Holomorphic curves with an infinite number of deficiencies

By Nobushige TODA[*)]

Professor Emeritus, Nagoya Institute of Technology

(Communicated by Shigefumi MORI, M. J. A., June 15, 2004)

Abstract: For each positive integer p, there exists a holomorphic curve of order p mean type with an infinite number of deficiencies, the sum of which to the a power is divergent, where $0 < a < 1/3$.

Key words: Holomorphic curve; deficiency.

1. Introduction. Let $f = [f_1, \dots, f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation $(f_1, \dots, f_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$, where n is a positive integer. We use the following notations:

$$\|f(z)\| = (|f_1(z)|^2 + \cdots + |f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a} = (a_1, \dots, a_{n+1}) \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$

$$||\boldsymbol{a}|| = (|a_1|^2 + \cdots + |a_{n+1}|^2)^{1/2},$$
$$(\boldsymbol{a}, f) = a_1 f_1 + \cdots + a_{n+1} f_{n+1},$$
$$(\boldsymbol{a}, f(z)) = a_1 f_1(z) + \cdots + a_{n+1} f_{n+1}(z).$$

The characteristic function of f is defined as follows (see [8]):

$$(1) \qquad T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log ||f(re^{i\theta})|| d\theta - \log ||f(0)||.$$

We suppose throughout the paper that f is transcendental; that is to say, $\lim_{r\to\infty} T(r, f)/\log r = \infty$ and f is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

For meromorphic functions in the complex plane we use the standard notation of Nevanlinna theory of meromorphic functions ([4, 5]).

For $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$, we write

$$m(r, \boldsymbol{a}, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{||\boldsymbol{a}|| ||f(re^{i\theta})||}{|(\boldsymbol{a}, f(re^{i\theta}))|} d\theta,$$

$$N(r, \boldsymbol{a}, f) = N(r, 1/(\boldsymbol{a}, f)).$$

We then have the first fundamental theorem:

$$(2) \qquad T(r, f) = m(r, \boldsymbol{a}, f) + N(r, \boldsymbol{a}, f) + O(1)$$

([8], p. 76). We call the quantity

$$\delta(\boldsymbol{a}, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \boldsymbol{a}, f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, \boldsymbol{a}, f)}{T(r, f)}$$

the deficiency (or defect) of $\boldsymbol{a}$ with respect to f. We have $0 \le \delta(\boldsymbol{a}, f) \le 1$ by (2).

Let X be a subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in N-subgeneral position; that is to say, $\#X \ge N + 1$ and any $N + 1$ elements of X generate $\boldsymbol{C}^{n+1}$, where N is an integer satisfying $N \ge n$. We say that X is in general position when X is in n-subgeneral position.

Cartan ([1], $N = n$) and Nochka ([6], $N > n$) gave the following:

Theorem A (Defect relation). *For any q elements $\boldsymbol{a}_j$ $(j = 1, \dots, q)$ of X,*

$$\sum\nolimits_{j=1}^{q} \delta(\boldsymbol{a}_j, f) \le 2N - n + 1,$$

where $2N - n + 1 \le q \le \infty$ (see also [2] or [3]).

Let Y be the set of $\boldsymbol{a} \in X$ satisfying $\delta(\boldsymbol{a}, f) > 0$. Then, as is well-known, Y is at most countable. When $n \ge 2$, it is not difficult to give holomorphic curves for which Y is finite, but it is not so easy to give those for which Y is infinite. It is of some interest to construct examples of holomorphic curves with an infinite number of deficiencies when $n \ge 2$.

The purpose of this paper is to prove the following theorem when $n \ge 2$ by applying the method given in Section 4.3 of [4].

Theorem. *For any positive integer p, there exists a holomorphic curve of order p mean type with an infinite number of deficiencies.*

2. Preliminary lemmas. In this section we prepare some lemmas for later use. Main idea of this section is given in Section 4.3 of [4]. Let $\{\eta_\nu\}$ be a decreasing sequence satisfying

2000 Mathematics Subject Classification. Primary 32H30; Secondary 30D35.

*) Present address: Chiyoda 3-16-15-302, Naka-ku, Nagoya, Aichi 460-0012.

