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BOSE-FERMI MIXTURES IN TWO OPTICAL LATTICES

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Abstract. We present stationary and travelling wave solutions for equations describing Bose-Fermi mixtures in an external potentials which are elliptic functions of modulus k. There are indications that such waves and localized objects may be observed in experiments with cold quantum degenerate gases.

1. Introduction

Recently, there has been a strong interest on quantum degenerate mixtures of bosons and fermions [3, 14, 16]. In this paper, we study a system of coupled nonlinear Schrödinger equations modelling a quantum degenerate mixture of bosons and fermions in optical lattice. Here we extend the results of our recent paper [10] and obtain new exact solutions in elliptic functions for the case when the boson and fermion ingredients are trapped by potentials with different strengths $V_{0,\mathrm{F}} \neq V_{0,\mathrm{B}}$.

2. Bose-Einstein Mixtures in Optical Lattice: Basic Equations in Mean Field Approximations

In this section we consider a mixture of BEC consisting of one boson and N_f fermion ingredients. In the one-dimensional approximation it is described by the following $N_f + 1$ coupled equations (see [16] and the references therein)

$$\mathrm{i}\hbar\frac{\partial\Psi^b}{\partial t} + \frac{1}{2m_\mathrm{B}}\frac{\partial^2\Psi^b}{\partial x^2} - V_\mathrm{B}\Psi^b - g_\mathrm{BB}|\Psi^b|^2\Psi^b - g_\mathrm{BF}\rho_f\Psi^b = 0 \tag{1}$$

$$\mathrm{i}\hbar\frac{\partial\Psi_j^f}{\partial t} + \frac{1}{2m_\mathrm{F}}\frac{\partial^2\Psi_j^f}{\partial x^2} - V_\mathrm{F}\Psi_j^f - g_\mathrm{BF}|\Psi^b|^2\Psi_j^f = 0 \tag{2}$$

where $\rho_f = \sum_{i=1}^{N_f} |\Psi_i^f|^2$ and

$$g_{\rm BB} = \frac{2a_{\rm BB}}{a_{\rm s}}, \qquad g_{\rm BF} = \frac{2a_{\rm BF}}{a_{\rm s}\alpha}, \qquad a_{\rm s} = \sqrt{\frac{\hbar}{m_{\rm B}\omega_\perp}}\,. \tag{3}$$

$a_{\rm BB}$ and $a_{\rm BF}$ are the scattering lengths for s-wave collisions for boson-boson and boson-fermion interactions, respectively. An appropriate class of periodic potentials to model the quasi-1D confinement produced by a standing light wave is given by [4]

$$V_{\rm B} = V_{0,{\rm B}}\,{\rm sn}^2(\alpha x, k), \qquad V_{\rm F} = V_{0,{\rm F}}\,{\rm sn}^2(\alpha x, k) \tag{4}$$

where ${\rm sn}(\alpha x, k)$ denotes the Jacobian elliptic sine function [2] with elliptic modulus $0 \leq k \leq 1$.

Experimental realization of two-component Bose-Einstein condensates have stimulated considerable attention in the quasi-1D regime [7] when the Gross-Pitaevskii equations for two interacting Bose-Einstein condensates reduce to coupled nonlinear Schrödinger (CNLS) equations with an external potential. In specific cases the two component CNLS equations [1,9,13] can be reduced to the Manakov system [12] with an external potential. Elliptic solutions for the CNLS and Manakov system were derived in [6,8,15].

In the presence of external elliptic potential explicit stationary solutions for NLS were derived in [4,5]. These results were generalized to the n-component CNLS in [7]. For two component CNLS explicit stationary solutions are derived in [11].

3. Type A Travelling Wave Solutions with Non-Trivial Phases

At first we restrict our attention to stationary solutions of these CNLS

$$\Psi^b(x,t) = q_0(x)\,{\rm e}^{-{\rm i}\frac{\omega_0}{\hbar}t + {\rm i}\Theta_0(x) + {\rm i}\kappa_0} \tag{5}$$

$$\Psi_j^f(x,t) = q_j(x)\,{\rm e}^{-{\rm i}\frac{\omega_j}{\hbar}t + {\rm i}\Theta_j(x) + {\rm i}\kappa_{0,j}} \tag{6}$$

where $j = 1,\ldots,N_f$, κ_0, $\kappa_{0,j}$, are constant phases, q_j and Θ_0, $\Theta_j(x)$ are real-valued functions connected by the relation

$$\Theta_0(x) = \mathcal{C}_0 \int_0^x \frac{{\rm d}x'}{q_0^2(x')}, \qquad \Theta_j(x) = \mathcal{C}_j \int_0^x \frac{{\rm d}x'}{q_j^2(x')}\,. \tag{7}$$

$\mathcal{C}_0, \mathcal{C}_j$, $j = 1,\ldots,N_f$ being constants of integration. Substituting the Ansatz (5), (6) in Equation (1) and separating the real and imaginary part we get

$$\frac{1}{2m_B} q_{0xx} - g_{\rm BB} q_0^3 - V_{\rm B} q_0 - g_{\rm BF}\left(\sum_{i=1}^{N_f} q_i^2\right) q_0 + \omega_0 q_0 = \frac{1}{2m_B}\frac{\mathcal{C}_0^2}{q_0^3} \tag{8}$$

$$\frac{1}{2m_F}q_{jxx} - g_{\rm BF}q_0^2q_j - V_{\rm F}q_j + \omega_j q_j = \frac{1}{2m_F}\frac{\mathcal{C}_j^2}{q_j^3}\,. \tag{9}$$

We seek solutions for q_0^2 and q_j^2, $j = 1, \ldots, N_f$ as a quadratic function of $\operatorname{sn}(\alpha x, k)$

$$q_0^2 = A_0 \operatorname{sn}^2(\alpha x, k) + B_0, \qquad q_j^2 = A_j \operatorname{sn}^2(\alpha x, k) + B_j. \tag{10}$$

Equating the coefficients of equal powers of $\operatorname{sn}(\alpha x, k)$ results in the following relations among the solution parameters ω_j, $\mathcal{C}_j$, A_j and B_j and the characteristic of the optical lattice V_0, α and k

$$\sum_{j=1}^{N_f} A_j = \frac{\alpha^2k^2}{g_{\rm BF}}\left(\frac{1}{m_{\rm B}} - \frac{g_{\rm BB}}{m_{\rm F}g_{\rm BF}}\right) - \frac{1}{g_{\rm BF}}\left(V_{0\rm B} - \frac{V_{0\rm F}g_{\rm BB}}{g_{\rm BF}}\right) \tag{11}$$

$$\begin{aligned} \omega_0 &= \frac{\alpha^2(k^2+1)}{2m_{\rm B}} + g_{\rm BB}B_0 + g_{\rm BF}\sum_{i=1}^{N_f}B_i + \frac{\alpha^2k^2}{2m_{\rm B}}\frac{B_0}{A_0} \\ \omega_j &= \frac{\alpha^2(k^2+1)}{2m_{\rm F}} + g_{\rm BF}B_0 + \frac{\alpha^2k^2}{2m_{\rm F}}\frac{B_j}{A_j}, \quad A_0 = \frac{\alpha^2k^2 - m_F V_{0\rm F}}{m_{\rm F}g_{\rm BF}} \end{aligned} \tag{12}$$

$$\mathcal{C}_0^2 = \frac{\alpha^2 B_0}{A_0}(A_0+B_0)(A_0+B_0k^2), \qquad \mathcal{C}_j^2 = \frac{\alpha^2 B_j}{A_j}(A_j+B_j)(A_j+B_jk^2) \tag{13}$$

where $j = 1, \ldots, N_f$. Next for convenience we introduce

$$B_0 = -\beta_0 A_0, \qquad B_j = -\beta_j A_j, \qquad j = 1, \ldots, N_f \tag{14}$$

then

$$\mathcal{C}_0^2 = \alpha^2 A_0^2 \beta_0(\beta_0 - 1)(1 - \beta_0 k^2) \tag{15}$$

$$\mathcal{C}_j^2 = \alpha^2 A_j^2 \beta_j(\beta_j - 1)(1 - \beta_j k^2). \tag{16}$$

