13. *Shifted Divisor Problem and Random Divisor Problem*

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(Communicated by Shokichi IYANAGA, M. J. A., March 12, 1993)

In this note, we consider the following sum which coincides with the sum of the Dirichlet divisor problem (see [1], [4]) when $\alpha = \beta = 1$;

$$D(x\,;\,\alpha,\,\beta) = \sum_{\substack{(m+\alpha)(n+\beta)\leq x \\ m,n\in N\cup\{0\}}}{}' 1, (0 < \alpha \leq 1,\ 0 < \beta \leq 1) \tag{1}$$

where the dash on Σ means that we count $\frac{1}{2}$ in place of 1 when $(m + \alpha)\cdot(n + \beta) = x$ in the sum. $D(x\,;\,\alpha,\,\beta)$ is the number of lattice points inside the region: $\{(u,\,v) \in \boldsymbol{R}^2 : 0 < u,\ 0 < v,\ uv \leq x\}$ where the lattice points are shifted by α in u-direction and by β in v-direction from their original sites.

About this sum, we obtain the following theorems of which detailed proofs will be given elsewhere together with related results. We only give here outlines of the proofs. First, we have the following Voronoï-type identity:

Theorem 1 (Voronoï-type identity).

$$D(x\,;\,\alpha,\,\beta) = x\log x + \left\{\left(-\frac{\Gamma'}{\Gamma}(\alpha)\right) + \left(-\frac{\Gamma'}{\Gamma}(\beta)\right) - 1\right\}x + \left(\frac{1}{2} - \alpha\right)\left(\frac{1}{2} - \beta\right) + \Delta(x\,;\,\alpha,\,\beta), \tag{2}$$

$$\begin{aligned}\Delta(x\,;\,\alpha,\,\beta) = &-\frac{\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_{\alpha,\beta}(n) + d_{-\alpha,-\beta}(n)}{\sqrt{n}}Y_1(4\pi\sqrt{nx}) - \\ &-\frac{\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{\alpha,-\beta}(n) + d_{-\alpha,\beta}(n)}{\sqrt{n}}K_1(4\pi\sqrt{nx}) - \\ &-i\frac{\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_{\alpha,-\beta}(n) - d_{-\alpha,\beta}(n)}{\sqrt{n}}J_1(4\pi\sqrt{nx}),\end{aligned} \tag{3}$$

where $\Gamma(\cdots)$ *denotes the gamma-function and* $Y_1(\cdots)$, $K_1(\cdots)$, $J_1(\cdots)$ *denote the Bessel functions in ordinary sense*:

$$\begin{aligned}Y_1(x) &= \frac{1}{\pi}\left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\left(-\frac{x^2}{4}\right)^k\cdot \\ &\quad\cdot\left\{2\log\left(\frac{x}{2}\right) - \psi(k+1) - \psi(k+2)\right\} - \frac{2}{\pi x}, \\ K_1(x) &= \left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\left(\frac{x^2}{4}\right)^k\cdot \\ &\quad\cdot\left\{\log\left(\frac{x}{2}\right) - \frac{1}{2}\psi(k+1) - \frac{1}{2}\psi(k+2)\right\} + \frac{1}{x},\end{aligned}$$

$$\left(\psi(x) = \frac{\Gamma'}{\Gamma}(x)\right),$$

$$J_1(x) = \left(\frac{x}{2}\right) \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left(-\frac{x^2}{4}\right)^k$$

(*see* [9]) *and*

$$d_{\alpha,\beta}(n) = \sum_{\substack{l,m \in N \\ lm=n}} e^{2\pi i(\alpha l + \beta m)}. \tag{4}$$

And the right hand side of $\Delta(x\,;\alpha,\beta)$ above converges boundedly in any finite range of x.

Second we have the truncated form of Theorem 1.