(3) $\eta_\nu > 0$ and $\sum_{\nu=1}^{\infty} \eta_\nu = 1, \quad \eta_0 = \eta_1$

and put

(4) $\theta_0 = 0, \quad \theta_k = \pi \sum_{\nu=0}^{k-1} \eta_\nu \quad (k = 1, 2, 3, \ldots).$

Then, $\{\theta_k\}$ is strictly increasing and it tends to

$$\pi \sum_{\nu=0}^{\infty} \eta_\nu = \pi\eta_0 + \pi \sum_{\nu=1}^{\infty} \eta_\nu \leq 2\pi$$

as $k \to \infty$.

Lemma 1 ([4], p. 99)**.** *For $k \geq 1$ if*

(5) $$\theta_k - \frac{1}{3}\pi\eta_k < \theta \leq \theta_k + \frac{1}{3}\pi\eta_k$$

and $z = re^{i\theta}$, then

(a) $\cos(\theta_\nu - \theta) \leq \cos\left(\frac{2}{3}\pi\eta_k\right) \quad (\nu \neq k)$;

(b) $|\exp\{ze^{-i\theta_\nu}\}| \leq \exp\left\{r \cos \frac{2}{3}\pi\eta_k\right\} \ (\nu \neq k)$.

Proof. (a) This inequality is given in [4], p. 99.

(b) From (a) we have the inequality

$$\begin{aligned} |\exp\{ze^{-i\theta_\nu}\}| &= |\exp\{re^{i(\theta-\theta_\nu)}\}| \\ &= \exp\{r\cos(\theta - \theta_\nu)\} \\ &\leq \exp\left\{r \cos \frac{2}{3}\pi\eta_k\right\} \quad (\nu \neq k). \qquad \square \end{aligned}$$

Let m be any positive integer, $\{a_k\}$ an arbitrary sequence of complex numbers such that at least two of $\{a_k\}_{k \geq m}$ are not equal to zero and are distinct, $\{b_k\}$ a sequence of positive numbers satisfying

$$s_1 = \sum_{k=1}^{\infty} b_k |a_k| < \infty, \quad s_2 = \sum_{k=1}^{\infty} b_k < \infty,$$

and we put

$$u(z) = \sum_{k=1}^{\infty} b_k a_k \exp\{ze^{-i\theta_k}\},$$

$$v_m(z) = \sum_{k=m}^{\infty} b_k \exp\{ze^{-i\theta_k}\}$$

and $w_0(z) \equiv 0$,

$$w_{m-1}(z) = \sum_{k=1}^{m-1} \alpha_k \exp\{ze^{-i\theta_k}\} \quad (m \geq 2)$$

for any complex numbers α_k. Further we put

$$A_0 \equiv 0, \qquad A_{m-1} = \sum_{k=1}^{m-1} |\alpha_k| \quad (m \geq 2).$$

Proposition 1. *For $z = re^{i\theta}$,*

1) $|u(z)| \leq s_1 e^r$; 2) $|v_m(z)| \leq s_2 e^r$;

3) $|u(z) + w_{m-1}(z)| \leq (s_1 + A_{m-1})e^r$;

4) $|v_m(z) + w_{m-1}(z)| \leq (s_2 + A_{m-1})e^r$.

Proof. It is easy to see this proposition, since

$$\begin{aligned} |\exp\{ze^{-i\theta_k}\}| &= |\exp\{re^{i(\theta-\theta_k)}\}| \\ &= \exp\{r\cos(\theta - \theta_k)\} \leq e^r. \qquad \square \end{aligned}$$

Lemma 2 (see [4], p. 99)**.** *When θ satisfies (5), for $z = re^{i\theta}$ and $k \geq m$ we have the inequalities:*

(6) $$\begin{aligned} &|u(z) + w_{m-1}(z) - b_k a_k \exp\{ze^{-i\theta_k}\}| \\ &\qquad \leq (s_1 + A_{m-1}) \exp\left\{r \cos \frac{2}{3}\pi\eta_k\right\}, \end{aligned}$$

