A base point free theorem of Reid type, II

By Shigetaka FUKUDA

Faculty of Education, Gifu Shotoku Gakuen University, Yanaizu-cho, Hashima-gun, Gifu 501-6194

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Abstract: Let X be a complete algebraic variety over $\mathbf{C}$. We consider a log variety (X, Δ) that is weakly Kawamata log terminal. We assume that $K_X + \Delta$ is a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor and that every irreducible component of $\lfloor \Delta \rfloor$ is $\mathbf{Q}$-Cartier. A nef and big $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor H on X is called *nef and log big* on (X, Δ) if $H|_B$ is nef and big for every center B of non-"Kawamata log terminal" singularities for (X, Δ). We prove that, if L is a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and log big on (X, Δ) for some $a \in \mathbf{N}$, then the complete linear system $|mL|$ is base point free for $m \gg 0$.

This paper is a continuation of [4].

We generally use the notation and terminology of [13].

Let X be a normal, complete algebraic variety over $\mathbf{C}$ and (X, Δ) a log variety that is log canonical. We assume that $K_X + \Delta$ is a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor. Let r be the smallest positive integer such that $r(K_X + \Delta)$ is Cartier (r is called the *singularity index* of (X, Δ)).

Definition (due to Reid [15]). Let $\Theta = \sum_{i=1}^{s} \Theta_i$ be a reduced divisor with only simple normal crossings on an n-dimensional non-singular complete variety over $\mathbf{C}$. We denote **Strata** $(\Theta) := \{\Gamma | 1 \leq k \leq n, 1 \leq i_1 < i_2 < \cdots < i_k \leq s, \quad \Gamma$ is an irreducible component of $\Theta_{i_1} \cap \Theta_{i_2} \cap \cdots \cap \Theta_{i_k} \neq \emptyset\}$. Let $f : Y \to X$ be a log resolution of (X, Δ) such that $K_Y = f^*(K_X + \Delta) + \sum a_j E_j$ (where $a_j \geq -1$). Let L be a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on X. L is called *nef and log big* on (X, Δ) if L is nef and big and $(L|_{f(\Gamma)})^{dim f(\Gamma)} > 0$ for any member Γ of $\mathbf{Strata}(\sum_{a_j=-1} E_j)$.

Remark. The set $\{f(\Gamma) | \Gamma \in \mathbf{Strata}\ (\sum_{a_j=-1} E_j)\}$ is the set of all the centers of log canonical (non-Kawamata log terminal) singularities $\mathrm{CLC}(X, \Delta)$ ([10, Definition 1.3]). Thus L is nef and log big on (X, Δ) if and only if L is nef and big and $(L|_B)^{dim B} > 0$ for any $B \in \mathrm{CLC}(X, \Delta)$. Therefore the definition of the notion of "nef and log big" does not depend on the choice of the log resolution f.

Remark. In the case in which (X, Δ) is Kawamata log terminal (klt), if L is nef and big, then L is nef and log big on (X, Δ).

In [15], M. Reid gave the following statement:

Let L be a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and log big on (X, Δ) for some $a \in \mathbf{N}$. Then $Bs|mL| = \emptyset$ for every $m \gg 0$.

In the case in which (X, Δ) is klt, this is the standard Kawamata-Shokurov result (cf. [8, Theorem 2.6]). While in the case in which (X, Δ) is weakly Kawamata log terminal, the proof in [4] of the statement needs the log minimal model program, which is still a conjecture in dimension ≥ 4 (the assumption that X is projective in [4] is not necessary. it suffices to assume that X is complete). On the other hand, the statement was proved when X is non-singular and Δ is a reduced divisor with only simple normal crossings in [2] and when dim $X = 2$ in [5]. We note that, if $aL - (K_X + \Delta)$ is nef and big but not nef and log big on (X, Δ), there exists a counterexample due to Zariski (cf. [11, Remark 3-1-2]).

