-k)!/i! & (i\geq 0, k<0) \\ C_{\varepsilon,K}\varepsilon^k C_K^{-i}(-i)!/k! & (i<0, k\geq 0) \\ C_K^{-i-k}(-k)!\,(-i)! & (i, k<0). \end{cases}$$

Y. Laurent [3] constructed the sheaf $\mathcal{E}_A^2$ of 2-microdifferential operators of finite order, which is a subsheaf of $\mathcal{E}_A^{2,\infty}$. For a section P of $\mathcal{E}_A^2$, $\sigma_A(P)$ denotes the principal symbol of P along A. Y. Laurent [3] also defined the sheaf of 2-microdifferential operators for general involutory submanifolds. See [3] for more details about 2-microdifferential operators.

2.2. *2-microfunctions.* Let M be a real analytic manifold with a complexification X. Let Σ be a regular involutory submanifold in $T_M^*X\backslash M$ with a complexification A in T^*X. Then, $\tilde{\Sigma}$ denotes the union of all bicharacteristic leaves of A issued from Σ. On $\tilde{\Sigma}$, there exists the sheaf $\mathcal{C}_{\tilde{\Sigma}}$ of microfunctions along $\tilde{\Sigma}$. $\tilde{\Sigma}$ is foliated by the canonical foliation of A and for any section u of $\mathcal{C}_{\tilde{\Sigma}}$, u has the unique continuation property along the leaves.

$T_{\Sigma}^*\tilde{\Sigma}$ is endowed with the sheaf $\mathcal{C}_{\Sigma}^2$ of 2-microfunctions along Σ, which is constructed by M. Kashiwara about in 1973 in Nice. The sheaf $\mathcal{C}_{\Sigma}^2$ plays a powerful role to study properties of microfunctions defined on Σ. Precisely, we have exact sequences

(10) $$0 \longrightarrow \mathcal{C}_{\tilde{\Sigma}}|_{\Sigma} \longrightarrow \mathcal{B}_{\Sigma}^2 \longrightarrow \pi_*(\mathcal{C}_{\Sigma}^2|_{T_{\Sigma}^*\tilde{\Sigma}\backslash\Sigma}) \longrightarrow 0$$

and

(11) $$0 \longrightarrow \mathcal{C}_M|_{\Sigma} \longrightarrow \mathcal{B}_{\Sigma}^2.$$

Here we set $\mathcal{B}_{\Sigma}^2 = \mathcal{C}^2{}_{\Sigma}|_{\Sigma}$ and $\pi: T_{\Sigma}^*\tilde{\Sigma}\backslash\Sigma \longrightarrow \Sigma$.

Moreover, there exists the canonical spectral map

(12) $$Sp_{\Sigma}^2 : \pi^{-1}\mathcal{B}_{\Sigma}^2 \longrightarrow \mathcal{C}_{\Sigma}^2.$$

For $u \in \mathcal{C}_M|_{\Sigma}$, we set $SS_{\Sigma}^2(u) = \mathrm{supp}\,(Sp_{\Sigma}^2(u))$, which is called the 2-singular spectrum of u along Σ. For details about 2-microfunctions, see M. Kashiwara and Y. Laurent [1].

§3. Statement of the main result. We follow the notation prepared in §1 and give

Theorem 1. *Let u be a microfunction solution to* (1) *defined in a neighborhood of ρ_0 and let Γ be the bicharacteristic leaf of Σ passing through ρ_0. Then there exist a neighborhood Ω of ρ_0 in T_M^*X and a family of integral manifolds $\{\gamma_t^{(1)}\}$ for the involutive system of vector fields (H_{p_1}, H_{p_2}) on $\Gamma \cap \Omega$ and a family of integral curves $\{\gamma_t^{(2)}\}$ of H_{p_3} on $\Gamma \cap \Omega$ such that*

$\operatorname{supp}(u)\cap\Gamma\cap\Omega=\bigcup_t\gamma_t^{(1)}\cup\bigcup_t\gamma_t^{(2)}\cup\{$*some of connected components of* $(\Gamma\cap\Omega)\setminus(\bigcup_t\gamma_t^{(1)}\cup\bigcup_t\gamma_t^{(2)})\}$.