(7) $$|v_m(z) - b_k \exp\{ze^{-i\theta_k}\}| \leq s_2 \exp\left\{r \cos \frac{2}{3}\pi\eta_k\right\},$$

(8) $$\begin{aligned} &|v_m(z) + w_{m-1}(z) - b_k \exp\{ze^{-i\theta_k}\}| \\ &\qquad \leq (s_2 + A_{m-1}) \exp\left\{r \cos \tfrac{2}{3}\pi\eta_k\right\} \end{aligned}$$

and for all sufficiently large r

(9) $$\begin{aligned} &|u(z) + w_{m-1}(z)| \\ &\qquad \geq \frac{1}{2} b_k |a_k| \exp\left\{r \cos \frac{1}{3}\pi\eta_k\right\} \quad (\textit{if } a_k \neq 0), \end{aligned}$$

(10) $$|v_m(z)| \geq \frac{1}{2} b_k \exp\left\{r \cos \frac{1}{3}\pi\eta_k\right\},$$

(11) $$|v_m(z) + w_{m-1}(z)| \geq \frac{1}{2} b_k \exp\left\{r \cos \frac{1}{3}\pi\eta_k\right\}.$$

Proof. We can prove these inequalities as in [4], p. 99 by Lemma 1 (b). For example, we prove (6).

$$\begin{aligned} &|u(z) + w_{m-1}(z) - b_k a_k \exp\{ze^{-i\theta_k}\}| \\ &\quad \leq |w_{m-1}(z)| + \sum_{\nu \neq k} b_\nu |a_\nu \exp\{ze^{-i\theta_\nu}\}| \\ &\quad \leq \sum_{\nu=1}^{m-1} |\alpha_\nu \exp\{ze^{-i\theta_\nu}\}| \\ &\qquad + \sum_{\nu \neq k} b_\nu |a_\nu \exp\{ze^{-i\theta_\nu}\}| \\ &\quad \leq (A_{m-1} + s_1) \exp\left\{r \cos \frac{2}{3}\pi\eta_k\right\}. \end{aligned}$$

Similarly we have (7) and (8). Next we prove (9). Suppose that $a_k \neq 0$. From (6) we have

$$\begin{aligned} &|u(z) + w_{m-1}(z)| \\ &\quad > b_k |a_k \exp\{ze^{-i\theta_k}\}| \\ &\qquad - (A_{m-1} + s_1) \exp\left\{r \cos \frac{2}{3}\pi\eta_k\right\} \\ &\quad = b_k |a_k| \exp\{r\cos(\theta - \theta_k)\} \\ &\qquad - (A_{m-1} + s_1) \exp\left\{r \cos \frac{2}{3}\pi\eta_k\right\} \\ &\quad \geq b_k |a_k| \exp\left\{r \cos \frac{1}{3}\pi\eta_k\right\} \\ &\qquad - (A_{m-1} + s_1) \exp\left\{r \cos \frac{2}{3}\pi\eta_k\right\} \end{aligned}$$

$$= \exp\left\{r\cos\frac{1}{3}\pi\eta_k\right\}\Big(b_k|a_k| - (A_{m-1}+s_1) \times \exp\left\{r\Big(\cos\frac{2}{3}\pi\eta_k - \cos\frac{1}{3}\pi\eta_k\Big)\right\}\Big)$$
$$\geq \frac{1}{2}b_k|a_k|\exp\left\{r\cos\frac{1}{3}\pi\eta_k\right\}$$

for all suficiently large r since

$$\cos\frac{2}{3}\pi\eta_k - \cos\frac{1}{3}\pi\eta_k = -2\sin\frac{\pi}{6}\eta_k\sin\frac{\pi}{2}\eta_k < 0.$$

Similarly we have (10) and (11). □

Lemma 3. *$u(z)+w_{m-1}(z)$ and $v_m(z)$ are linearly independent over $\boldsymbol{C}$.*

