26. Notes on Quasi-polarized Varieties

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(Communicated by Kunihiko KODAIRA, M. J. A., March 14, 1988)

0. Let V be a variety, which means, an irreducible reduced projective scheme over an algebraically closed field $\mathfrak{K}$ of any characteristic. A line bundle L on V is said to be *nef* if $LC \geqq 0$ for any curve C in V. It is said to be *big* if $\kappa(L)=n=\dim V$. In case L is nef, it is big if and only if $L^n>0$ (cf. [2; (6.5)]). When L is nef and big, the pair (V, L) will be called a *quasi-polarized variety*. In this note we report several generalizations of results on polarized manifolds. For details see [4].

1. We have $\chi(V, tL)=\sum_{j=0}^{n} \chi_j t^{[j]}/j!$ for some integers $\chi_0, \chi_1, \cdots, \chi_n$ where $t^{[j]}=t(t+1)\cdots(t+j-1)$ and $t^{[0]}=1$. By the Riemann-Roch theorem we have $\chi_n=L^n$. Moreover, if V is normal, we have

$$-2\chi_{n-1}=(\omega+(n-1)L)L^{n-1}$$

for the canonical divisor ω of V. We set $g(V, L)=1-\chi_{n-1}$, which is called the *sectional genus* of (V, L). We set $\Delta(V, L)=n+L^n-h^0(V, L)$, which is called the *$\Delta$-genus* of (V, L). We conjecture:

Both the Δ-genus and the sectional genus are non-negative for any quasi-polarized variety. Moreover, $\Delta=0$ if and only if $g=0$.

We expect further that we can classify somehow (V, L)'s with small Δ and g.

2. First of all we have the following

Theorem. *$\Delta(V, L) \geqq 0$ for any quasi-polarized variety. Moreover, if $\Delta=0$, there are a polarized variety (W, H) and a birational morphism $f: V \to W$ such that $L=f^*H$ and $\Delta(W, H)=0$.*

We have a complete classification of polarized varieties of Δ-genus zero (cf. [1]). In particular $g(W, H)=0$ and H is very ample. Hence $g(V, L)=0$ and $\mathrm{Bs}|L|=\varnothing$ if $\Delta(V, L)=0$.

3. From now on, we assume $\mathrm{char}(\mathfrak{K})=0$, since we need vanishing theorems of Kodaira-Kawamata-Viehweg type. Using the above theorem we obtain the following

Theorem. *Let (V, L) be a normal quasi-polarized variety with* $\dim V=n$. *Suppose that $h^n(V, -tL)=0$ for any t such that $0<t\leqq n$. Then there is a birational morphism $f: V \to \boldsymbol{P}^n$ such that $L=f^*\mathcal{O}(1)$.*

4. Next we improve results in [3]. An element of $\mathrm{Pic}(V)\otimes\boldsymbol{Q}$ is called a *$\boldsymbol{Q}$-bundle* on V. We define $\boldsymbol{Q}$-valued intersection numbers of $\boldsymbol{Q}$-bundles and the nefness of them in the natural way.

Let $\pi: M \to V$ be a desingularization of a normal variety V and set $S=\{x \in V \mid \dim \pi^{-1}(x)>0\}$ and $E=\pi^{-1}(S)$. Then π is said to be nice if E is a divisor having no singularity other than simple normal crossings. Thus $E=\sum E_i$ and each prime component E_i is smooth.

V is said to have only *log-terminal* singularities if it is normal and there is a nice desingularization $\pi: M \to V$ such that $K=\pi^*\omega+\sum a_i E_i$ for some $\boldsymbol{Q}$-bundle ω on V and some rational numbers a_i with $a_i>-1$, where K is the canonical bundle of M. Since E_i's are π-exceptional, this implies that ω corresponds to the canonical sheaf of V and hence V is $\boldsymbol{Q}$-Gorenstein, which means, some positive multiple of a canonical Weil divisor of V is Cartier. If furthermore $a_i \geqq 0$ (resp. $a_i>0$) for every i, then V has only *canonical* (resp. *terminal*) singularities. Note that log-terminal singularities are rational (cf. [5; § 1–3]).

We say that V is Gorenstein in codimension k if there is a subset X of V such that $\operatorname{codim} X>k$ and $V-X$ has only Gorenstein singularities.

