N-WAVE TYPE SYSTEMS AND THEIR GAUGE EQUIVALENT RELATED TO THE ORTHOGONAL ALGEBRAS

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Abstract. The reductions of the integrable N-wave type equations solvable by the inverse scattering method with the generalized Zakharov–Shabat system L and related to some simple Lie algebra $\mathfrak{g}$ are analyzed. Special attention is paid to the $\mathbb{Z}_2$-reductions including ones that can be embedded also in the Weyl group of $\mathfrak{g}$. The consequences of these restrictions on the structure of the dresing factors are outlined. An example of 4-wave equations (with application to nonlinear optics) and its gauge equivalent are given.

1. Introduction

The aim of the present paper is to study the class of N-wave equations [1, 8, 11–13], their generalizations to simple Lie algebras [2, 5] and their gauge equivalent ones extending the results in [6]. We describe their scattering data, dressing factors, 1-soliton solutions and outline some of their reductions.

The N-wave type equations related to the simple Lie algebras can be solved by applying the inverse scattering method for the generalized Zakharov–Shabat system [5]:

$$L(\lambda)\psi \equiv \left(i\frac{\mathrm{d}}{\mathrm{d}x} + [J, Q(x,t)] - \lambda J \right) \psi(x,t,\lambda) = 0 \tag{1}$$

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where $J = \sum_{k=1}^{r} J_k H_{e_k}$ belongs to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the potential matrix

$$Q(x,t) = \sum_{\alpha>0} q_\alpha(x,t)E_\alpha + p_\alpha(x,t)E_{-\alpha} \qquad (2)$$

takes values in $\mathfrak{g}/\mathfrak{h}$. Here $E_{\pm\alpha}$ are the root vectors of the simple Lie algebra $\mathfrak{g}$, $r = \text{rank}\,\mathfrak{g}$ and Δ_+ is the set of positive roots of $\mathfrak{g}$. The ISM allows us to write the considered N-wave system as a compatibility condition

$$[L(\lambda), M(\lambda)] = 0 \qquad (3)$$

for the pair of Lax operators $L(\lambda)$ and $M(\lambda)$, where

$$M(\lambda)\psi \equiv \left(i\frac{\mathrm{d}}{\mathrm{d}t} + [I, Q(x,t)] - \lambda I\right)\psi(x,t,\lambda) = 0, \qquad (4)$$

and $I = \sum_{k=1}^{r} I_k H_{e_k} \in \mathfrak{h}$. The N-wave system related to $\mathfrak{g}$ has the form:

$$i[J, Q_t] - i[I, Q_x] + [[I, Q], [J, Q]] = 0. \qquad (5)$$

The zero-curvature condition (3) is invariant under the action of the group of gauge transformations [15]. Therefore the gauge equivalent systems are again completely integrable, posses hierarchy of Hamiltonian structures, etc ([1, 6, 13, 15]).

In Section 2 we describe the general form of the gauge equivalent N-wave systems. In Section 3 we reformulate the Riemann–Hilbert problem (RHP) for the gauge equivalent systems, introduce the scattering data and describe their time evolution. In Section 4 we outline folowing [3, 4] the consequences of the $\mathbb{Z}_2$ reductions for the gauge equivalent systems. In Section 5 we extend the Zakharov–Shabat dressing method [6, 14, 12, 13] for the gauge equivalent systems related to the orthogonal algebras and provide the general form of their 1-soliton solutions. These results are applied on the example of 4-wave system related to the algebra $\mathbf{B}_2 \simeq so(5)$ in Section 6.

2. General Form of the Gauge Equivalent Systems

Let us first fix the notation and the normalizations of the basis of $\mathfrak{g}$. By Δ_+ (Δ_-) we denote the set of positive (negative) roots of the algebra with respect to the ordering provided by J, namely $\alpha \in \Delta_\pm$ if $\alpha(J) \gtrless 0$. By $\{E_\alpha, H_i\}$, $\alpha \in \Delta$, $i = 1, \ldots, r$ we denote the Cartan–Weyl basis of $\mathfrak{g}$ with the standard commutation relations, see [7].