In order that our results (10) are consistent with the parametrization (5), (6), (7) we must ensure that both $q_0(x)$ and $\Theta_0(x)$ are real-valued, and also $q_j(x)$ and $\Theta_j(x)$ are real-valued; this means that $C_0^2 \geq 0$ and $q_0^2(x) \geq 0$ and also $C_j^2 \geq 0$ and $q_j^2(x) \geq 0$. An elementary analysis shows that one of the following conditions

$$\text{a)} \quad A_l \geq 0, \quad \beta_l \leq 0 \qquad\qquad \text{b)} \quad A_l \leq 0, \quad 1 \leq \beta_l \leq \frac{1}{k^2} \tag{17}$$

for $l = 0, \ldots, N_f$ must hold. Using the well known transformation $x \to x - c_j t$, $j = 0, \ldots, N_f$ it is easy to obtain travelling wave solutions with different velocities c_j

$$\begin{aligned} \Psi^b(x,t) &= q_0(x - c_0 t)\, \mathrm{e}^{-\mathrm{i}\frac{\omega_0}{\hbar}t - \mathrm{i}\hbar\, m_{\rm B}\left(\frac{1}{2}c_0^2 t + c_0 x\right) + \mathrm{i}\Theta_0(x) + \mathrm{i}\kappa_0} \\ \Psi_j^f(x,t) &= q_j(x - c_j t)\, \mathrm{e}^{-\mathrm{i}\frac{\omega_j}{\hbar}t - \mathrm{i}\hbar\, m_{\rm B}\left(\frac{1}{2}c_j^2 t + c_j x\right) + \mathrm{i}\Theta_j(x) + \mathrm{i}\kappa_{0,j}} \end{aligned}$$

where $j = 1, \ldots, N_f$, κ_0, $\kappa_{0,j}$, are constant phases, q_j and Θ_0, $\Theta_j(x)$ are real-valued functions connected by the relation

$$\Theta_0(x,t) = \mathcal{C}_0 \int_0^{x-c_0 t} \frac{\mathrm{d}x'}{q_0^2(x')}, \quad \Theta_j(x,t) = \mathcal{C}_j \int_0^{x-c_j t} \frac{\mathrm{d}x'}{q_j^2(x')}, \quad j = 1 \ldots, N_f.$$

We display also mixed type solutions for which the boson part has trivial phase while the fermions have nontrivial phases and vice versa. These are obtained with

1. generic C_j and $B_0 = -A_0$, $B_0 = 0$ or $B_0 = -A_0/k^2$.
2. C_0 generic and $B_j = -A_j$, $B_0 = 0$ or $B_j = -A_j/k^2$.

Under certain conditions $Q_j(x,t)$ become periodic functions of x, see [10, 11]. If the periods T_0, T_j satisfy

$$\Theta_0(x+T_0) - \Theta_0(x) = 2\pi p_0, \qquad \Theta_j(x+T_j) - \Theta_j(x) = 2\pi p_j \tag{18}$$

for $j = 1, \ldots, N_f$ then Ψ^b, Ψ_j^f will be periodic in x with periods $T_0 = 2m_0\omega/\alpha$, $T_j = 2m_j\omega/\alpha$. This holds true provided there exist pairs of integers m_0, p_0, and m_j, p_j, such that

$$\frac{m_0}{p_0} = -\pi \left[\alpha v_0 \zeta(\omega) + \omega\tau_0/\alpha\right]^{-1}, \qquad \frac{m_j}{p_j} = -\pi \left[\alpha v_j \zeta(\omega) + \omega\tau_j/\alpha\right]^{-1}$$

where ω (and ω') are the half-periods of the Weierstrass functions ζ.

When $V_{0,\mathrm{F}} = V_{0,\mathrm{B}} = V_0$ and inserting (10) in (8) and equating the coefficients of equal powers of $\mathrm{sn}(\alpha x, k)$ results in the following relations among the parameters ω_j, $\mathcal{C}_j$, A_j and B_j and the characteristic of the optical lattice V_0, α and k

$$\sum_{j=1}^{N_f} A_j = \frac{\alpha^2 k^2}{g_{\mathrm{BF}}} \left(\frac{1}{m_{\mathrm{B}}} - \frac{g_{\mathrm{BB}}}{m_{\mathrm{F}} g_{\mathrm{BF}}}\right) - \frac{V_0}{g_{\mathrm{BF}}} \left(1 - \frac{g_{\mathrm{BB}}}{g_{\mathrm{BF}}}\right) \tag{19}$$

$$\begin{aligned} \omega_0 &= \frac{\alpha^2(k^2+1)}{2m_{\mathrm{B}}} + g_{\mathrm{BB}} B_0 + g_{\mathrm{BF}} \sum_{i=1}^{N_f} B_i + \frac{\alpha^2 k^2}{2m_{\mathrm{B}}} \frac{B_0}{A_0} \\ A_0 &= \frac{\alpha^2 k^2 - m_F V_0}{m_{\mathrm{F}} g_{\mathrm{BF}}}, \qquad \omega_j = \frac{\alpha^2(k^2+1)}{2m_{\mathrm{F}}} + g_{\mathrm{BF}} B_0 + \frac{\alpha^2 k^2}{2m_{\mathrm{F}}} \frac{B_j}{A_j} \end{aligned} \tag{20}$$

$$\mathcal{C}_0^2 = \frac{\alpha^2 B_0}{A_0}(A_0+B_0)(A_0+B_0 k^2), \qquad \mathcal{C}_j^2 = \frac{\alpha^2 B_j}{A_j}(A_j+B_j)(A_j+B_j k^2) \tag{21}$$

where $j = 1, \ldots, N_f$. Next for convenience we introduce

$$B_0 = -\beta_0 A_0, \qquad B_j = -\beta_j A_j, \qquad j = 1, \ldots, N_f$$

then

$$\mathcal{C}_0^2 = \alpha^2 A_0^2 \beta_0(\beta_0 - 1)(1 - \beta_0 k^2), \qquad \mathcal{C}_j^2 = \alpha^2 A_j^2 \beta_j(\beta_j - 1)(1 - \beta_j k^2).$$

Table 1. Constraints ensuring the existence of type A solutions. Here $W = g_{\rm BF} m_{\rm F} W_{\rm B}/(m_{\rm B} W_{\rm F})$.