Proof. First of all we note that neither $u(z)+w_{m-1}(z)$ nor $v_m(z)$ is identically equal to zero by (9) and (10). Supose that they are linearly dependent over $\boldsymbol{C}$. Then there is a non-zero constant a satisfying $(u(z)+w_{m-1}(z))/v_m(z) \equiv a$. By the choice of $\{a_k\}$, there is at least one $k \geq m$ such that $a_k \neq 0,\ a$. For this k, $z = re^{i\theta}$ with θ satisfying (5) and all suficiently large r we have

$$0 \neq |a - a_k| = \left|\frac{u(z)+w_{m-1}(z) - a_k v_m(z)}{v_m(z)}\right|$$
$$\leq \frac{(s_1 + A_{m-1} + |a_k|s_2)\exp\left\{r\big(\cos\frac{2}{3}\pi\eta_k\big)\right\}}{\frac{1}{2}b_k\exp\left\{r\cos\frac{1}{3}\pi\eta_k\right\}}$$
$$= 2\frac{(s_1 + A_{m-1} + |a_k|s_2)}{b_k} \times \exp\Big\{r\Big(\cos\frac{2}{3}\pi\eta_k - \cos\frac{1}{3}\pi\eta_k\Big)\Big\}$$
$$= 2\frac{(s_1 + A_{m-1} + |a_k|s_2)}{b_k} \times \exp\Big\{-2r\sin\frac{\pi}{6}\eta_k\sin\frac{\pi}{2}\eta_k\Big\},$$

which tends to zero as $r \to \infty$ since $\sin(\pi/6)\eta_k\sin(\pi/2)\eta_k > 0$. This is a contradiction. We have our lemma. □

Let $f = [f_1, \ldots, f_{n+1}]$ be a transcendental holomorphic curve and for any positive integer p, we put $P(z) = z^p$. We consider the holomorphic curve

$$f \circ P = [f_1 \circ P, \ldots, f_{n+1} \circ P].$$

Note that $f_1 \circ P, \ldots, f_{n+1} \circ P$ have no common zero and are linearly independent over $\boldsymbol{C}$.

We put

$$\rho(f) = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}\ \text{(: the order of } f).$$

Lemma 4. *For any $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$*

[1] *$T(r, f\circ P) = T(r^p, f)$ and $\rho(f\circ P) = p\rho(f)$;*
[2] *$m(r, \boldsymbol{a}, f\circ P) = m(r^p, \boldsymbol{a}, f)$;*
[3] *$\delta(\boldsymbol{a}, f\circ P) = \delta(\boldsymbol{a}, f)$.*

Proof. [1] By the definition (1) and as $||f\circ P(z)|| = ||f(z^p)||$ we have

$$T(r, f\circ P) = \frac{1}{2\pi}\int_0^{2\pi}\log||f(r^pe^{ip\theta})||d\theta - \log||f(0)||$$
$$= \frac{1}{2p\pi}\int_0^{2p\pi}\log||f(r^pe^{i\phi})||d\phi - \log||f(0)||$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\log||f(r^pe^{i\phi})||d\phi - \log||f(0)||$$
$$= T(r^p, f).$$

The second assertion can easily be obtained from this relation.

[2] From the definition of $m(r, \boldsymbol{a}, f\circ P)$, we easily obtain this relation by the same way as in [1].

[3] From both [1] and [2], we have

$$\delta(\boldsymbol{a}, f\circ P) = \liminf_{r\to\infty}\frac{m(r, \boldsymbol{a}, f\circ P)}{T(r, f\circ P)} = \liminf_{r\to\infty}\frac{m(r^p, \boldsymbol{a}, f)}{T(r^p, f)} = \delta(\boldsymbol{a}, f).$$ □

3. Examples of holomorphic curve with an infinite number of deficiencies. We shall give examples of holomorphic curve with an infinite number of deficiencies in this section. Suppose that $n \geq 2$ throughout this section. Let $\{\eta_k\}$ and $\{\theta_k\}$ be those given in (3) and (4) of Section 2 respectively. Let $Y = \{\boldsymbol{a}_k = (a_{1k}, \ldots, a_{nk}, 1) \in \boldsymbol{C}^{n+1}\}$ be in general position and $\{c_{jk}\}_{k=1}^{\infty}$ $(j = 1, \ldots, n)$ be sequences of positive numbers satisfying

$$\det(c_{jk})\ \ (j, k = 1, \ldots, n) \neq 0,$$
$$c_{1k} = c_{2k} = \cdots = c_{nk} = c_k\ \ (k = n, n+1, \ldots)$$

and

$$S_j = \sum\nolimits_{k=1}^{\infty} c_{jk} < \infty\ \ (j = 1, \ldots, n),$$
$$S_{n+1} = \sum\nolimits_{k=1}^{\infty}\Big(\sum\nolimits_{j=1}^{n} c_{jk}|a_{jk}|\Big) < \infty.$$

