Discreteness criteria and algebraic convergence theorem for subgroups in $PU(1,n;C)$

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Abstract: In this paper, we will study the discreteness criterion for non-elementary subgroups in $PU(1,n;C)$. Several discreteness criteria are obtained. As an application, the convergence theorem of discrete subgroups in $PU(1,n;C)$ is discussed.

Key words: Discreteness criterion; convergence theorem; subgroup in $PU(1,n;C)$.

1. Introduction. Throughout this paper, we will adopt the same notations and definitions as in [3, 11, 12, 13] such as H^n_C, $U(1,n;C)$, $PU(1,n;C)$, discrete groups, limit sets and so on. For example, a subgroup G in $PU(1,n;C)$ is called *non-elementary* if it contains two non-elliptic elements of infinite order with distinct fixed points. Otherwise G is called *elementary*. See [3, 7, 8, 11, 12, 13, 14, 17, 18] etc. for more details of complex hyperbolic space H^n_C.

In 1976, Jørgensen ([9]) proved a necessary condition for a non-elementary two generator subgroup of $SL(2,C)$ to be discrete, which is called Jørgensen's inequality. By using this inequality, Jørgensen discussed the discreteness criterion and proved that

Theorem J_1 ([9]). *A non-elementary subgroup G of $SL(2,C)$ is discrete if and only if all its two-generator subgroups are discrete.*

Theorem J_2 ([10]). *A non-elementary subgroup G of $SL(2,R)$ is discrete if and only if each one-generator subgroup of G is discrete.*

See [1, 5, 15, 16, 19, 20, 21] etc. for generalizations of Theorems J_1 and J_2 in n-dimensional hyperbolic space.

In complex hyperbolic space, Kamiya ([13]) proved that

Theorem *K*. *If G is a non-elementary finitely generated subgroup of $PU(1,n;C)$, then G is discrete if and only if $\langle f,g\rangle$ is discrete for any f and g in G.*

Dai etc. ([4]) generalized Theorem K as follows:

Theorem *DFN*. *If G is a non-elementary subgroup of $PU(1,n;C)$ with Condition A, then G is discrete if and only if $\langle f,g\rangle$ is discrete for any f and g in G.*

Here, G is said to satisfy *Condition A* if it has no sequence $\{g_j\}$ of distinct elements of finite order such that $\mathrm{Card}(\mathrm{fix}(g_j))=\infty$ and $g_j\to I$ as $j\to\infty$, where $\mathrm{fix}(g_j)=\{x\in\partial H^n_C\colon g_j(x)=x\}$.

As the first main aim of this paper, we will study the discreteness criterion further and prove.

Theorem 1.1. *Let $G\subset PU(1,n;C)$ be non-elementary. Then G is discrete if and only if $W(G)$ is discrete (i.e., finite) and all non-elementary subgroups generated by two loxodromic elements of G are discrete.*

Here

$$W(G)=\bigcap_{f\in h(G)} G_{\mathrm{fix}(f)},$$

where $h(G)$ is the set of all loxodromic elements in G and $G_{\mathrm{fix}(f)}=\{g\in G\colon \mathrm{fix}(f)\subset\mathrm{fix}(g)\}$.

Following Theorem 1.1, we have

Corollary 1.1. *If a non-elementary subgroup $G\subset PU(1,n;C)$ is not discrete, then either there exists a sequence in G consisting of elliptic elements such that it converges to I or there exists a two-generator subgroup of G which is non-elementary and non-discrete.*

Theorem 1.2. *Let $G\subset PU(1,n;C)$ be non-elementary and $\dim[M(G)]$ be even. Then G is discrete if and only if $W(G)$ is finite and each one-generator subgroup of G is discrete.*

Here $M(G)$ denotes the smallest G-invariant, totally geodesic sub-manifold (cf. [3]).

