Smoothing effect and exponential time decay of solutions of Schrödinger equations

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Abstract: We seek a sufficient condition on initial data in order that the solution of Schrödinger equation has both analytically smoothing effect and exponential decay in time.

Key words: Exponential decay; Schrödinger equation; smoothing effect.

1. Introduction. We shall construct the solutions to the Cauchy problem for Schrödinger equation which have the properties both of the analytically smoothing effect and of the exponential time decay, if the initial data belong to the image of Sobolev spaces by a Fourier integral operator.

We consider the following Cauchy problem,

(1.1)
$$\begin{cases} \partial_t u(t,x) = i\triangle u(t,x) & \text{for} \quad t \in (0,\infty),\ x \in \mathbf{R}^n, \\ u(0,x) = u_0(x) & \text{for} \quad x \in \mathbf{R}^n, \end{cases}$$

where $i = \sqrt{-1}$, $\partial_t = (\partial/\partial t)$ and $\triangle$ is Laplacian in $\mathbf{R}^n$ which is defined by

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

In this paper we investigate the sufficient conditions of u_0 in order that the solutions u of (1.1) are real analytic with respect to the space variable x and decay exponentially with respect to the time variable t, that is, for any compact set K in $\mathbf{R}^n$ there are positive constants ε and C such that the solution u of (1.1) satisfies

$$\sup_{x\in K} |D_x^\alpha u(t,x)| \leqq C(\rho|t|)^{-|\alpha|}|\alpha|!e^{-\varepsilon t}$$

for any compact set K in $\mathbf{R}^n$ and $\alpha \in \mathbf{Z}_+^n = (\mathbf{N}\bigcup\{0\})^n$. As we can see in Rauch [2] the solution of (1.1) does not necessarily decay exponentially, even if its initial datum $u_0(x)$ decays exponentially. For example, for $u_0(x) = e^{-|x|^2}$ we have the solution of (1.1)

$$u(t,x) = \frac{1}{\sqrt{1+it}^n} e^{-\frac{|x|^2}{4(1+it)}},$$

which has no exponential decay. It is well known that if $u_0 \in L^1(\mathbf{R}^n)$, then u is given by the integral

$$u(t,x) = ct^{-\frac{n}{2}} \int e^{\frac{i|x-y|^2}{4t}} u_0(y)dy$$

and it follows that u decays as

$$|u(t,x)| \leqq Ct^{-\frac{n}{2}} \|u_0\|_{L^1(\mathbf{R}^n)}.$$

However the solution $u(t,x)$ decays exponentially in t, if u_0 belongs to the image of Sobolev spaces by a Fourier integral operator below. For $\mu \in \mathbf{C}$, denote

$$\varphi(x,\xi) = x\left(\xi - \frac{i\mu\xi}{|\xi|}\right)$$

and define

$$I_\varphi(x,D)u(x) = \frac{1}{\sqrt{2\pi}^n} \int e^{i\varphi(x,\xi)} \hat{u}(\xi)d\xi,$$

where $\hat{u}$ stands for the Fourier transform of u. We can prove the following result.

Theorem. *Let* $\psi \in H^{[(n/2)]+1}(\mathbf{R}^n)$ *and* $\varphi(x,\xi) = x(\xi - (i\mu\xi/|\xi|))$. *If* $u_0 = I_\varphi(x,D)\psi$ *and* $\operatorname{Re}\mu > 0$, $\operatorname{Im}\mu > 0$, *then for any* $\delta > 0$ *there are* $C > 0$ *such that the solution* u *of* (1.1) *satisfies*

(1.2)
$$|D_x^\alpha u(t,x)| \leqq C\Big(|\mu| + \frac{1}{2\operatorname{Re}\mu t}\Big)^{|\alpha|} |\alpha|!||\psi||_s e^{-2\operatorname{Re}\mu \operatorname{Im}\mu\, t + (\operatorname{Re}\mu+\delta)\langle x\rangle},$$

for $t > 0, x \in \mathbf{R}^n$ *and* $\alpha \in \mathbf{Z}_+^n = (\mathbf{N}\bigcup\{0\})^n$.

We remark that in [1] it is showed there is a solution of (1.1) having the analytically smoothing effect, if the initial value u_0 decays exponential with respect to space variables. The above Theorem says that the analytically smoothing effect of solutions of Schrödinger equations occurs, even if the initial data do not decay exponentially. As an example we can

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give a function $\psi(x) = e^{-|x|^2}$. If $n = 1$, we can calculate $u_0(x) = I_\varphi(x, D)\psi(x)$ as follows:

$$u_0(x) = \frac{e^{-x^2}}{2}\left\{e^{\mu x} + e^{-\mu x} + \frac{i\int_0^x e^{\tau^2}\,d\tau}{\sqrt{2\pi}}(e^{\mu x} - e^{-\mu x})\right\},$$

which increases exponentially.