Theorem 2 (Truncated Voronoï-type identity).

$$\begin{aligned} \Delta(x\,;\alpha,\beta) = &-\frac{\sqrt{x}}{2} \sum_{n \le N} \frac{d_{\alpha,\beta}(n) + d_{-\alpha,-\beta}(n)}{\sqrt{n}} Y_1(4\pi\sqrt{nx}) - \\ &- \frac{\sqrt{x}}{\pi} \sum_{n \le N} \frac{d_{\alpha,-\beta}(n) + d_{-\alpha,\beta}(n)}{\sqrt{n}} K_1(4\pi\sqrt{nx}) - \\ &- i\frac{\sqrt{x}}{2} \sum_{n \le N} \frac{d_{\alpha,-\beta}(n) - d_{-\alpha,\beta}(n)}{\sqrt{n}} J_1(4\pi\sqrt{nx}) +. \\ &+ O(x^{\varepsilon}) + O(x^{\frac{1}{2}+\varepsilon} N^{-\frac{1}{2}}) \ \textit{for} \ \forall\, \varepsilon > 0. \end{aligned} \tag{5}$$

We give the outlines of proofs of Theorems 1 and 2. By Perron's summation formula we have

$$D(x\,;\alpha,\beta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s,\alpha)\zeta(s,\beta) \frac{x^s}{s}\, ds,\ (c > 1), \tag{6}$$

where $\zeta(s,\alpha)$ is the Hurwitz zeta-function: $\zeta(s,\alpha) = \sum_{n=0}^{\infty} \dfrac{1}{(n+\alpha)^s}$, ($\Re s > 1$) (see [9]), and moving the path of integration in (6) from $s = c + it$, $-\infty < t < +\infty$, to the path $s = -a + it$, $-\infty < t < +\infty$, $a > 0$, we have

$$\begin{aligned} D(x\,;\alpha,\beta) = x\log x + \left\{\left(-\frac{\Gamma'}{\Gamma}(\alpha)\right) + \left(-\frac{\Gamma'}{\Gamma}(\beta)\right) - 1\right\} x + \\ + \zeta(0,\alpha)\zeta(0,\beta) + \Delta(x\,;\alpha,\beta) \end{aligned} \tag{7}$$

where

$$\Delta(x\,;\alpha,\beta) = \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} \zeta(s,\alpha)\zeta(s,\beta) \frac{x^s}{s}\, ds,\ (a > 0). \tag{8}$$

Applying the functional equation of $\zeta(s,\alpha)$:

$$\zeta(1-s,\alpha) = \frac{\Gamma(s)}{(2\pi)^s} \{e^{-\frac{\pi}{2}is}\zeta_\alpha(s) + e^{\frac{\pi}{2}is}\zeta_{-\alpha}(s)\},$$

where $\zeta_\alpha(s)$ is Lerch's zeta-function: $\zeta_\alpha(s) = \sum_{n=1}^{\infty} \dfrac{e^{2\pi i n\alpha}}{n^s}$, ($\Re s > 1$) (see [9]) to (8), we have

$$\begin{aligned} \Delta(x\,;\alpha,\beta) &= \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} \left\{\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\right\}^2 \Big\{e^{-\pi i(1-s)}\zeta_\alpha(1-s)\zeta_\beta(1-s) + \\ &\quad + \zeta_{-\alpha}(1-s)\zeta_\beta(1-s) + \zeta_\alpha(1-s)\zeta_{-\beta}(1-s) + \\ &\quad + e^{\pi i(1-s)}\zeta_{-\alpha}(1-s)\zeta_{-\beta}(1-s)\Big\} \frac{x^s}{s}\, ds \\ &= I_1 + I_2 + I_3 + I_4, \quad \text{say}, \end{aligned} \tag{9}$$

where

(10) $$I_1 = 2\pi i x \sum_{n=1}^{\infty} d_{\alpha,\beta}(n) \frac{1}{8\pi^2} \int_{2+2a-i\infty}^{2+2a+i\infty} \cos\left(\frac{\pi}{2}s\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-1\right) \cdot \cdot \left(\frac{4\pi\sqrt{nx}}{2}\right)^{-s} ds -$$

$$- 2\pi x \sum_{n=1}^{\infty} d_{\alpha,\beta}(n) \frac{1}{8\pi^2} \int_{2+2\alpha-i\infty}^{2+2a+i\infty} \sin\left(\frac{\pi}{2}s\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-1\right) \left(\frac{4\pi\sqrt{nx}}{2}\right)^{-s} ds,$$

$I_2 = \cdots, \cdots,$ etc,

following the method of [1] and we obtain Theorem 1.