As the second main aim of this paper, by using Theorem 1.1, we will discuss the convergence theo-

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rem in $PU(1,n;C)$ and prove

Theorem 1.3. *Let G_0 be a non-elementary and discrete group of $PU(1,n;C)$. For all positive integers m, let ρ_m be an isomorphism of G_0 onto a discrete group $G_m \subset PU(1,n;C)$. Assume that*

$$\rho_m(g) \to \rho(g)\ (m \to \infty)\ \forall g \in G_0,\ \rho(g) \in PU(1,n;C).$$

Then the group $G = \{\rho(g) \colon g \in G_0\}$ is discrete and ρ is an isomorphism of G_0 onto G.

Remark 1.1. Theorems 1.1 is a generalization of Theorems K and DFN.

Remark 1.2. Corollary 1.1 is a generalization of Theorem 1.3 in [4].

Remark 1.3. Theorem 1.2 is a generalization of Theorem J_2 and Theorem 2 in [1] into the case of $PU(1,n;C)$.

Remark 1.4. Theorem 1.3 is a generalization of Theorem 1 in [9] and Theorem 1.6 in [20] into the case of $PU(1,n;C)$.

2. The proofs of the main results.

2.1. The proof of Theorem 1.1.

The following lemmas are crucial for us.

Lemma 2.1 ([4, 13]). *Suppose that f and $g \in PU(1,n;C)$ generate a discrete and non-elementary group. Then*

i) *if f is parabolic or loxodromic, we have*

$$\max\{N(f), N([f,g])\} \geq 2 - \sqrt{3},$$

where $[f,g] = fgf^{-1}g^{-1}$ is the commutator of f and g, $N(f) = \|f - I\|$.

ii) *if f is elliptic, we have*

$$\max\{N(f), N([f,g^i]) \colon i = 1,2,\dots,n+1\} \geq 2 - \sqrt{3}.$$

Lemma 2.2 ([2]). *If g is a loxodromic element in $PU(1,n;C)$ and $f \in PU(1,n;C)$ does not interchange the two fixed points of g, then for all large enough j, the elements $g^j f$ or $g^{-j} f$ are loxodromic.*

The proof of Theorem 1.1. The necessity is obvious. For the converse, we suppose that $W(G)$ is finite and each non-elementary subgroup generated by two loxodromic elements of G is discrete, but G itself is not discrete. Then there is a sequence $\{f_m\} \in G$ such that

$$f_m \to I \quad (m \to \infty).$$

Since G is non-elementary, there exist two loxodromic elements $g_j \in G$ $(j = 1,2)$ which have no common fixed point. Then for large enough m,

$$N(f_m) + \sum_{k=1}^{n+1} N([f_m, g_j^k]) < 2 - \sqrt{3} \quad (j = 1,2).$$

We may assume that for large enough m, f_m doesn't interchange the fixed points of g_j $(j = 1,2)$ since $f_m \to I$ $(m \to \infty)$. Therefore $\langle f_m, g_j \rangle$ $(j = 1,2)$ are elementary for large enough m by Lemma 2.1. Hence $\mathrm{fix}(g_j) \subset \mathrm{fix}(f_m)$ holds for each $j = 1,2$ and sufficiently large m. Let $T(k_1) = \bigcap_{m \geq k_1} \mathrm{fix}(f_m)$. Then $T(k_1)$ contains the linear span of the fixed points of g_j and so has dimension at least 1 for large positive integer k_1. Thus by passing to a subsequence of $\{f_m\}$ (denoted by the same manner), we have

$$T(k_1) \neq \emptyset \quad \text{and} \quad 1 \leq \dim[T(k_1)] \leq n-1.$$

Suppose that there exists some loxodromic element $g \in G$ such that

$$\mathrm{fix}(g) \cap T(k_1) = \emptyset.$$

Then (if needed, passing to a subsequence) there exists k_2 $(> k_1)$ such that

$$\mathrm{fix}(g) \subset T(k_2)$$

and

$$\dim[T(k_1)] + 1 \leq \dim[T(k_2)] \leq n-1.$$

By repeating this step finite times, we can find k such that

$$\mathrm{fix}(h) \subset T(k)$$

holds for any loxodromic element $h \in G$. Then $f_m \in W(G)$ for all $m > k$. This contradiction completes the proof. □

2.2. The proof of Theorem 1.2.

The following lemma takes an important role in the proof of Theorem 1.2. Its proof follows from Corollary 4.5.3 in [3].