1	$\beta_0 \le 0$	$\beta_j \le 0$	$A_0 \ge 0$	$A_j \ge 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr W$
2	$\beta_0 \le 0$	$1 \le \beta_j \le 1/k^2$	$A_0 \ge 0$	$A_j \le 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \gtrless W$
3	$1 \le \beta_0 \le 1/k^2$	$\beta_j \le 0$	$A_0 \le 0$	$A_j \ge 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \gtrless W$
4	$1 \le \beta_0 \le 1/k^2$	$1 \le \beta_j \le 1/k^2$	$A_0 \le 0$	$A_j \le 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr W$

In order for our results (10) to be consistent with the parametrization (5)–(7) we must ensure that both $q_0(x)$ and $\Theta_0(x)$ are real-valued, and also $q_j(x)$ and $\Theta_j(x)$ are real-valued; this means that $C_0^2 \ge 0$ and $q_0^2(x) \ge 0$ and also $C_j^2 \ge 0$ and $q_j^2(x) \ge 0$ (see Table 1, $W_{\rm B} = (\alpha^2 k^2 - m_{\rm B} V_0)$, $W_{\rm F} = (\alpha^2 k^2 - m_{\rm F} V_0)$). An elementary analysis shows that with $l = 0, \dots, N_f$ one of the following conditions must hold

$$\text{a)} \quad A_l \ge 0, \quad \beta_l \le 0 \qquad \text{b)} \quad A_l \le 0, \quad 1 \le \beta_l \le \frac{1}{k^2} \cdot$$

4. Type B Nontrivial Phase Solutions

For the first time solutions of this type were derived in [4,5] for the case of nonlinear Schrödinger equation and in [7] for the n-component CNLSE. For Bose–Fermi mixtures solutions of this type are possible

- when we have two lattices $V_{\rm B}$ and $V_{\rm F}$.
- when $m_{\rm B} = m_{\rm F}$.

We seek the solutions in one of the following forms:

$$q_0^2 = A_0 \,\mathrm{sn}(\alpha x, k) + B_0, \qquad q_j^2 = A_j \,\mathrm{sn}(\alpha x, k) + B_j, \quad j = 1, \dots, N_f \tag{22}$$

$$q_0^2 = A_0 \,\mathrm{cn}(\alpha x, k) + B_0, \qquad q_j^2 = A_j \,\mathrm{cn}(\alpha x, k) + B_j \tag{23}$$

$$q_0^2 = A_0 \,\mathrm{dn}(\alpha x, k) + B_0, \qquad q_j^2 = A_j \,\mathrm{dn}(\alpha x, k) + B_j. \tag{24}$$

In the first case (22) we have

$$V_{\rm B} = \frac{3\alpha^2 k^2}{8 m_{\rm B}}, \qquad V_{\rm F} = \frac{3\alpha^2 k^2}{8 m_{\rm F}}$$

$$A_0 = -\frac{\alpha^2 k^2}{4 m_{\rm F} g_{\rm BF}} \frac{B_j}{A_j}, \qquad \frac{B_1}{A_1} = \dots = \frac{B_{N_f}}{A_{N_f}}$$

$$\sum_j A_j = -\frac{\alpha^2 k^2}{4 m_{\rm B} g_{\rm BF}} \frac{B_0}{A_0} - \frac{A_0 g_{\rm BB}}{g_{\rm BF}}$$

$$\omega_0 = \frac{\alpha^2 (k^2 + 1)}{8 m_{\rm B}} + g_{\rm BB} B_0 + g_{\rm BF} B_1 - \frac{\alpha^2 k^2}{8 m_{\rm B}} \frac{B_0^2}{A_0^2}$$

$$\omega_j = \frac{\alpha^2(k^2+1)}{8m_{\rm F}} + g_{\rm BF}B_0 - \frac{\alpha^2 k^2}{8m_{\rm F}}\frac{B_j^2}{A_j^2}$$

$$\mathcal{C}_0^2 = \frac{\alpha^2}{4A_0^2}(B_0^2 - A_0^2)(A_0^2 - B_0^2 k^2), \qquad \mathcal{C}_j^2 = \frac{\alpha^2}{4A_j^2}(B_j^2 - A_j^2)(A_j^2 - B_j^2 k^2).$$

We remark that due to relations $\frac{B_1}{A_1} = \cdots = \frac{B_{N_f}}{A_{N_f}}$ we have that all q_j of the fermion fields are proportional to q_1.

5. Examples of Elliptic Solutions

Using the general solution equations (11)–(13) we have the following special cases (these solutions are possible only when we have some restrictions on $g_{\rm BB}$, $g_{\rm BF}$, and V_0, see Table 1):

Example 1. Suppose that $B_0 = B_j = 0$. Therefore we have

$$q_0(x) = \sqrt{A_0}\,\mathrm{sn}(\alpha x, k), \qquad q_j = \sqrt{A_j}\,\mathrm{sn}(\alpha x, k) \tag{25}$$

$$A_0 = \frac{\alpha^2 k^2 - m_{\rm F}V_0}{m_{\rm F}g_{\rm BF}}, \qquad \sum_j A_j = \frac{\alpha^2 k^2}{g_{\rm BF}}\left(\frac{1}{m_{\rm B}} - \frac{g_{\rm BB}}{m_F g_{\rm FB}}\right) - \frac{V_0}{g_{\rm BF}}\left(1 - \frac{g_{\rm BB}}{g_{\rm BF}}\right)\cdot \tag{26}$$

For the frequencies ω_0 and ω_j we have

$$\omega_0 = \frac{\alpha^2(1+k^2)}{2m_{\rm B}}, \qquad \omega_j = \frac{\alpha^2(1+k^2)}{2m_{\rm F}}\cdot$$

as well as $\mathcal{C}_0 = \mathcal{C}_j = 0$.

Example 2. Let $B_0 = -A_0$ and $B_j = -A_j$ hold true. Then we have

$$q_0(x) = \sqrt{-A_0}\,\mathrm{cn}(\alpha x, k), \qquad q_j(x) = \sqrt{-A_j}\,\mathrm{cn}(\alpha x, k). \tag{27}$$

The coefficients A_0 and A_j have the same form as (26). The frequencies ω_0 and ω_j now look as follows

$$\omega_0 = \frac{\alpha^2(1-2k^2)}{2m_{\rm B}} + V_0, \qquad \omega_j = \frac{\alpha^2(1-2k^2)}{2m_{\rm F}} + V_0.$$

The constants $\mathcal{C}_0$ and $\mathcal{C}_j$ are equal to zero again.

Example 3. $B_0 = -A_0/k^2$ and $B_j = -A_j/k^2$. In this case we obtain

$$\begin{aligned} q_0(x) &= \frac{\sqrt{-A_0}}{k}\mathrm{dn}(\alpha x, k), \qquad & q_j(x) &= \frac{\sqrt{-A_j}}{k}\mathrm{dn}(\alpha x, k) \\ \omega_0 &= \frac{\alpha^2(k^2-2)}{2m_{\rm B}} + \frac{V_0}{k^2}, \qquad & \omega_j &= \frac{\alpha^2(k^2-2)}{2m_{\rm F}} + \frac{V_0}{k^2}\cdot \end{aligned} \tag{28}$$

As before $\mathcal{C}_0 = \mathcal{C}_j = 0$.