2. Notation. For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$, an inner product and a norm of vectors denote by

$$x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n,$$
$$|x| = \sqrt{x \cdot x}$$

respectively and we also use $\langle x\rangle = \sqrt{1 + |x|^2}$. Throughout this paper,

$$\tilde{\xi} := \frac{\xi}{|\xi|} \quad \text{for} \quad \xi \neq 0.$$

We define the Fourier transform of $u \in L^2(\mathbf{R}^n)$ as follows:

$$\hat{u}(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} u(x)\, đx,$$

where $đx = (2\pi)^{-n/2}dx$. Under this notation, the inverse Fourier transform can be wrriten as:

$$u(x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi}\hat{u}(\xi)\, đ\xi.$$

For a set Ω in $\mathbf{R}^n$ $L^2(\Omega)$ denotes a set of functions u satisfying

$$\int_\Omega |u(x)|^2\, đx < \infty$$

and $\|u\|_{L^2(\Omega)}$ denotes the norm of $L^2(\Omega)$ which is defined the above quantity. We also use $L^2 = L^2(\mathbf{R}^n)$ and $\|u\| = \|u\|_{L^2(\mathbf{R}^n)}$ for simplicity.

For a real number s define $\|u\|_s = \|\langle\xi\rangle^s\hat{u}(\xi)\|$ and H^s is the set of functions whose Sobolev norms $\|\cdot\|_s$ are finite.

Let $m \in \mathbf{R}$, $1 \geq \rho \geq \delta$ and $\delta \neq 1$. $S^m_{\rho,\delta}$ means the symbol class of pseudo-differential operators which is defined by

$$S^m_{\rho,\delta} = \{p(x,\xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n);$$
$$|p^{(\alpha)}_{(\beta)}(x,\xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}, \forall\alpha,\beta \in \mathbf{Z}_\pm\}$$

where $p^{(\alpha)}_{(\beta)}(x,\xi) = \partial^\alpha_\xi D^\beta_x p(x,\xi)$ and

$$\partial^\alpha_\xi = \partial^{\alpha_1}_{\xi_1}\cdots\partial^{\alpha_n}_{\xi_n}$$
$$D^\beta_x = (-i)^{|\beta|}\partial^{\beta_1}_{x_1}\cdots\partial^{\beta_n}_{x_n},$$

for both $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{Z}^n_+ = (\mathbf{N}\bigcup\{0\})^n$. It is well known that for $p \in S^0_{\rho,\delta}$ the pseudo differential operator $p(x,D)$ of which symbol is $p(x,\xi)$ operates continuously in $L^2(\mathbf{R}^n)$.

3. Fourier integral operators. In this section, we shall define Fourier integral operators which are a modification of Fourier integral operators introduced in Kajitani [1] and investigate the properties of these tools, so that we prove the main theorem. To begin with, we give the definition of the function space on which Fourier integral operators act.

Definition 1. For a real number δ and a non negative number s, we define weighted Sobolev space as follows:

$$H^s_\delta = \{u \in L^2_{\mathrm{loc}}(\mathbf{R}^n);\ e^{\delta\langle x\rangle}u(x) \in H^s\}.$$

At the next stage, we define a Fourier integral operator I_φ, which is applied to transform the original equation (1.1) to another equation with respect to a new unknown function.

Definition 2. Let $\varphi(x,\xi) = x\cdot\xi - i\mu x\cdot\tilde{\xi}$ for a complex number $\mu \in \mathbf{C}$. For $v \in H^s$, we denote by I_φ as follows:

$$I_\varphi v(x) = \int_{\mathbf{R}^n} e^{i\varphi(x,\xi)}\hat{v}(\xi)\, đ\xi, \tag{3.1}$$

where $đ\xi = (2\pi)^{-n/2}d\xi$ and $\hat{v}$ stands for the Fourier transform of v.

Then, we obtain the following result.

Lemma 1. *Let s be a non-negative integer and $\varphi(x,\xi) = x\cdot\xi - i\mu x\cdot\tilde{\xi}$ for $\mu \in \mathbf{C}$. Then, I_φ operates continuously from H^s to $H^s_{-(|\mathrm{Re}\,\mu|+\delta)}$ for all $\delta > 0$ and satisfies that there is $C > 0$ such that*

$$\|I_\varphi v\|_{H^s_{-(|\mathrm{Re}\,\mu|+\delta)}} \leqq C\|v\|_s, \tag{3.2}$$

for any $v \in H^s$.