Lemma 2.3. *Let $G \subset PU(1,n;F)$ be non-elementary and* $\dim[M(G)]$ *be even. If the identity is not an accumulation point of the elliptic elements in G, then G is discrete.*

Lemma 2.4. *Let $G_0 = G|_{M(G)} \subset PU(1,n;C)$ be non-elementary and $\dim[M(G)]$ be even. Then G_0 is discrete if and only if each one-generator subgroup of G_0 is discrete.*

Proof. The necessity is obvious. For the converse, we suppose that each one-generator subgroup of G_0 is discrete, but G_0 itself is not discrete. Apply Theorem 1.1 to get a two-generator subgroup $\langle f, g \rangle$

of G_0 which is non-elementary and non-discrete. By choosing finitely many elements $f_1, \cdots, f_r$, we can get a subgroup $G_1 = \langle f, g, f_1, \cdots, f_r\rangle$ of G_0 such that G_1 is non-elementary and non-discrete and $M(G_1) = M(G_0)$. Selberg's lemma tells us that G_1 contains a torsion-free subgroup G_2 with finite index. Then G_2 is also non-discrete and $M(G_2) = M(G_1)$. Since each one-generator subgroup of G_0 is discrete, we know that G_2 contains no elliptic element. It follows from $M(G_2) = M(G_0)$ and Lemma 2.3 that G_2 is discrete. This contradiction completes the proof. □

The following lemma is obvious (cf. [22]).

Lemma 2.5. *Let $G \subset PU(1, n; C)$ be non-elementary. Then G is discrete if and only if both $G|_{M(G)}$ and $W(G)$ are discrete.*

The proof of Theorem 1.2. Since $G|_{M(G)}$ is non-elementary and $dim[M(G)]$ is even, by Lemma 2.4, $G|_{M(G)}$ is discrete if and only if each one-generator subgroup of $G|_{M(G)}$ is discrete. The proof follows from Lemma 2.5. □

2.3. The proof of Theorem 1.3.

Lemma 2.6. *Let G be a non-elementary subgroup of $PU(1, n; C)$ and $\{\rho_m\}$ be a sequence of isomorphisms of G onto discrete subgroups $\rho_m(G) \subset PU(1, n; C)$. If*

$$\rho_m(g) \to g \quad (m \to \infty) \quad \forall g \in G,$$

then G is either discrete or $W(G)$ is infinite.

Proof. By Theorem 1.1, we only need to show that $\langle f, g\rangle$ is discrete for any two loxodromic elements f, g in G which have no common fixed points under the condition $W(G)$ is finite.

Suppose that there are two loxodromic elements $f, g \in G$ with no common fixed points such that the two-generator subgroup $\langle f, g\rangle$ is not discrete. It follows from Selberg's lemma that there exists a torsion free subgroup G_1 of $\langle f, g\rangle$ with finite index. Therefore G_1 is non-elementary and there is a sequence $\{h_j\}$ of G_1 such that $h_j \to I$ as $j \to \infty$. For any two loxodromic elements f_1, f_2 in G_1 which have no common fixed point, it follows by continuity that

$$N(\rho_m(h_j)) + \sum_{k=1}^{n+1} N([\rho_m(h_j), \rho_m(f_i^k)]) < 2 - \sqrt{3}, \qquad i = 1, 2$$

for large j and m. Thus, $\langle \rho_m(h_j), \rho_m(f_i)\rangle$ is elementary by Lemma 2.1 and the discreteness of $\rho_m(G)$. Thus

$$\text{fix}(f_i) \subset \text{fix}(h_j), \quad i = 1, 2$$

for large j. Hence we may assume that all h_j are elliptic and we can find an integer L such that

$$\text{fix}(q) \subset T_h(L)$$

for all loxodromic elements $q \in G_1$, where $T_h(L) = \bigcap_{m\ge L} \text{fix}(h_m)$.