Example 4. $B_0 = 0$ and $B_j = -A_j$. The result reads

$$\begin{aligned} q_0(x) &= \sqrt{A_0}\,\mathrm{sn}(\alpha x, k), & q_j(x) &= \sqrt{-A_j}\,\mathrm{cn}(\alpha x, k) \\ \omega_0 &= \frac{\alpha^2(1-k^2)}{2m_{\mathrm{B}}} + V_0 + A_0 g_{\mathrm{BB}}, & \omega_j &= \frac{\alpha^2}{2m_{\mathrm{F}}}\,. \end{aligned} \tag{29}$$

By analogy with the previous examples the constants A_0, A_j, $\mathcal{C}_0$ and $\mathcal{C}_j$ are given by formulae (26) and $\mathcal{C}_0$, $\mathcal{C}_j$ are all zero.

Example 5. $B_0 = 0$ and $B_j = -A_j/k^2$. Thus, one gets

$$\begin{aligned} q_0(x) &= \sqrt{A_0}\,\mathrm{sn}(\alpha x, k), & q_j(x) &= \frac{\sqrt{-A_j}}{k}\,\mathrm{dn}(\alpha x, k) \\ \omega_0 &= \frac{\alpha^2(k^2-1)}{2m_{\mathrm{B}}} + \frac{V_0}{k^2} + \frac{A_0 g_{\mathrm{BB}}}{k^2}, & \omega_j &= \frac{\alpha^2 k^2}{2m_{\mathrm{F}}}\,. \end{aligned} \tag{30}$$

Example 6. Let $B_0 = -A_0$ and $B_j = 0$. Hence we have

$$\begin{aligned} q_0(x) &= \sqrt{-A_0}\,\mathrm{cn}(\alpha x, k), & q_j(x) &= \sqrt{A_j}\,\mathrm{sn}(\alpha x, k) \\ \omega_0 &= \frac{\alpha^2}{2m_{\mathrm{B}}} - g_{\mathrm{BB}} A_0, & \omega_j &= \frac{\alpha^2(1-k^2)}{2m_{\mathrm{F}}} + V_0. \end{aligned}$$

Example 7. Let $B_0 = -A_0$ and $B_j = -A_j/k^2$. We obtain

$$\begin{aligned} q_0(x) &= \sqrt{-A_0}\,\mathrm{cn}(\alpha x, k), & q_j(x) &= \frac{\sqrt{-A_j}}{k}\,\mathrm{dn}(\alpha x, k) \\ \omega_0 &= \frac{V_0}{k^2} - \frac{\alpha^2}{2m_{\mathrm{B}}} + \frac{1-k^2}{k^2} A_0 g_{\mathrm{BB}}, & \omega_j &= V_0 - \frac{\alpha^2 k^2}{2m_{\mathrm{F}}}\,. \end{aligned}$$

Example 8. Suppose $B_0 = -A_0/k^2$ and $B_j = 0$. Then

$$\begin{aligned} q_0(x) &= \frac{\sqrt{-A_0}}{k}\,\mathrm{dn}(\alpha x, k), & q_j(x) &= \sqrt{A_j}\,\mathrm{sn}(\alpha x, k) \\ \omega_0 &= \frac{\alpha^2 k^2}{2m_{\mathrm{B}}} - \frac{g_{\mathrm{BB}} A_0}{k^2}, & \omega_j &= \frac{\alpha^2(k^2-1)}{2m_{\mathrm{F}}} + \frac{V_0}{k^2}\,. \end{aligned}$$

Example 9. Let $B_0 = -A_0/k^2$ and $B_j = -A_j$. Thus

$$\begin{aligned} q_0(x) &= \frac{\sqrt{-A_0}}{k}\,\mathrm{dn}(\alpha x, k), & q_j(x) &= \sqrt{-A_j}\,\mathrm{cn}(\alpha x, k) \\ \omega_0 &= V_0 - \frac{\alpha^2 k^2}{2m_{\mathrm{B}}} + \frac{k^2-1}{k^2} g_{\mathrm{BB}} A_0, & \omega_j &= \frac{V_0}{k^2} - \frac{\alpha^2}{2m_{\mathrm{F}}}\,. \end{aligned}$$

All these cases when $V_0 = 0$ and $j = 2$ are derived for the first time in [3].

Table 2. Constraints ensuring the existence of generic type B trivial phase solutions. Here $W = g_{\rm BF} m_{\rm F} W_{\rm B}/(m_{\rm B} W_{\rm F})$.

1	$q_0 = \sqrt{A_0}\,\mathrm{sn}(\alpha x, k)$ $q_j = \sqrt{A_j}\,\mathrm{sn}(\alpha x, k)$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr W$	$V_0 \lessgtr \alpha^2 k^2/m_{\rm F}$
2	$q_0 = \sqrt{-A_0}\,\mathrm{cn}(\alpha x, k)$ $q_j = \sqrt{-A_j}\,\mathrm{cn}(\alpha x, k)$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr W$	$V_0 \gtrless \alpha^2 k^2/m_{\rm F}$
3	$q_0 = \sqrt{-A_0}\,\mathrm{dn}(\alpha x, k)/k$ $q_j = \sqrt{-A_j}\,\mathrm{dn}(\alpha x, k)/k$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr W$	$V_0 \gtrless \alpha^2 k^2/m_{\rm F}$
4	$q_0 = \sqrt{A_0}\,\mathrm{sn}(\alpha x, k)$ $q_j = \sqrt{-A_j}\,\mathrm{cn}(\alpha x, k)$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \gtrless W$	$V_0 \lessgtr \alpha^2 k^2/m_{\rm F}$
5	$q_0 = \sqrt{A_0}\,\mathrm{sn}(\alpha x, k)$ $q_j = \sqrt{-A_j}\,\mathrm{dn}(\alpha x, k)/k$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \gtrless W$	$V_0 \lessgtr \alpha^2 k^2/m_{\rm F}$
6	$q_0 = \sqrt{-A_0}\,\mathrm{cn}(\alpha x, k)$ $q_j = \sqrt{A_j}\,\mathrm{sn}(\alpha x, k)$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \gtrless W$	$V_0 \gtrless \alpha^2 k^2/m_{\rm F}$
7	$q_0 = \sqrt{-A_0}\,\mathrm{cn}(\alpha x, k)$ $q_j = \sqrt{-A_j}\,\mathrm{dn}(\alpha x, k)/k$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr W$	$V_0 \gtrless \alpha^2 k^2/m_{\rm F}$
8	$q_0 = \sqrt{-A_0}\,\mathrm{dn}(\alpha x, k)/k$ $q_j = \sqrt{A_j}\,\mathrm{sn}(\alpha x, k)$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \gtrless W$	$V_0 \gtrless \alpha^2 k^2/m_{\rm F}$
9	$q_0 = \sqrt{-A_0}\,\mathrm{dn}(\alpha x, k)/k$ $q_j = \sqrt{-A_j}\,\mathrm{cn}(\alpha x, k)$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr W$	$V_0 \gtrless \alpha^2 k^2/m_{\rm F}$

5.1. Mixed Trivial Phase Solution

Example 10. When

$$B_0 = 0, \qquad B_1 = 0, \qquad B_2 = -A_2, \qquad B_j = -A_j/k^2, \quad j = 3, \ldots, N_f.$$

the solutions obtain the form

$$q_0 = \sqrt{A_0}\,\mathrm{sn}(\alpha x, k), \qquad q_1 = \sqrt{A_1}\,\mathrm{sn}(\alpha x, k)$$
$$q_2 = \sqrt{-A_2}\,\mathrm{cn}(\alpha x, k), \qquad q_j = \sqrt{-A_j}\,\mathrm{dn}(\alpha x, k)/k.$$