§4. Proof of Theorem 1. By finding a suitable real quantized contact transformation, we can reduce the problem to studying the equation

(13) $$\{(D_1+\sqrt{-1}D_2)P_3+(\text{lower order})\}u=0$$

defined in a neighborhood of $\rho_0=(0,\sqrt{-1}dx_n)\in\sqrt{-1}T^*\boldsymbol{R}^n$, which satisfies the conditions analogous to those for (1). Moreover, we may assume

(14) $$\Sigma=\{(x,\sqrt{-1}\xi\cdot dx)\,;\,\xi_1=\xi_2=\xi_3=0\}.$$

Here we take a coordinate of $\sqrt{-1}T^*\boldsymbol{R}^n$ as $(x,\sqrt{-1}\xi\cdot dx)$ with $x,\xi\in\boldsymbol{R}^n$. We set $x'=(x_1,x_2,x_3)$, $\xi'=(\xi_1,\xi_2,\xi_3)\in\boldsymbol{R}^3$ and $x''=(x_4,\cdots,x_n)$, $\xi''=(\xi_4,\cdots\xi_n)\in\boldsymbol{R}^{n-3}$ and take a coordinate of $T^*_\Sigma\tilde{\Sigma}$ as $(x\,;\sqrt{-1}\xi''\,;\sqrt{-1}x'^*)$ with $x'^*=(x_1^*,x_2^*,x_3^*)$. We put as a complexification of Σ

(15) $$\Lambda=\{(z,\xi\cdot dz)\in T^*\boldsymbol{C}^n\,;\,\zeta_1=\zeta_2=\zeta_3=0\}$$

where we take a coordinate of $T^*\boldsymbol{C}^n$ as $(z,\zeta\cdot dz)$ with $z,\zeta\in\boldsymbol{C}^n$. We study the equation (13) 2-microlocally along Σ and then see easily that

(16) $$\sigma_\Lambda(P_3)(\tau)\neq 0$$

for $\tau\in C_1=\{(x\,;\sqrt{-1}\xi''\,;\sqrt{-1}x'^*)\in T^*_\Sigma\tilde{\Sigma}\setminus\Sigma\,;\,x_1^*=x_2^*=0\}$. Thus 2-microlocally in a neighborhood of $\tau\in C_1$, it suffices to study the 2-microdifferential equation $\{(D_1+\sqrt{-1}D_2)+P_3^{-1}Q\}u=0$. Here Q satisfies the condition

(17) $$\{(j,i)\in\boldsymbol{Z}^2\,;\,(P_3^{-1}Q)_{ij}\not\equiv 0\}\subset\{j\leq 0,\,j-1\leq i\}.$$

Then, by Theorem 3.1 of N. Tose [6] (see also [8], [9] and [11]), we can find an invertible section R of $\mathcal{E}_\Lambda^{2,\infty}$ defined in a neighborhood of $\tau\in C_1$ and satisfying

(18) $$R\{(D_1+\sqrt{-1}D_2)+Q\}=\{(D_1+\sqrt{-1}D_2)\}R.$$

By (18) and the unique continuation properties of 2-microfunctions with holomorphic parameters (see N. Tose [7]), we see that for any 2-microfunction solution u to (16),

(19) $\operatorname{supp}(u)\cap C_1$ is a union of integral manifolds for $(\partial/\partial x_1,\partial/\partial x_2)$.

On the other hand, we can find a real quantized contact transformation which transforms the equation (1) into

(20) $$\{(P_1+\sqrt{-1}P_2)D_3+(\text{lower order})\}u=0$$

defined in a neighborhood of $\rho_0=(0,\sqrt{-1}dx_n)$. Here the equation (20) satisfies the conditions analogous to those for (1). Moreover, we may assume the condition (14). Then, in the same way as in studying (13) 2-microlocally, we have

(21) $$\sigma_\Lambda(P_1+\sqrt{-1}P_2)(\tau)\neq 0$$

for $\tau\in C_2=\{(x\,;\sqrt{-1}\xi''\,;\sqrt{-1}x'^*)\,;\,x_1^*=0\}$. Further, for any 2-microfunction solution u to the equation (20), we can show

(22) $\operatorname{supp}(u)\cap C_2$ is invariant under $\partial/\partial x_3$.

We get back to the original situation in §1 and set Λ to be a complexification of Σ in T^*X. Since 2-microdifferential operators of finite order are invertible at 2-elliptic point, we have

(23) $$SS^2_\Sigma(u)\setminus\Sigma\subset\tilde{C}_1\cup\tilde{C}_2.$$

Here $\tilde{C}_1=\{\tau\in T^*_\Sigma\tilde{\Sigma}\setminus\Sigma\,;\,\sigma_\Lambda(P_3)=0\}$ and $\tilde{C}_2=\{\tau\in T^*_\Sigma\tilde{\Sigma}\setminus\Sigma\,;\,\sigma_\Lambda(P_1+\sqrt{-1}P_2)=0\}$.

Moreover $\tilde{C}_1$ and $\tilde{C}_2$ are disjoint to each other in $T^*_{\tilde{\Sigma}}\tilde{\Sigma}\backslash\Sigma$. By (19), (22) and (23), we can show the assertion of Theorem 1 if we consult the fundamental exact sequences (10) and (11) and the unique continuation properties of $\mathcal{C}_{\tilde{\Sigma}}$.

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