Since the conditions $h_j \to I$ $(j \to \infty)$ and $\text{ord}(h_j) < m$ imply that $h_j = I$ for all large enough j, we may assume that there is a purely elliptic sequence $\{g_j\}$ of G such that for all j, $\text{ord}(g_j) = \infty$ and

$$g_j \to I \quad (j \to \infty).$$

For any loxodromic element $h \in G$, considering the two-generator group $\rho_m(\langle h, g_j\rangle) = \rho_m(\langle h^l g_j, h\rangle)$, we can find an integer M such that

$$\text{fix}(p) \subset T_g(M)$$

for all loxodromic elements $p \in G$.

It means that $g_j \in W(G)$ for $j > M$. The finiteness of $W(G)$ implies that there exists j_0 such that for all $j > j_0$, $g_j = I$. This is the desired contradiction. □

The proof of Theorem 1.3. We can prove that the map ρ is an isomorphism as the proof of Theorem 5.10 in [15].

Since G_0 is discrete and non-elementary, there are loxodromic elements $f, g \in G_0$ which have no common fixed point such that $\langle f, g\rangle$ is discrete, non-elementary and isomorphic to the free group of rank two. Similarly, we can show that $\langle \rho_m(f), \rho_m(g)\rangle$ is discrete and non-elementary by similar reasoning as that in [15], and then $\langle \rho(f), \rho(g)\rangle$ is non-elementary. Thus, G is non-elementary.

We claim that $W(G)$ is finite.

At first, we prove that every nontrivial element $\rho(h)$ in $W(G)$ is an element of finite order. Suppose that $ord(\rho(h)) = \infty$. Then $\langle \rho(h)\rangle$ is infinite and there is a sequence $\{\rho(h^k)\}$ of $\langle \rho(h)\rangle$ such that $\rho(h^k) \to I$ as $k \to \infty$. Hence $\rho(h^k) \to I$ as $k \to \infty$. We know $ord(h) = \infty$ since $ord(\rho(h)) = \infty$. It follows from the discreteness of G_0 that h is not elliptic. There is a loxodromic element $q \in G_0$ such that $\langle h, q\rangle$ is non-elementary and isomorphic to a free group of rank two. So is $\langle h^k, q\rangle$ for every integer k, and $\langle \rho_m(h^k), \rho_m(q)\rangle$ is discrete and non-elementary. By Lemma 2.1, we have

$$N(\rho_m(h^k)) + \sum_{l=1}^{n+1} N([\rho_m(h^k), \rho_m(q^l)]) > 2 - \sqrt{3}$$

for all m and k. This contradicts the facts $\rho_m(h^k) \to \rho(h^k)$ as $m \to \infty$ and $\rho(h^k) \to I$ as $k \to \infty$. Therefore, every nontrivial element $\rho(h) \in W(G)$ is an element of finite order.

By Gehring and Martin [6], $\rho^{-1}[W(G)] \subset G_0$ is finite. Thus $W(G)$ is finite.

Consider the sequence of isomorphisms $\psi_m : G \to G_m$ defined by

$$\psi_m(g) = \rho_m(\rho^{-1}(g)) \quad \forall g \in G, \ m \in \mathbf{N}.$$

Then $\psi_m(g) \to g$ as $m \to \infty$ for each $g \in G$. It follows from Lemma 2.6 that G is discrete. □

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References

[1] W. Abikoff and A. Haas, Nondiscrete groups of hyperbolic motions, Bull. London Math. Soc. **22** (1990), no. 3, 233–238.