As for the proof of Theorem 2, we only start from the truncated form of the formula in place of Perron's summation formula to find easy to treat without the difficulty of convergence.

Applying the asymptotic expansion of Y_1, K_1 and J_1:

$$Y_1(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{2} - \frac{\pi}{4}\right) + O(x^{-\frac{3}{2}}),$$

$$K_1(x) = \sqrt{\frac{\pi}{2x}} e^{-x}(1 + O(x^{-1})),$$

$$J_1(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2} - \frac{\pi}{4}\right) + O(x^{-\frac{3}{2}})$$

to the above identity (5), we have, with the choice of $N = x^{\frac{1}{3}}$,

Corollary 1 (Voronoï-type estimate).

(11) $$\Delta(x\,;\alpha,\beta) \ll x^{\frac{1}{3}+\varepsilon} \textit{ for } \forall\, \varepsilon > 0.$$

Substituting $\Delta(x\,;\alpha,\beta)$ with (3) or (5) in $\int_0^X |\Delta(x\,;\alpha,\beta)|^2 dx$, we have

Theorem 3 (Hardy-type estimate).

(12) $$\Delta(x\,;\alpha,\beta) = \Omega(x^{\frac{1}{4}}).$$

Next, we regard α, β as the random variables with uniform distribution and express $\Delta(x\,;\alpha,\beta)$ by its Fourier series using the fact $\Delta(x\,;\alpha \pm 1, \beta \pm 1) = \Delta(x\,;\alpha,\beta)$.

By Parseval's identity, we have the mean-value and the variance of $\Delta(x\,;\alpha,\beta)$.

Theorem 4 (Kendall-type estimate).

(13) $$E[\Delta(x\,;\alpha,\beta)] = \int_0^1\int_0^1 \Delta(x\,;\alpha,\beta)\,d\alpha d\beta = 0,$$

(14) $$V[\Delta(x\,;\alpha,\beta)] = \int_0^1\int_0^1 |\Delta(x\,;\alpha,\beta)|^2 d\alpha d\beta = \\ = \frac{x}{2}\left\{\sum_{n=1}^{\infty}\frac{d(n)}{n} Y_1(4\pi\sqrt{nx})^2 + \sum_{n=1}^{\infty}\frac{d(n)}{n} J_1(4\pi\sqrt{nx})^2 + \right. \\ \left. + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{d(n)}{n} K_1(4\pi\sqrt{nx})^2\right\} \\ \ll x^{\frac{1}{2}},$$

where $d(n)$ is the divisor function of n.

Lastly we have

Theorem 5. *For any small $\varepsilon > 0$ and any $C > 0$, there exists $x(\varepsilon, C) > 0$ such that*

(15) $\mu\{(\alpha, \beta) \in (0, 1]^2 : |\Delta(x; \alpha, \beta)| \leq Cx^{\frac{1}{4}} \log x\} \geq 1 - \varepsilon$ *for* $\forall x \geq x(\varepsilon, C)$,

where $\mu\{\cdots\}$ *denotes the Lebesgue measure.*

Theorems 4 and 5 give us an answer for the Dirichlet divisor problem in a statistical sense.

Acknowledgements. The author is indebted to Professors Yukinao Isokawa, Masanori Katsurada, Yukio Ohkubo and Makoto Ishibashi in Kagoshima for valuable discussions and especially to Professor Isokawa who informed the author of the exsistence of the works of D. G. Kendall and R. A. Rankin [5, 6, 7].

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