We shall prove the following result in this paper:

Main theorem. *Assume that (X, Δ) is weakly Kawamata log terminal (wklt) and that every irreducible component of $\lfloor \Delta \rfloor$ is $\mathbf{Q}$-Cartier. Let L be a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and log big on (X, Δ) for some $a \in \mathbf{N}$. Then $Bs|mL| = \emptyset$ for every $m \gg 0$.*

This implies a kind of "log abundance theorem":

Corollary. *If (X, Δ) has only $\mathbf{Q}$-factorial weakly Kawamata log terminal singularities and $K_X + \Delta$ is nef and log big on (X, Δ), then $Bs|mr(K_X + \Delta)| = \emptyset$ for every $m \gg 0$.*

We note that, concerning the log abundance conjecture, the following facts are known:

(1) If (X, Δ) is klt and $K_X + \Delta$ is nef and big,

then $\mathrm{Bs}|mr(K_X+\Delta)|=\emptyset$ for every $m \gg 0$ (cf. [8, Theorem 2.6]).

(2) If $\dim X \le 3$ and $K_X+\Delta$ is nef, then $K_X+\Delta$ is semi-ample ([7], [1], [9], [13, 8.4], [12]).

1. Preliminaries. We collect some results that will be needed in the next section.

Proposition 1 ([17])**.** *(X,Δ) is wklt if and only if it is divisorial log terminal.*

Proposition 2 (Shokurov's Connectedness Lemma, [6, Lemma 2.2], [10, Theorem 1.4], cf. [16, 5.7], [13, 17.4])**.** *Let W be a normal, complete algebraic variety over $\mathbf{C}$, (W,Γ) a log variety that is log canonical and $g: V \to W$ a log resolution of (W,Γ). Then (the support of the effective part of $\lfloor(g^*(K_W+\Gamma)-K_V)\rfloor)\cap g^{-1}(s)$ is connected for every $s \in W$.*

Proposition 3 (Reid Type Vanishing, cf. [3], [4, Proposition 1])**.** *Assume that (X,Δ) is wklt. Let D be a $\mathbf{Q}$-Cartier integral Weil divisor. If $D-(K_X+\Delta)$ is nef and log big on (X,Δ), then $H^i(X,\mathcal{O}_X(D))=0$ for every $i>0$.*

Proposition 4 (cf. [9, the proof of Lemma 3], [8, Theorem 2.6])**.** *Assume that (X,Δ) is wklt. Let L be a nef Cartier divisor such that $aL-(K_X+\Delta)$ is nef and big for some $a \in \mathbf{N}$. If $Bs|mL| \cap \lfloor\Delta\rfloor=\emptyset$ for every $m \gg 0$, then $Bs|mL|=\emptyset$ for every $m \gg 0$.*

2. Proof of main theorem. We proceed along the lines of that in [4].

Let S be an irreducible component of $\lfloor\Delta\rfloor$. From [13, 17.5] (cf. [16, 3.8]), S is normal.

Let $f: Y \to X$ be a log resolution of (X,Δ) such that the following conditions are satisfied:

(1) $\mathrm{Exc}(f)$ consists of divisors,

(2) $K_Y + f_*^{-1}\Delta + F = f^*(K_X+\Delta)+E$,

(3) E and F are f-exceptional effective $\mathbf{Q}$-divisors such that $\mathrm{Supp}(E)$ and $\mathrm{Supp}(F)$ do not have common irreducible components,

(4) $\lfloor F\rfloor = 0$.

Claim 1 ([16, p.99])**.** For any member $G \in$ **Strata** $(f_*^{-1}\lfloor\Delta\rfloor)$, $\mathrm{Exc}(f)$ does not include G.

We put $S_0 := f_*^{-1}S$ and $\mathrm{Diff}(\Delta-S) := (f|_{S_0})_*(f^*(K_X+\Delta)|_{S_0}-(K_Y+S_0)|_{S_0})$.

We note that $(K_X+\Delta)|_S = K_S+\mathrm{Diff}(\Delta-S)$.

Claim 2 (cf. [14, Corollary 5.62])**.** $(S,\mathrm{Diff}(\Delta-S))$ is wklt and every irreducible component of $\lfloor\mathrm{Diff}(\Delta-S)\rfloor$ is $\mathbf{Q}$-Cartier.