Proof. It suffices to show that (3.2) holds. In the first place, we have from the definition of I_φ,

$$\begin{aligned} &e^{-(|\mathrm{Re}\,\mu|+\delta)\langle x\rangle}I_\varphi v(x) \\ &= e^{-(|\mathrm{Re}\,\mu|+\delta)\langle x\rangle}\int_{\mathbf{R}^n} e^{ix\cdot\xi+\mu x\cdot\tilde{\xi}}\hat{v}(\xi)\, đ\xi \\ &= \int_{\mathbf{R}^n} e^{ix\cdot\xi}e^{\mu x\cdot\tilde{\xi}-(|\mathrm{Re}\,\mu|+\delta)\langle x\rangle}\hat{v}(\xi)\, đ\xi \\ &= \int_{\mathbf{R}^n} e^{ix\cdot\xi}p(x,\xi)\hat{v}(\xi)\, đ\xi, \end{aligned} \tag{3.3}$$

where we put $p(x,\xi) := e^{\mu x\cdot\tilde{\xi}-(|\mathrm{Re}\,\mu|+\delta)\langle x\rangle}$. We may denote the last integral by $Pv(x)$ using the corresponding capital letter $P = p(x,D)$ as pseudo-differential operators. Remember in mind for the further discussion that $p(x,\xi)$ decays exponentially with respect to the space variable $x \in \mathbf{R}^n$, that is,

$$|p(x,\xi)| = e^{(\operatorname{Re}\mu)x\cdot\tilde{\xi}-(|\operatorname{Re}\mu|+\delta)\langle x\rangle} \leqq e^{-\delta\langle x\rangle},$$

because the estimate $x\cdot\tilde{\xi} \leqq |x| \leqq \langle x\rangle$ holds for every $\xi \in \mathbf{R}^n \setminus \{0\}$. For the purpose of avoiding effects of the singularity at the origin $\xi = 0$, we should divide integral region $\mathbf{R}^n$ into two parts, namely the unit ball in $\mathbf{R}^n$ and the other. Let take $\chi_0 \in C_0^\infty(\mathbf{R}^n)$ such that $\operatorname{supp}\chi_0 \subset \{\xi \in \mathbf{R}^n;\ |\xi| \leqq 1\}$, $0 \leqq \chi_0(\xi) \leqq 1$ for all $\xi \in \mathbf{R}^n$, and $\chi_0(\xi) = 1$ in some neighborhood of $\xi = 0$. Then, we can divide p as follows:

$$\begin{aligned} p(x,\xi) = {} & p(x,\xi)\chi_0(\xi) + p(x,\xi)(1-\chi_0(\xi)) \\ =: {} & p_0(x,\xi) \quad + p_1(x,\xi). \end{aligned}$$

First, we shall examine $p_0(x,\xi)$ which is supported in a compact set in the neighborhood of $\xi = 0$. Here, we have

$$\begin{aligned} P_0v(x) &= \int_{\mathbf{R}^n} e^{ix\cdot\xi} p_0(x,\xi)\hat{v}(\xi)\, đ\xi \\ &= \int_{|\xi|\leqq 1} e^{ix\cdot\xi} p(x,\xi)\chi_0(\xi)\hat{v}(\xi)\, đ\xi. \end{aligned}$$

Hence,

$$\begin{aligned} &|D_x^\alpha P_0v(x)| \\ &\leqq \int_{|\xi|\leqq 1} \left|D_x^\alpha \left[e^{ix\cdot\xi}p(x,\xi)\right]\hat{v}(\xi)\right|\, đ\xi \\ &\leqq \sum_{\alpha'\leqq\alpha} \binom{\alpha}{\alpha'} \int_{|\xi|\leqq 1} |\xi^{\alpha-\alpha'} D_x^{\alpha'} p(x,\xi)\hat{v}(\xi)|\, đ\xi. \end{aligned}$$

Since

$$\begin{aligned} |D_x^{\alpha'} p(x,\xi)| &= \left|D_x^{\alpha'}\left[e^{\mu x\cdot\tilde{\xi}-(|\operatorname{Re}\mu|+\delta)\langle x\rangle}\right]\right| \\ &\leqq C_{\alpha'} e^{-\delta\langle x\rangle}, \end{aligned}$$

we obtain

$$\begin{aligned} |D_x^\alpha P_0v(x)| &\leqq C_\alpha e^{-\delta\langle x\rangle} \int_{|\xi|\leqq 1} 1\cdot|\hat{v}(\xi)|\, đ\xi \\ &\leqq \tilde{C}_\alpha e^{-\delta\langle x\rangle}\|v\| \end{aligned}$$

for $|\alpha| \leqq s$. Thus, it is proved that

$$\|P_0v\|_s \leqq C_0\|v\| \tag{3.4}$$

for some positive constant C_0. Next, we shall examine p_1. Recall that we discuss the integral (3.3) on the region except for the neighborhood of $\xi = 0$, then we have

$$\begin{aligned} \left|p_{1\,(\beta)}^{\,(\alpha)}(x,\xi)\right| &\leqq \tilde{C}_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{|\alpha|}e^{-\delta\langle x\rangle} \\ &\leqq C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}. \end{aligned}$$