Put

$$\varphi_j(z) = \sum\nolimits_{k=1}^{\infty} c_{jk}\exp\{ze^{-i\theta_k}\}\ \ (j = 1, \ldots, n),$$
$$\varphi_{n+1}(z) = -\sum\nolimits_{k=1}^{\infty}\Big(\sum\nolimits_{j=1}^{n} c_{jk}a_{jk}\Big)\exp\{ze^{-i\theta_k}\},$$
$$\psi_1(z) = \sum\nolimits_{k=n}^{\infty} c_k\exp\{ze^{-i\theta_k}\},$$

and $\varphi_j - \psi_1 = h_j \quad (j = 1, \ldots, n)$.

Note that if we put $a_k = \sum_{j=1}^{n} a_{jk} \ (k = 1, 2, \ldots)$, then $\{a_k\}$ satisfies the condition on $\{a_k\}$ given in Section 2 since Y is in general position.

Proposition 2. *For $|z| = r$,*

$$|\varphi_j(z)| < S_j e^r \quad (j = 1, 2, \ldots, n+1).$$

Proof. For any k and $z = re^{i\theta}$, we have the inequality

$$|\exp\{ze^{-i\theta_k}\}| = |\exp\{re^{i(\theta-\theta_k)}\}| \\ = \exp\{Re(re^{i(\theta-\theta_k)})\} \leq e^r,$$

so that we easily have our proposition. □

Proposition 3. *$\varphi_1, \ldots, \varphi_{n+1}$ have no common zeros.*

Proof. We have only to prove that $\varphi_1, \ldots, \varphi_n$ have no common zeros. Suppose that they have a common zero at $z = z_o$. Then, as

$$\varphi_j(z) = \sum_{k=1}^{n-1} c_{jk} \exp\{ze^{-i\theta_k}\} + \psi_1(z) \ \ (j = 1, \ldots, n),$$

it holds that

$$0 = \sum_{k=1}^{n-1} c_{jk} \exp\{z_o e^{-i\theta_k}\} + \psi_1(z_o) \ \ (j = 1, \ldots, n),$$

from which we have for $j = 1, \ldots, n-1$

$$0 = \sum_{k=1}^{n-1} (c_{jk} - c_{nk}) \exp\{z_o e^{-i\theta_k}\}. \tag{12}$$

Here, by the choice of $\{c_{jk}\}$ it holds that

$$0 \neq \det(c_{jk}) \ \ (j, k = 1, \ldots, n) \\ = c_{nn} \det(c_{jk} - c_{nk}) \ \ (j, k = 1, \ldots, n-1),$$

$c_{nn} \neq 0$, so that we have from (12) that

$$\exp\{z_o e^{-i\theta_k}\} = 0 \ \ (k = 1, \ldots, n-1),$$

which is a contradiction. We have our proposition. □

Proposition 4. *$\varphi_1, \ldots, \varphi_{n+1}$ are linearly independent over $\boldsymbol{C}$.*

Proof. Put $\alpha_1\varphi_1 + \cdots + \alpha_{n+1}\varphi_{n+1} = 0$. Then we have

$$\alpha_1 h_1 + \cdots + \alpha_n h_n + \alpha_{n+1}\varphi_{n+1} \\ + (\alpha_1 + \cdots + \alpha_n)\psi_1 = 0. \tag{13}$$