[2] W. Cao and X. Wang, Geometric characterizations for subgroups of $PU(1, n; C)$, Northeast. Math. J. **21** (2005), no. 1, 45–53.

[3] S. S. Chen and L. Greenberg, Hyperbolic spaces, in *Contributions to analysis* (*a collection of papers dedicated to Lipman Bers*), Academic Press, New York, 1974, pp. 49–87.

[4] B. Dai, A. Fang and B. Nai, Discreteness criteria for subgroups in complex hyperbolic space, Proc. Japan Acad. Ser. A Math. Sci. **77** (2001), no. 10, 168–172.

[5] A. Fang and B. Nai, On the discreteness and convergence in n-dimensional Möbius groups, J. London Math. Soc. (2) **61** (2000), no. 3, 761–773.

[6] F. W. Gehring and G. J. Martin, Discrete quasiconformal groups. I, Proc. London Math. Soc. (3) **55** (1987), no. 2, 331–358.

[7] W. M. Goldman, *Complex hyperbolic geometry*, Oxford Univ. Press, New York, 1999.

[8] W. M. Goldman and J. R. Parker, Dirichlet polyhedra for dihedral groups acting on complex hyperbolic space, J. Geom. Anal. **2** (1992), no. 6, 517–554.

[9] T. Jørgensen, On discrete groups of Möbius transformations, Amer. J. Math. **98** (1976), no. 3, 739–749.

[10] T. Jørgensen and P. Klein, Algebraic convergence of finitely generated Kleinian groups, Quart. J. Math. Oxford Ser. (2) **33** (1982), no. 131, 325–332.

[11] S. Kamiya, Notes on elements of $U(1, n; C)$, Hiroshima Math. J. **21** (1991), no. 1, 23–45.

[12] S. Kamiya, Notes on some classical series associated with discrete subgroups of $U(1, n; C)$ on $\partial B^n \times \partial B^n \times \partial B^n$, Proc. Japan Acad. Ser. A Math. Sci. **68** (1992), no. 6, 137–139.

[13] S. Kamiya, Chordal and matrix norms of unitary transformations, First Korean-Japanese Colloquium on Finite or infinite dimensional complex analysis (eds. J. Kajiwara, H. Kazama and K. H. Shon), 1993, 121–125.

[14] S. Kamiya, On discrete subgroups of $PU(1, 2; C)$ with Heisenberg translations, J. London Math. Soc. (2) **62** (2000), no. 3, 827–842.

[15] G. J. Martin, On discrete Möbius groups in all dimensions: a generalization of Jørgensen's inequality, Acta Math. **163** (1989), no. 3-4, 253–289.

[16] G. J. Martin, On discrete isometry groups of negative curvature, Pacific J. Math. **160** (1993), no. 1, 109–127.

[17] J. R. Parker, On Ford isometric spheres in complex hyperbolic space, Math. Proc. Cambridge Philos. Soc. **115** (1994), no. 3, 501–512.

[18] J. R. Parker, Uniform discreteness and Heisenberg translations, Math. Z. **225** (1997), no. 3, 485–505.

[19] X. Wang and W. Yang, Discreteness criterions for subgroups in $SL(2, C)$, Math. Proc. Cambridge Philos. Soc. **124** (1998), no. 1, 51–55.

[20] X. Wang and W. Yang, Discreteness criteria of Möbius groups of high dimensions and convergence theorems of Kleinian groups, Adv. Math. **159** (2001), no. 1, 68–82.

[21] X. Wang, L. Li and W. Cao, Discreteness criteria for Möbius groups on $\bar{R}^n$, Israel J of Math. **150** (2005), 357–368.

[22] X. Wang, Dense subgroups of n-dimensional Möbius groups, Math. Z. **243** (2003), no. 4, 643–651.