Remark. After completing the original version (math/9801113) of this paper, the author was informed that Claim 2 was in Kollár-Mori's book ([14, Corollary 5.62]). But there they prove it in the case in which $(K_X+S)|_S = K_S$. So we give a proof to the claim, for the convenience of the reader.

Proof of Claim 2. By the Subadjunction Lemma ([16, 3.2.2], cf. [11, Lemma 5-1-9]), $\mathrm{Diff}(\Delta-S) \ge 0$. Here $\lfloor\mathrm{Diff}(\Delta-S)\rfloor = (f|_{S_0})_*((f_*^{-1}\lfloor\Delta-S\rfloor)|_{S_0})$. From [16, 3.2.3] and Proposition 1 or from [9, Lemma 4], $(S,\mathrm{Diff}(\Delta-S))$ is wklt.

Let D be an irreducible component of $\lfloor\Delta-S\rfloor$. For $x \in D\cap S$, there exist $y_1 \in f_*^{-1}D$ and $y_2 \in S_0$ such that $f(y_1)=f(y_2)=x$. Applying Proposition 2 to $(X,\{\Delta\}+S+D)$ and f, we obtain $y_3 \in f_*^{-1}D\cap S_0$ such that $f(y_3)=x$. Thus $(f|_{S_0})_*(f_*^{-1}D|_{S_0}) = \mathrm{Supp}\,(D|_S)$ by Claim 1.

We put $\mathrm{Diff}(\{\Delta\}+D) := (f|_{S_0})_*(f^*(K_X+S+\{\Delta\}+D)|_{S_0}-(K_Y+S_0)|_{S_0})$.

We note that $(K_X+S+\{\Delta\}+D)|_S = K_S+\mathrm{Diff}(\{\Delta\}+D)$.

Here $\mathrm{Diff}(\{\Delta\}+D) \ge 0$ from [16, 3.2.2] (cf. [11, Lemma 5-1-9]) and (the effective part of $\lfloor((f|_{S_0})^*\,(K_S+\mathrm{Diff}(\{\Delta\}+D))-K_{S_0})\rfloor) = f_*^{-1}D|_{S_0}$. Applying Proposition 2 to $(S,\,\mathrm{Diff}(\{\Delta\}+D))$ and $f|_{S_0}$, from the fact that every connected component of $f_*^{-1}D|_{S_0}$ is irreducible, we deduce that every connected component of $(f|_{S_0})_*(f_*^{-1}D|_{S_0})$ is irreducible. As a result every irreducible component of $(f|_{S_0})_*(f_*^{-1}D|_{S_0})$ is $\mathbf{Q}$-Cartier, because $D|_S$ is $\mathbf{Q}$-Cartier. □

Claim 3. $aL|_S-(K_S+\mathrm{Diff}(\Delta-S))$ is nef and log big on $(S,\mathrm{Diff}(\Delta-S))$.

Proof of Claim 3. The assertion follows from the fact that (the effective part of $\lfloor((f|_{S_0})^*\,(K_S+\mathrm{Diff}\,(\Delta-S))-K_{S_0})\rfloor) = (f_*^{-1}\,\lfloor\Delta-S\rfloor)|_{S_0}$. □

Claim 4. $|mL|\,\big|_S = \big|mL|_S\big|$ for $m \ge a$.

Proof of Claim 4. We note that $mL-S$ is a $\mathbf{Q}$-Cartier integral divisor, $(X,\Delta-S)$ is wklt and $mL-S-(K_X+\Delta-S)$ is nef and log big on $(X,\Delta-S)$. Thus $H^1(X,\mathcal{O}_X(mL-S))=0$ from Proposition 3. □

We complete the proof of the theorem by induction on $\dim X$.

By Claim 2 and Claim 3 and by induction hypothesis, $\mathrm{Bs}\big|mL|_S\big|=\emptyset$ for $m \gg 0$. Thus, by Claim 4, $\mathrm{Bs}|mL|\cap\lfloor\Delta\rfloor=\emptyset$ for every $m \gg 0$. Consequently Proposition 4 implies the assertion.

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