Now, suppose that $\alpha_{n+1} \neq 0$. Then, by the definition of φ_{n+1}, ψ_1 and $h_1, \ldots, h_n$ we can take $m = n$,

$$u = \varphi_{n+1}, \quad w_{n-1} = (\alpha_1 h_1 + \cdots + \alpha_n h_n)/\alpha_{n+1}$$

and $v_n = \psi_1$ in Lemma 3 to obtain that

$$(\alpha_1 h_1 + \cdots + \alpha_n h_n)/\alpha_{n+1} + \varphi_{n+1} \quad \text{and} \quad \psi_1$$

are linearly independent over $\boldsymbol{C}$. But the relation (13) reduces to the relation

$$\alpha_{n+1}\{(\alpha_1 h_1 + \cdots + \alpha_n h_n)/\alpha_{n+1} + \varphi_{n+1}\} \\ + (\alpha_1 + \cdots + \alpha_n)\psi_1 = 0,$$

which means that $(\alpha_1 h_1 + \cdots + \alpha_n h_n)/\alpha_{n+1} + \varphi_{n+1}$ and ψ_1 are linearly dependent over $\boldsymbol{C}$ since $\alpha_{n+1} \neq 0$. This is a contradiction. α_{n+1} must be equal to zero. So we have from (13)

$$\alpha_1 h_1 + \cdots + \alpha_n h_n + (\alpha_1 + \cdots + \alpha_n)\psi_1 = 0. \tag{14}$$

Next suppose that $\alpha_1 + \cdots + \alpha_n \neq 0$. Then we have from (14)

$$\left(\sum_{j=1}^{n} \alpha_j h_j\right) / (\alpha_1 + \cdots + \alpha_n) + \psi_1 = 0.$$

But, by applying (11) in Lemma 2 to $m = n$, $v_n = \psi_1$ and $w_{n-1} = \left(\sum_{j=1}^{n} \alpha_j h_j\right)/(\alpha_1 + \cdots + \alpha_n)$ we have that

$$\left(\sum_{j=1}^{n} \alpha_j h_j\right) / (\alpha_1 + \cdots + \alpha_n) + \psi_1 \neq 0,$$

which is a contradiction. This means that $\alpha_1 + \cdots + \alpha_n$ must be equal to zero. As $\alpha_n = -\alpha_1 - \cdots - \alpha_{n-1}$, we have from (14) that

$$\alpha_1(h_1 - h_n) + \cdots + \alpha_{n-1}(h_{n-1} - h_n) = 0. \tag{15}$$

Here,

$$h_j(z) - h_n(z) = \sum_{k=1}^{n-1} (c_{jk} - c_{nk}) \exp\{ze^{-i\theta_k}\}$$

$(j = 1, \ldots, n-1)$, $\det(c_{jk} - c_{nk}) \neq 0$ (see the proof of Proposition 3) and $\exp\{ze^{-i\theta_1}\}, \ldots, \exp\{ze^{-i\theta_{n-1}}\}$ are linearly independent over $\boldsymbol{C}$ since $0 < \theta_1 < \cdots < \theta_{n-1} < 2\pi$, so that $h_1 - h_n, \ldots, h_{n-1} - h_n$ are linearly independent over $\boldsymbol{C}$. We have from (15) that $\alpha_1 = \cdots = \alpha_{n-1} = 0$, and so $\alpha_n = 0$. We have that $\varphi_1, \ldots, \varphi_{n+1}$ are linearly independent over $\boldsymbol{C}$. □

We put $\varphi = [\varphi_1, \ldots, \varphi_{n+1}]$. Then, φ is a non-degenerate holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ by Propositions 3 and 4.

Proposition 5. *$T(r, \varphi) < r + O(1)$.*

Proof. As

$$\|\varphi(re^{i\theta})\| = (|\varphi_1(re^{i\theta})|^2 + \cdots + |\varphi_{n+1}(re^{i\theta})|^2)^{1/2} \\ \leq \left(\sum_{j=1}^{n+1} S_j^2\right)^{1/2} e^r$$

by Proposition 2, we have this proposition by the definition of $T(r, \varphi)$. □

As in the case of Lemma 2, we have the following estimates.