Thus, p_1 belongs to the symbol class $S^0_{1,0}$. Therefore the boundedness theorem for pseudo-differential operators implies

$$\|P_1v\|_s \leqq C_1\|v\|_s. \tag{3.5}$$

Summing up (3.4) and (3.5), we obtain after all that

$$\|I_\varphi v\|_{H^s_{-(|\operatorname{Re}\mu|+\delta)}} = \|Pv\|_s \leqq C\|v\|_s$$

for some positive constant C, which implies (3.2). This completes the proof of Lemma 1. □

4. Proof of the main theorem. Let $\varphi(x,\xi) = x\cdot\xi - i\mu x\cdot\tilde{\xi}$. Transform the unknown function u to a new one v by $u = I_\varphi v$ in the original equation (1.1), then

$$\begin{aligned} i\triangle u &= i\triangle I_\varphi v \\ &= i\int_{\mathbf{R}^n} \triangle e^{ix\cdot(\xi-i\mu\tilde{\xi})}\hat{v}(\xi)\, đ\xi \\ &= -i\int_{\mathbf{R}^n} e^{i\varphi(x,\xi)} \sum_{j=1}^n (\xi_j - i\mu\tilde{\xi}_j)^2\hat{v}(\xi)\, đ\xi \\ &= -i\int_{\mathbf{R}^n} e^{i\varphi(x,\xi)}(|\xi| - i\mu)^2\hat{v}(\xi)\, đ\xi \\ &= -iI_\varphi(|D| - i\mu)^2 v(t,x). \end{aligned} \tag{4.1}$$

Hence we obtain the following equation.

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = -i\left(|D| - i\mu\right)^2 v, \\ v(0,x) = \psi(x). \end{cases}$$

We can solve the above equation as follows:

$$v(t,x) = \int e^{ix\xi - i(|\xi|-i\mu)^2 t}\hat{\psi}(\xi)\, đ\xi.$$

Hence if $\operatorname{Re}\mu > 0$ and $\operatorname{Im}\mu > 0$, we get

$$\begin{aligned} ||D_x^\alpha v(t,x)||_s &= ||e^{-i(|\xi|-i\mu)^2 t}\xi^\alpha\hat{\psi}(\xi)||_s \\ &\leqq |||\xi|^{|\alpha|}e^{-2\operatorname{Re}\mu|\xi|t - 2\operatorname{Re}\mu\operatorname{Im}\mu t}\hat{\psi}||_s \\ &\leqq \frac{|\alpha|!e^{-\operatorname{Re}\mu\operatorname{Im}\mu t}||\psi||_s}{(2\operatorname{Re}\mu t)^{|\alpha|}}, \end{aligned} \tag{4.2}$$

for any $\alpha \in \mathbf{Z}^n_+$. On the other hand we have

$$\begin{aligned} D_x^\alpha u(t,x) &= \int e^{ix(\xi-i\mu\tilde{\xi})}(\xi - i\mu\tilde{\xi})^\alpha\hat{v}(t,\xi)\, đ\xi \\ &= I_\phi(D - i\mu\tilde{D})^\alpha v(t,x). \end{aligned}$$

Therefore it follows from Lemma 1 and Sobolev's lemma that for any $\delta > 0$ and $s = [(n/2)] + 1$

$$
\begin{aligned}
&e^{-(\operatorname{Re}\mu+\delta)\langle x\rangle}|D_x^\alpha u(t,x)| \\
&\qquad \leqq C_s\|e^{-(\operatorname{Re}\mu+\delta)\langle x\rangle} D_x^\alpha u(t,x)\|_s \\
&\qquad = C_s\|D_x^\alpha u(t,x)\|_{H^s_{-(\operatorname{Re}\mu+\delta)}} \\
&\qquad = C_s\|I_\varphi(D - i\mu\tilde{D})^\alpha v(t,x)\|_{H^s_{-(\operatorname{Re}\mu+\delta)}} \\
&\qquad \leqq \tilde{C}_s\|(D - i\mu\tilde{D})^\alpha v(t,x)\|_s,
\end{aligned}
$$

which yields (1.2) together with (4.2). Thus we have completed the proof of Theorem.

References

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