Using equations (11)–(13) we have

$$A_0 = \frac{\alpha^2 k^2 - V_0 m_F}{m_F g_{\rm BF}}, \ \sum_{j=1}^{N_f} A_j = \alpha^2 k^2 \left(\frac{1}{m_B g_{\rm BF}} - \frac{g_{\rm BB}}{m_F g_{\rm BF}^2}\right) - V_0 \left(\frac{1}{g_{\rm BF}} - \frac{g_{\rm BB}}{g_{\rm BF}^2}\right)$$

$$\omega_0 = \frac{\alpha^2 (k^2 - 1)}{2m_B} + \frac{g_{\rm BF}}{k^2}\Big(A_1 + (1 - k^2) A_2\Big) + \frac{g_{\rm BB} A_0}{k^2} + \frac{V_0}{k^2}$$

$$\omega_1 = \frac{\alpha^2(1+k^2)}{2m_F}, \qquad \omega_2 = \frac{1}{2m_F}\alpha^2, \qquad \omega_j = \frac{\alpha^2 k^2}{2m_F}, \qquad j = 3, \ldots, N_F.$$

Example 11. Let $B_0 = B_1 = 0$ and $B_j = -A_j$ where $j = 2, \ldots, N_f$. Therefore, the solutions read

$$\begin{aligned} q_0(x) &= \sqrt{A_0}\,\mathrm{sn}(\alpha x, k) \\ q_1(x) &= \sqrt{A_1}\,\mathrm{sn}(\alpha x, k) \\ q_j(x) &= \sqrt{-A_j}\,\mathrm{cn}(\alpha x, k). \end{aligned}$$

Then we obtain for frequencies the following results

$$\omega_0 = \frac{\alpha^2(1-k^2)}{2m_\mathrm{B}} + V_0 + g_\mathrm{BB}A_0 + g_\mathrm{BF}A_1, \quad \omega_1 = \frac{\alpha^2(1+k^2)}{2m_\mathrm{F}}, \quad \omega_j = \frac{\alpha^2}{2m_\mathrm{F}}\,\cdot$$

Example 12. Suppose $B_0 = -A_0$, $B_1 = 0$, $B_2 = -A_2$ and $B_j = -A_j/k^2$ where $j = 3, \ldots, N_f$. The solutions have the form

$$\begin{aligned} q_0(x) &= \sqrt{-A_0}\,\mathrm{cn}(\alpha x, k), \qquad & q_1(x) &= \sqrt{A_1}\,\mathrm{sn}(\alpha x, k) \\ q_2(x) &= \sqrt{-A_2}\,\mathrm{cn}(\alpha x, k), \qquad & q_j(x) &= \sqrt{-A_j}\,\mathrm{dn}(\alpha x, k)/k. \end{aligned}$$

The frequencies are

$$\begin{aligned} \omega_0 &= \frac{V_0}{k^2} - \frac{\alpha^2}{2m_\mathrm{B}} + \frac{1-k^2}{k^2}(g_\mathrm{BB}A_0 + g_\mathrm{BF}A_2) + \frac{g_\mathrm{BF}}{k^2}A_1 \\ \omega_1 &= V_0 + \frac{\alpha^2(1-k^2)}{2m_\mathrm{F}}, \qquad \omega_2 = V_0 + \frac{\alpha^2(1-2k^2)}{2m_\mathrm{F}}, \qquad \omega_j = V_0 - \frac{\alpha^2 k^2}{2m_\mathrm{F}}\,\cdot \end{aligned}$$

Example 13. Let $B_0 = -A_0$, $B_1 = -A_1$ and $B_j = -A_j/k^2$ for $j = 2, \ldots, N_f$. Then

$$\begin{aligned} q_0(x) &= \sqrt{-A_0}\,\mathrm{cn}(\alpha x, k) \\ q_1(x) &= \sqrt{-A_1}\,\mathrm{cn}(\alpha x, k) \\ q_j(x) &= \sqrt{-A_j}\,\mathrm{dn}(\alpha x, k)/k \\ \omega_0 &= \frac{V_0}{k^2} - \frac{\alpha^2}{2m_\mathrm{B}} + \frac{1-k^2}{k^2}\left(g_\mathrm{BB}A_0 + g_\mathrm{BF}A_1\right) \\ \omega_1 &= V_0 + \frac{\alpha^2(1-2k^2)}{2m_\mathrm{F}}, \qquad \omega_j = V_0 - \frac{\alpha^2 k^2}{2m_\mathrm{F}}\,\cdot \end{aligned}$$

Example 14. Let $B_0 = -A_0/k^2$, $B_1 = -A_1$ and $B_j = -A_j/k^2$ for $j = 2, \ldots, N_f$. Hence

$$\begin{aligned}
q_0(x) &= \sqrt{-A_0}\,\mathrm{dn}(\alpha x, k)/k \\
q_1(x) &= \sqrt{-A_1}\,\mathrm{cn}(\alpha x, k) \\
q_j(x) &= \sqrt{-A_j}\,\mathrm{dn}(\alpha x, k)/k
\end{aligned}$$

$$\omega_0 = \frac{\alpha^2(k^2-2)}{2m_{\mathrm{B}}} + \frac{V_0}{k^2} + \frac{1-k^2}{k^2}(g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}A_1)$$

$$\omega_1 = \frac{V_0}{k^2} - \frac{\alpha^2}{2m_{\mathrm{F}}}, \qquad \omega_j = \frac{V_0}{k^2} + \frac{\alpha^2(k^2-2)}{2m_{\mathrm{F}}}\,.$$

Certainly these examples do not exhaust all possible combinations of solutions and it is easy to it.

6. Vector Soliton Solutions

6.1. Vector Bright-Bright Soliton Solutions

When $k \to 1$, $\mathrm{sn}(\alpha x, 1) = \tanh(\alpha x)$ and $B_0 = -A_0$, $B_j = -A_j$ we obtain that the solutions read

$$q_0 = \sqrt{-A_0}\frac{1}{\cosh(\alpha x)}, \qquad q_j = \sqrt{-A_j}\frac{1}{\cosh(\alpha x)}$$

where $A_0 \leq 0$ as well as $A_j \leq 0$. Using equations (11)–(13) we have

$$A_0 = \frac{\alpha^2 - V_0 m_{\mathrm{F}}}{m_{\mathrm{F}} g_{\mathrm{BF}}}, \qquad V = V_0 \tanh^2(\alpha x)$$

$$\sum_{j=1}^{N_f} A_j = \frac{\alpha^2}{g_{\mathrm{BF}}}\left(\frac{1}{m_{\mathrm{B}}} - \frac{g_{\mathrm{BB}}}{m_{\mathrm{F}} g_{\mathrm{BF}}}\right) - \frac{V_0}{g_{\mathrm{BF}}}\left(1 - \frac{g_{\mathrm{BB}}}{g_{\mathrm{BF}}}\right)$$

$$\omega_0 = V_0 - \frac{1}{2m_B}\alpha^2, \qquad \omega_j = V_0 - \frac{1}{2m_F}\alpha^2.$$

As a consequence of the restrictions on A_0 and A_j one can get the following unequalities

$$g_{\mathrm{BF}} > 0, \qquad V_0 \geq \frac{\alpha^2}{m_{\mathrm{F}}}, \qquad g_{\mathrm{BB}} \leq \frac{(\alpha^2 - m_{\mathrm{B}} V_0) m_{\mathrm{F}}}{(\alpha^2 - m_{\mathrm{F}} V_0) m_{\mathrm{B}}} g_{\mathrm{BF}}$$

$$g_{\mathrm{BF}} < 0, \qquad V_0 \leq \frac{\alpha^2}{m_{\mathrm{F}}}, \qquad g_{\mathrm{BB}} \geq \frac{(\alpha^2 - m_{\mathrm{B}} V_0) m_{\mathrm{F}}}{(\alpha^2 - m_{\mathrm{F}} V_0) m_{\mathrm{B}}} g_{\mathrm{BF}}.$$

Vector bright soliton solution when $V_0 = 0$ is derived for the first time in [3].