Proposition 6. *When θ satisfies* (5), *for* $|z| = r$

$$\left|\varphi_{n+1}(z) + \left(\sum\nolimits_{j=1}^{n} c_{jk}a_{jk}\right)\exp\{ze^{-i\theta_k}\}\right|$$

$$\tag{16} \leq \sum\nolimits_{\nu\neq k}\left|\left(\sum\nolimits_{j=1}^{n} c_{j\nu}a_{j\nu}\right)\exp\{ze^{-i\theta_k}\}\right|$$

$$\leq S_{n+1}\exp\left\{r\cos\frac{2}{3}\pi\eta_k\right\},$$

$$\tag{17} |\varphi_j(z) - c_{jk}\exp\{ze^{-i\theta_k}\}| \leq S_j\exp\left\{r\cos\frac{2}{3}\pi\eta_k\right\}$$

$(j = 1,\dots,n)$ *and for all sufficiently large* r

$$\tag{18} |\varphi_j(z)| \geq \frac{1}{2}c_{jk}\exp\left\{r\cos\frac{1}{3}\pi\eta_k\right\}$$

$(j = 1,\dots,n)$.

Proposition 7. *When $z = re^{i\theta}$ and r is any sufficiently large number, we have uniformly for θ satisfying* (5) *in Lemma* 1

$$\frac{||\boldsymbol{a}_k||\,||\varphi(re^{i\theta})||}{|(\boldsymbol{a}_k,\varphi(re^{i\theta}))|} \geq \frac{||\boldsymbol{a}_k||(\max_{1\leq j\leq n}c_{jk})\exp\left\{r\cos\frac{1}{3}\pi\eta_k\right\}}{2(S_{n+1}+\sum_{j=1}^{n}|a_{jk}|S_j)\exp\left\{r\cos\frac{2}{3}\pi\eta_k\right\}}.$$

Proof. First we note that $(\boldsymbol{a}_k,\varphi(re^{i\theta})) \not\equiv 0$ for any $\boldsymbol{a}_k \in Y$ due to Proposition 4.

From (18) for all suficiently large r and for θ satisfying (5) in Lemma 1 we have the inequality

$$||\boldsymbol{a}_k||\,||\varphi(re^{i\theta})|| \geq ||\boldsymbol{a}_k||\max_{1\leq j\leq n}|\varphi_j(re^{i\theta})| \geq \frac{||\boldsymbol{a}_k||}{2}\max_{1\leq j\leq n}c_{jk}\exp\left\{r\cos\frac{1}{3}\pi\eta_k\right\}.$$

From (16) and (17) for θ satisfying (5) in Lemma 1 we have the inequality

$$\begin{aligned}|(\boldsymbol{a}_k,\varphi(z))| &\leq \left|\varphi_{n+1}(z) + \left(\sum\nolimits_{j=1}^{n} c_{jk}a_{jk}\right)\exp\{ze^{-i\theta_k}\}\right| \\ &\quad + \sum\nolimits_{j=1}^{n}|a_{jk}(\varphi_j(z) - c_{jk}\exp\{ze^{-i\theta_k}\})| \\ &\leq S_{n+1}\exp\left\{r\cos\frac{2}{3}\pi\eta_k\right\} \\ &\quad + \left(\sum\nolimits_{j=1}^{n}|a_{jk}|S_j\right)\exp\left\{r\cos\frac{2}{3}\pi\eta_k\right\} \\ &= \left(S_{n+1} + \sum\nolimits_{j=1}^{n}|a_{jk}|S_j\right)\exp\left\{r\cos\frac{2}{3}\pi\eta_k\right\}.\end{aligned}$$

From these two inequalities we have our proposition. □

Proposition 8. *For all sufficiently large r, we have the inequality*

$$m(r,\boldsymbol{a}_k,\varphi) \geq \frac{2}{9}r\eta_k^3 + O(1).$$

Proof. From the definition of $m(r,\boldsymbol{a}_k,\varphi)$, we have by Proposition 7 for $\beta_k = \pi\eta_k/3$,