Theorem (compare [3; Theorem 2]). *Let (V, L) be a polarized variety of dimension n having only log-terminal singularities. Suppose further that V is Gorenstein in codimension 2. Then $\omega+(n-1)L$ is nef unless $(V, L) \simeq (\boldsymbol{P}^n, \mathcal{O}(1))$, $(\boldsymbol{P}^2, \mathcal{O}(2))$, a scroll over a smooth curve, or V is a (possibly singular) hyperquadric in $\boldsymbol{P}^{n+1}$ with $L=\mathcal{O}(1)$.*

Here, (V, L) is said to be a *scroll* over C if there is a vector bundle $\mathcal{E}$ on C such that $V \simeq \boldsymbol{P}_C(\mathcal{E})$ with $L=\mathcal{O}(1)$. Note that V is Gorenstein in codimension 2 if it has only canonical singularities.

Corollary. *Let (V, L) be as in the theorem. Then $g(V, L) \geqq 0$. Moreover $g=0$ implies $\Delta(V, L)=0$.*

Corollary. *Let (V, L) be as in the theorem and suppose $g(V, L)=1$. Then $\omega=(1-n)L$ unless (V, L) is a scroll over a smooth elliptic curve.*

5. When $\operatorname{Bs}|L|=\varnothing$, the nefness of $\omega+tL$ for $t>0$ can be proved occasionally by induction on n. This approach was used by Sommese effectively in various papers. The theorem below improves upon a result in [7; (2.1)] and will be useful in this method.

Theorem. *Let A be an irreducible reduced ample Cartier divisor on a normal $\boldsymbol{Q}$-Gorenstein variety V. Suppose that $V-X$ has only log-terminal singularities for some finite set X, the double dual of $\omega^{\otimes m}$ is invertible in a neighborhood of A for some positive integer m and that $(\omega+tA)_A$ is nef for some $t \geqq 2-m^{-1}$. Then $\omega+tA$ is nef on V unless $(V, \mathcal{O}(A))$ is a scroll over a smooth curve with $\dim V=2$.*

6. Here we assume $n=\dim V \leqq 3$, since we need Mori's flip theorem in dimension 3 (cf. [6]).

We say that quasi-polarized varieties (V_1, L_1) and (V_2, L_2) are birationally equivalent if there is a variety X together with birational morphisms $f_i: X \to V_i$ such that $f_1^* L_1=f_2^* L_2$.

Theorem. *Let (V, L) be a quasi-polarized variety with $n=\dim V \leqq 3$. Then there is a quasi-polarized variety (V', L') which is birationally equivalent to (V, L), has only $\boldsymbol{Q}$-factorial terminal singularities, and further satisfies one of the following conditions:*

1) *$\omega'+(n-1)L'$ is nef for the canonical sheaf ω' of V'.*

2) *$\Delta(V', L')=0$.*

3) *(V', L') is a scroll over a smooth curve.*

Here $\boldsymbol{Q}$-factorial means that every Weil divisor on V' is a rational multiple of a Cartier divisor.

Corollary. *$g(V, L)\geqq 0$ for any quasi-polarized variety of dimension $\leqq 3$. Moreover, $g=0$ implies $\Delta(V, L)=0$ if V is normal.*

Corollary. *Suppose further that $g(V, L)=1$ and V is normal. Then $\omega'=(1-n)L'$ or (V', L') is a scroll over an elliptic curve, where (V', L') is as in the theorem.*

These results will follow from the Flip conjecture in higher dimension too.

References

[1] T. Fujita: On polarized varieties of small Δ-genera. Tôhoku Math. J., **34**, 319–341 (1982).

[2] ——: Semipositive line bundles. J. Fac. Sci. Univ. of Tokyo, **30**, 353–378 (1983).

[3] ——: On polarized manifolds whose adjoint bundles are not semipositive. Algebraic Geometry Sendai 1985. Advanced Studies in Pure Math., Kinokuniya, **10**, 167–178 (1987).

[4] ——: On quasi-polarized varieties (in preparation).

[5] Y. Kawamata, K. Matsuda, and K. Matsuki: Introduction to the minimal model problem. Algebraic Geometry Sendai 1985. Advanced Studies in Pure Math., Kinokuniya, **10**, 283–360 (1987).

[6] S. Mori: Flip theorem and the existence of minimal models for threefolds (preprint).

[7] A. J. Sommese: On the adjunction theoretic structure of projective varieties. Complex Analysis and Algebraic Geometry. Lecture Notes in Math., vol. 1194, Springer, pp. 175–213 (1986).