6.2. Vector Dark-Dark Soliton Solutions

When $k \to 1$ and $B_0 = B_j = 0$ are satisfied the solutions read

$$q_0(x) = \sqrt{A_0}\tanh(\alpha x), \qquad q_j(x) = \sqrt{A_j}\tanh(\alpha x).$$

The natural restrictions $A_0 \geq 0$ and $A_j \geq 0$ lead to

$$\begin{aligned} &g_{\mathrm{BF}} > 0, \qquad g_{\mathrm{BB}} \leq \frac{(\alpha^2 - m_{\mathrm{B}}V_0)m_{\mathrm{F}}}{(\alpha^2 - m_{\mathrm{F}}V_0)m_{\mathrm{B}}} g_{\mathrm{BF}}, \qquad V_0 \leq \alpha^2/m_{\mathrm{F}} \\ &g_{\mathrm{BF}} < 0, \qquad g_{\mathrm{BB}} \geq \frac{(\alpha^2 - m_{\mathrm{B}}V_0)m_{\mathrm{F}}}{(\alpha^2 - m_{\mathrm{F}}V_0)m_{\mathrm{B}}} g_{\mathrm{BF}}, \qquad V_0 \geq \alpha^2/m_{\mathrm{F}} \\ &A_0 = \frac{\alpha^2 - m_{\mathrm{F}}V_0}{m_{\mathrm{F}}g_{\mathrm{BF}}}, \quad \sum_j A_j = \frac{\alpha^2}{g_{\mathrm{BF}}}\left(\frac{1}{m_{\mathrm{B}}} - \frac{g_{\mathrm{BB}}}{m_F g_{\mathrm{FB}}}\right) - \frac{V_0}{g_{\mathrm{BF}}}\left(1 - \frac{g_{\mathrm{BB}}}{g_{\mathrm{BF}}}\right). \end{aligned} \tag{31}$$

For the frequencies ω_0 and ω_j and the constants $\mathcal{C}_0$ and $\mathcal{C}_j$ we have

$$\omega_0 = \frac{\alpha^2}{m_{\mathrm{B}}}, \qquad \omega_j = \frac{\alpha^2}{m_{\mathrm{F}}}, \qquad \mathcal{C}_0 = \mathcal{C}_j = 0. \tag{32}$$

6.3. Vector Bright-Dark Soliton Solutions

When $k \to 1$, $B_0 = -A_0$ and $B_j = 0$, we have

$$\begin{aligned} q_0(x) &= \frac{\sqrt{-A_0}}{\cosh(\alpha x)}, \qquad q_j(x) = \sqrt{A_j}\tanh(\alpha x) \\ \omega_0 &= \frac{\alpha^2}{2m_{\mathrm{B}}} - g_{\mathrm{BB}}A_0, \qquad \omega_j = V_0, \qquad \mathcal{C}_0 = \mathcal{C}_j = 0. \end{aligned}$$

The parameters A_0 and A_j are given by (31). In this case we have the following restrictions

$$\begin{aligned} &g_{\mathrm{BF}} > 0, \qquad g_{\mathrm{BB}} \geq \frac{(\alpha^2 - m_{\mathrm{B}}V_0)m_{\mathrm{F}}}{(\alpha^2 - m_{\mathrm{F}}V_0)m_{\mathrm{B}}} g_{\mathrm{BF}}, \qquad V_0 \geq \alpha^2/m_{\mathrm{F}} \\ &g_{\mathrm{BF}} < 0, \qquad g_{\mathrm{BB}} \leq \frac{(\alpha^2 - m_{\mathrm{B}}V_0)m_{\mathrm{F}}}{(\alpha^2 - m_{\mathrm{F}}V_0)m_{\mathrm{B}}} g_{\mathrm{BF}}, \qquad V_0 \leq \alpha^2/m_{\mathrm{F}}. \end{aligned}$$

6.4. Vector Dark-Bright Soliton Solutions

When $k \to 1$ and provided that $B_0 = 0$ and $B_j = -A_j$ the result is

$$q_0(x) = \sqrt{A_0}\tanh(\alpha x), \quad q_j(x) = \frac{\sqrt{-A_j}}{\cosh(\alpha x)}, \quad \omega_0 = V_0 + A_0 g_{\mathrm{BB}}, \quad \omega_j = \frac{\alpha^2}{2m_{\mathrm{F}}}.$$

By analogy with the previous examples the constants A_0, A_j, $\mathcal{C}_0$ and $\mathcal{C}_j$ are given by formulae (31) and (32), respectively. The restrictions now are

$$g_{\rm BF} > 0, \qquad g_{\rm BB} \geq \frac{(\alpha^2 - m_{\rm B}V_0)m_{\rm F}}{(\alpha^2 - m_{\rm F}V_0)m_{\rm B}} g_{\rm BF}, \qquad V_0 \leq \alpha^2/m_{\rm F}$$

$$g_{\rm BF} < 0, \qquad g_{\rm BB} \leq \frac{(\alpha^2 - m_{\rm B}V_0)m_{\rm F}}{(\alpha^2 - m_{\rm F}V_0)m_{\rm B}} g_{\rm BF}, \qquad V_0 \geq \alpha^2/m_{\rm F}.$$

6.5. Vector Dark-Dark-Bright Soliton Solutions

Let $B_0 = B_1 = 0$ and $B_j = -A_j$ where $j = 2, \ldots, N_f$. Therefore the solutions read

$$q_0(x) = \sqrt{A_0}\tanh(\alpha x), \quad q_1(x) = \sqrt{A_1}\tanh(\alpha x), \quad q_j(x) = \sqrt{-A_j}\,\mathrm{sech}(\alpha x).$$

Then we obtain for frequencies the following results

$$\omega_0 = V_0 + g_{\rm BB}A_0 + g_{\rm BF}A_1, \qquad \omega_1 = \frac{\alpha^2}{m_{\rm F}}, \qquad \omega_j = \frac{\alpha^2}{2m_{\rm F}} \cdot$$

These examples are by no means exhaustive.

6.6. Nontrivial Phase, Trigonometric Limit

In this section we consider a trap potential of the form $V_{\rm trap} = V_0\cos(2\alpha x)$, as a model for an optical lattice. Our potential V is similar and differs only with additive constant. When $k \to 0$, $\mathrm{sn}(\alpha x, 0) = \sin(\alpha x)$

$$q_0^2 = A_0\sin^2(\alpha x) + B_0, \qquad q_j^2 = A_j\sin^2(\alpha x) + B_j \tag{33}$$

$$V = V_0\sin^2(\alpha x) = \frac{1}{2}(V_0 - V_0\cos(2\alpha x)). \tag{34}$$

Using equations (11)–(13) again we obtain the following result when (see Table 3)

$$A_0 = -\frac{V_0}{g_{\rm BF}}, \qquad \sum_{j=1}^{N_f} A_j = -\frac{V_0}{g_{\rm BF}}\left(1 - \frac{g_{\rm BB}}{g_{\rm BF}}\right)$$

Table 3. $W = g_{\rm BF}m_{\rm F}W_{\rm B}/(m_{\rm B}W_{\rm F})$.