$$\begin{aligned}&m(r,\boldsymbol{a}_k,\varphi) \\ &\geq \frac{1}{2\pi}\int_{\theta_k-\beta_k}^{\theta_k+\beta_k}\log\frac{||\boldsymbol{a}_k||||\varphi(re^{i\theta})||}{|(\boldsymbol{a}_k,\varphi(re^{i\theta}))|}d\theta \\ &\geq \frac{r}{2\pi}\int_{\theta_k-\beta_k}^{\theta_k+\beta_k}\left(\cos\frac{\pi}{3}\eta_k - \cos\frac{2\pi}{3}\eta_k\right)d\theta + O(1) \\ &= \left(\frac{r}{2\pi}2\sin\frac{\pi}{6}\eta_k\sin\frac{\pi}{2}\eta_k\right)\frac{2\pi}{3}\eta_k + O(1) \\ &\geq \frac{2r}{3}\eta_k\cdot\frac{2}{\pi}\frac{\pi}{6}\eta_k\cdot\frac{2}{\pi}\frac{\pi}{2}\eta_k + O(1) = \frac{2}{9}r\eta_k^3 + O(1),\end{aligned}$$

since $\sin x \geq (2/\pi)x$ for $0 \leq x \leq (\pi/2)$. □

Combining Propositions 5 and 8, we have the following

Theorem 1. (I) *φ is of order* 1 *mean type.*
(II) $\delta(\boldsymbol{a}_k,\varphi) \geq (2/9)\eta_k^3$ $(k = 1,2,3,\dots)$.

Proof. (I) From Propositions 5 and 8 we have

$$\frac{2}{9}r\eta_1^3 + O(1) \leq T(r,\varphi) < r + O(1).$$

(II) From Propositions 5 and 8 we have

$$\delta(\boldsymbol{a}_k,\varphi) = \liminf_{r\to\infty}\frac{m(r,\boldsymbol{a}_k,\varphi)}{T(r,\varphi)} \geq \frac{2}{9}\eta_k^3.$$

□

Remark. Let φ, $Y = \{\boldsymbol{a}_k\}$ and η_k etc. be those given in this section and for any positive integer p put $P(z) = z^p$.

A. Put $\varphi\circ P = [\varphi_1\circ P,\dots,\varphi_{n+1}\circ P]$. Then, we obtain the following theorem from Theorem 1 and Lemma 4.

Theorem 2. (I) *$\varphi\circ P$ is of order p mean type*; (II) $\delta(\boldsymbol{a}_k,\varphi\circ P) \geq (2/9)\eta_k^3$ $(k = 1,2,3,\dots)$.

B. Put

$$Y_1 = Y \cup \{\boldsymbol{b}_m = (m+1)\boldsymbol{a}_1 \mid 1 \leq m \leq N-n\},$$

where N is a positive integer larger than n. Then, Y_1 is in N-subgeneral position but not in N'-subgeneral position for any positive integer $N' < N$. It is easy to see the following

Corollary 1. *For our $\varphi\circ P$ given in* **A** *we have*

$$\delta(\boldsymbol{a}_k,\varphi\circ P) \geq \frac{2}{9}\eta_k^3 \quad (k = 1,2,3,\dots)$$

and

$$\delta(\boldsymbol{b}_m, \varphi \circ P) \geq \frac{2}{9}\eta_1^3 \quad (m = 1, \ldots, N - n).$$

C. As in the case of meromorphic function ([4], p. 98), we have the following

Corollary 2. *For any* $0 < \epsilon < 1/3$*, there exist a holomorphic curve* $\varphi \circ P$ *of order* p *mean type and* $\{\boldsymbol{a}_k\}$ $(k = 1, 2, \ldots)$ *in general position satisfying*

$$\sum_{k=1}^{\infty} \delta(\boldsymbol{a}_k, \varphi \circ P)^{1/3-\epsilon} = \infty. \tag{19}$$

Taking the result of Weitsman ([7]) and this corollary into consideration, we would like to know whether the inequality

$$\sum_{\boldsymbol{a} \in X} \delta(\boldsymbol{a}, f)^{1/3} < \infty \tag{20}$$

holds or not when the (lower) order of f is finite.

Added in proof. After our original submission, we found three papers: [9, 10] and [11] relating to our paper. [10] and [11] give holomorphic curves with an infinite number of deficiencies, which are different from ours. Those in [10] satisfy (19). (20) is given in [9] as an open problem.

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