1	$\beta_0 \leq 0$	$\beta_j \leq 0$	$A_0 \geq 0$	$A_j \geq 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr g_{\rm BF}$	$V_0 \lessgtr 0$
2	$\beta_0 \leq 0$	$\beta_j \geq 1$	$A_0 \geq 0$	$A_j \leq 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \gtrless g_{\rm BF}$	$V_0 \lessgtr 0$
3	$\beta_0 \geq 1$	$\beta_j \leq 0$	$A_0 \leq 0$	$A_j \geq 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \gtrless g_{\rm BF}$	$V_0 \gtrless 0$
4	$\beta_0 \geq 1$	$\beta_j \geq 1$	$A_0 \leq 0$	$A_j \leq 0$	$g_{\rm BF} \gtrless 0$	$g_{\rm BB} \lessgtr g_{\rm BF}$	$V_0 \gtrless 0$

$$\omega_0 = \frac{1}{2m_{\rm B}}\alpha^2 + B_0 g_{\rm BB} + g_{\rm BF}\sum_{i=1}^{N_f} B_i, \qquad \omega_j = \frac{1}{2m_{\rm F}}\alpha^2 + g_{\rm BF}B_0$$
$$\mathcal{C}_0^2 = \alpha^2 B_0(A_0+B_0), \qquad \mathcal{C}_j^2 = \alpha^2 B_j(A_j+B_j)$$

where

$$\Theta_0(x) = \arctan\left(\sqrt{\frac{A_0+B_0}{B_0}}\tan(\alpha x)\right)$$
$$\Theta_j(x) = \arctan\left(\sqrt{\frac{A_j+B_j}{B_j}}\tan(\alpha x)\right).$$

This solution is the most important from the physical point of view [16].

7. Linear Stability, Preliminary Results

To analyze linear stability of our initial system of equations we seek solutions in the form

$$\psi_0(x,t) = (q_0(x) + \varepsilon\phi_0(x,t))\exp\left(-\frac{\mathrm{i}\omega_0}{\hbar}t + \mathrm{i}\Theta_0(x) + \mathrm{i}\kappa_0\right)$$
$$\psi_j(x,t) = (q_1(x) + \varepsilon\phi_j(x,t))\exp\left(-\frac{\mathrm{i}\omega_j}{\hbar}t + \mathrm{i}\Theta_1(x) + \mathrm{i}\kappa_1\right)$$

and obtain the following linearized equations

$$\hbar\begin{pmatrix}\mathbf{\Phi}_0\\ \mathbf{\Phi}_1\\ \vdots\\ \mathbf{\Phi}_{N_f}\end{pmatrix}_{,t} = \begin{pmatrix}\mathbf{\Lambda}_0 & \mathbf{U_1} & \mathbf{U_2} & \dots & \mathbf{U_{N_f}}\\ \mathbf{V_1} & \mathbf{\Lambda}_1 & 0 & \dots & 0\\ \mathbf{V_2} & 0 & \mathbf{\Lambda}_2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{V_{N_f}} & 0 & 0 & \dots & \mathbf{\Lambda}_{N_f}\end{pmatrix}\begin{pmatrix}\mathbf{\Phi}_0\\ \mathbf{\Phi}_1\\ \vdots\\ \mathbf{\Phi}_{N_f}\end{pmatrix}$$

$$\mathbf{\Phi}_0 = \begin{pmatrix}\phi_0^{\rm R}\\ \phi_0^{\rm I},\end{pmatrix}, \qquad \mathbf{\Phi}_j = \begin{pmatrix}\phi_j^{\rm R}\\ \phi_j^{\rm I}\end{pmatrix}$$

where

$$\mathbf{\Lambda}_0 = \begin{pmatrix}S_0 & L_{0,-}\\ L_{0,+} & S_0\end{pmatrix}, \qquad \mathbf{U}_j = \begin{pmatrix}0 & 0\\ U_{0,j} & 0\end{pmatrix}$$

and

$$\mathbf{\Lambda}_j = \begin{pmatrix}S_j & L_{j,-}\\ L_{j,+} & S_j\end{pmatrix}, \qquad \mathbf{V}_j = \begin{pmatrix}0 & 0\\ U_{1,j} & 0\end{pmatrix}$$

$$S_0 = -\frac{\mathcal{C}_0}{m_{\rm B}q_0}\partial_x\left(\frac{1}{q_0}\right), \qquad S_j = -\frac{\mathcal{C}_j}{m_{\rm F}q_j}\partial_x\left(\frac{1}{q_j}\right)$$

$$
\begin{aligned}
L_{0,-} &= -\frac{1}{2m_{\mathrm{B}}}\left(\partial_{xx}^2 - \frac{\mathcal{C}_0^2}{q_0^4}\right) + V + g_{\mathrm{BB}}q_0^2 + g_{\mathrm{BF}}q_1^2 - \omega_0 \\
L_{0,+} &= \frac{1}{2m_{\mathrm{B}}}\left(\partial_{xx}^2 - \frac{\mathcal{C}_0^2}{q_0^4}\right) - V - 3g_{\mathrm{BB}}q_0^2 - g_{\mathrm{BF}}q_1^2 + \omega_0 \\
L_{j,-} &= -\frac{1}{2m_{\mathrm{F}}}\left(\partial_{xx}^2 - \frac{\mathcal{C}_j^2}{q_0^4}\right) + V + g_{\mathrm{BF}}q_0^2 - \omega_j \\
L_{j,+} &= \frac{1}{2m_{\mathrm{F}}}\left(\partial_{xx}^2 - \frac{\mathcal{C}_j^2}{q_0^4}\right) - V - g_{\mathrm{BF}}q_0^2 + \omega_j \\
& U_{0,j} = -2g_{\mathrm{BF}}q_0^2, \qquad U_{1,j} = -2g_{\mathrm{BF}}q_0q_j.
\end{aligned}
$$

The analysis of the latter matrix system is a difficult problem and only numerical simulations are possible. Recently a great progress was achieved for analysis of linear stability of periodic solutions of type (5), (6) (see, e.g., [4, 5, 7, 11] and references therein). Nevertheless the stability analysis is known only for solutions of type (25)–(30) and solutions with nontrivial phase of type (33) and (34). Linear analysis of soliton solutions is well developed, but it is out scope of the present paper.

Finally we discuss three special cases:

Case I. Let $B_0 = B_j = 0$ then for $j = 1, \ldots, N_f$ and $q_0 = \sqrt{A_0}\,\mathrm{sn}(\alpha x, k)$, $q_j = \sqrt{A_j}\,\mathrm{sn}(\alpha x, k)$ we have the following linearized equations

$$
\begin{aligned}
\hbar\phi_{0,t}^{\mathrm{R}} &= -\frac{1}{2m_{\mathrm{B}}}\partial_{xx}^2\phi_0^{\mathrm{I}} + \left(V_0 + g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j\right)\mathrm{sn}^2(\alpha x, k)\phi_0^{\mathrm{I}} - \omega_0\phi_0^{\mathrm{I}} \\
\hbar\phi_{0,t}^{\mathrm{I}} &= \frac{1}{2m_{\mathrm{B}}}\partial_{xx}^2\phi_0^{\mathrm{R}} - \left(V_0 + 3g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j\right)\mathrm{sn}^2(\alpha x, k)\phi_0^{\mathrm{R}} \\
&\quad + \omega_0\phi_0^{\mathrm{R}} - 2g_{\mathrm{BF}}A_0\,\mathrm{sn}^2(\alpha x, k)\sum_j \phi_j^{\mathrm{R}} \\
\hbar\phi_{j,t}^{\mathrm{R}} &= -\frac{1}{2m_{\mathrm{F}}}\partial_{xx}^2\phi_j^{\mathrm{I}} + (V_0 + g_{\mathrm{BF}}A_0)\,\mathrm{sn}^2(\alpha x, k)\phi_j^{\mathrm{I}} - \omega_j\phi_j^{\mathrm{I}} \\
\hbar\phi_{j,t}^{\mathrm{I}} &= \frac{1}{2m_{\mathrm{F}}}\partial_{xx}^2\phi_j^{\mathrm{R}} - (V_0 + g_{\mathrm{BF}}A_0)\,\mathrm{sn}^2(\alpha x, k)\phi_j^{\mathrm{R}} + \omega_j\phi^{\mathrm{R}} \\
&\quad - 2g_{\mathrm{BF}}\sqrt{A_0A_j}\,\mathrm{sn}^2(\alpha x, k)\phi_0^{\mathrm{R}}.
\end{aligned}
$$

Case II. Let $B_0 = -A_0$, $B_j = -A_j$ then for $q_0 = \sqrt{-A_0}\,\mathrm{cn}(\alpha x, k)$, $q_j = \sqrt{-A_j}\,\mathrm{cn}(\alpha x, k)$ we obtain the following linearized equations

$$
\begin{aligned}
\hbar\phi^{\mathrm{R}}_{0,t} = & -\frac{1}{2m_{\mathrm{B}}}\partial^2_{xx}\phi^{\mathrm{I}}_0 + \left(V_0 + g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j\right)\mathrm{sn}^2(\alpha x, k)\phi^{\mathrm{I}}_0 \\
& - \left(g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j + \omega_0\right)\phi^{\mathrm{I}}_0 \\
\hbar\phi^{\mathrm{I}}_{0,t} = & \frac{1}{2m_{\mathrm{B}}}\partial^2_{xx}\phi^{\mathrm{R}}_0 + \left(3g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j + \omega_0\right)\phi^{\mathrm{R}}_0 \\
& - \left(V_0 + 3g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j\right)\mathrm{sn}^2(\alpha x, k)\phi^{\mathrm{R}}_0 \\
& + 2g_{\mathrm{BF}}A_0(1 - \mathrm{sn}^2(\alpha x, k))\sum_j \phi^{\mathrm{R}}_j \\
\hbar\phi^{\mathrm{R}}_{j,t} = & -\frac{1}{2m_{\mathrm{F}}}\partial^2_{xx}\phi^{\mathrm{I}}_j + (V_0 + g_{\mathrm{BF}}A_0)\,\mathrm{sn}^2(\alpha x, k)\phi^{\mathrm{I}}_j - (g_{\mathrm{BF}}A_0 + \omega_j)\phi^{\mathrm{I}}_j \\
\hbar\phi^{\mathrm{I}}_{j,t} = & \frac{1}{2m_{\mathrm{F}}}\partial^2_{xx}\phi^{\mathrm{R}}_j - (V_0 + g_{\mathrm{BF}}A_0)\,\mathrm{sn}^2(\alpha x, k)\phi^{\mathrm{R}}_j + (g_{\mathrm{BF}}A_0 + \omega_j)\phi^{\mathrm{R}}_j \\
& - 2g_{\mathrm{BF}}\sqrt{A_0A_j}(1 - \mathrm{sn}^2(\alpha x, k))\phi^{\mathrm{R}}_0, \qquad j = 1, \ldots, N_f.
\end{aligned}
$$

Case III. Let $B_0 = -A_0/k^2$, $B_j = -A_j/k^2$ therefore the solutions are

$$
q_0 = \sqrt{-A_0}\,\mathrm{dn}(\alpha x, k)/k, \qquad q_j = \sqrt{-A_j}\,\mathrm{dn}(\alpha x, k)/k
$$

and we obtain the following linearized equations

$$
\begin{aligned}
\hbar\phi^{\mathrm{R}}_{0,t} = & -\frac{1}{2m_{\mathrm{B}}}\partial^2_{xx}\phi^{\mathrm{I}}_0 + \left(V_0 + g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j\right)\mathrm{sn}^2(\alpha x, k)\phi^{\mathrm{I}}_0 \\
& - \left(g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j + k^2\omega_0\right)\frac{\phi^{\mathrm{I}}_0}{k^2} \\
\hbar\phi^{\mathrm{I}}_{0,t} = & \frac{1}{2m_{\mathrm{B}}}\partial^2_{xx}\phi^{\mathrm{R}}_0 + \left(3g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j + k^2\omega_0\right)\frac{\phi^{\mathrm{R}}_0}{k^2} \\
& - \left(V_0 + 3g_{\mathrm{BB}}A_0 + g_{\mathrm{BF}}\sum_j A_j\right)\mathrm{sn}^2(\alpha x, k)\phi^{\mathrm{R}}_0
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2g_{\mathrm{BF}}A_0(1-k^2\,\mathrm{sn}^2(\alpha,k))}{k^2}\sum_j \phi_j^{\mathrm{R}} \\
\hbar\phi_{j,t}^{\mathrm{R}} =& -\frac{1}{2m_{\mathrm{F}}}\partial_{xx}^2\phi_j^{\mathrm{I}} + (V_0+g_{\mathrm{BF}}A_0)\,\mathrm{sn}^2(\alpha x,k)\phi_j^{\mathrm{I}} \\
&-\frac{g_{\mathrm{BF}}A_0+k^2\omega_j}{k^2}\phi_j^{\mathrm{I}} \\
\hbar\phi_{j,t}^{\mathrm{I}} =& \frac{1}{2m_{\mathrm{F}}}\partial_{xx}^2\phi_j^{\mathrm{R}} - (V_0+g_{\mathrm{BF}}A_0)\,\mathrm{sn}^2(\alpha x,k)\phi_j^{\mathrm{R}} + \frac{g_{\mathrm{BF}}A_0+k^2\omega_j}{k^2}\phi_j^{\mathrm{R}} \\
&-\frac{2g_{\mathrm{BF}}\sqrt{A_0A_j}(1-k^2\,\mathrm{sn}^2(\alpha,k))\phi_0^{\mathrm{R}}}{k^2}, \qquad j=1,\ldots,N_f.
\end{aligned}
$$

These cases are by no means exhaustive.

8. Conclusions

In conclusion, we have considered the mean field model for boson-fermion mixtures in two optical lattices. Classes of quasi-periodic, periodic, elliptic solutions have been analyzed. These solutions can be used as initial states which can generate localized matter waves (solitons) through the modulational instability mechanism. This important problem is under consideration.

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