Let us now go to the gauge equivalent systems. The notion of gauge equivalence allows one to associate with the N-wave system an equivalent one [6] solvable by the inverse scattering method for the gauge equivalent linear problem:

$$\tilde{L}\tilde{\psi}(x,t,\lambda) \equiv \left(i\frac{d}{dx} - \lambda S\right)\tilde{\psi}(x,t,\lambda) = 0,$$

$$\tilde{M}\tilde{\psi}(x,t,\lambda) \equiv \left(i\frac{d}{dt} - \lambda f(S)\right)\tilde{\psi}(x,t,\lambda) = 0 \tag{6}$$

where $\tilde{\psi}(x,t,\lambda) = g^{-1}(x,t)\psi(x,t,\lambda)$,

$$S = \text{Ad}_g J \equiv g^{-1}(x,t)Jg(x,t), \tag{7}$$

and $g(x,t) = \psi(x,t,0)$ is the Jost solution at $\lambda = 0$. The zero-curvature condition $[\tilde{L}, \tilde{M}] = 0$ gives:

$$S_t - \frac{d}{dx}f(S) = 0 \tag{8}$$

where $f(S) = \sum_{p=0}^{r-1} \alpha_p S^{2p+1}$ is an odd polynomial of S. It is natural that $f(S) = g^{-1}(x,t)Ig(x,t)$, i. e., it is uniquely determined by I. Both J and I belong to the Cartan subalgebra $\mathfrak{h}$ so they have common set of eigenspaces. In order to express $f(S)$ through their eigenvalues J_k and I_k we introduce the diagonal matrix-valued functions:

$$f_k(J) = \frac{J}{J_k}\prod_{s\neq k}\frac{J^2 - J_s^2}{J_k^2 - J_s^2} = H_{e_k} \in \mathfrak{h} \tag{9}$$

where by H_{e_k} we denote the element in $\mathfrak{h}$ dual to the basis vector e_k in the root space of $\mathfrak{g}$. Using (9) and applying Ad_g we get:

$$I = \sum_{k=1}^{r} I_k f_k(J), \tag{10}$$

$$f(S) \equiv g^{-1}(x,t)Ig(x,t) = \sum_{k=1}^{r} I_k f_k(S). \tag{11}$$

In addition $S(x,t)$ satisfies the characteristic equations:

$$S^{\kappa_0}\prod_{k=1}^{r}(S^2 - J_k^2) = 0, \tag{12}$$

where $\kappa_0 = 0$ if $\mathfrak{g} \simeq C_r$ or D_r and $\kappa_0 = 1$, if $\mathfrak{g} \simeq B_r$.

Then the equation gauge equivalent to (1) becomes:

$$S_t - \alpha_0 S_x - \sum_{p=1}^{r-1} \alpha_p (S^{2p+1})_x = 0 \,. \tag{13}$$

The function $S(x, t) \in \mathfrak{g}$ is also subject to constraints; one of them is provided by (12). To construct the others we assume that $\mathfrak{g} \simeq B_r$ or D_r and use the typical representation of $\mathfrak{g}$. It this settings we easily see that all odd powers of H_{e_k} also belong to the Cartan subalgebra $\mathfrak{h}$. Thus we conclude that all odd powers of S also belong to $\mathfrak{g}$. The invariance properties of the trace lead to:

$$\mathrm{trace}(J^{2k}) \equiv 2 \sum_{k=1}^{r} J_k^{2k} = \mathrm{trace}(S)^{2k} \,, \tag{14}$$

for $k = 1, \ldots, r$. The conditions (14) are precisely r independent algebraic constraints on S. Solving for them we conclude that the number of independent coefficients in S is equal to the number of roots $|\Delta|$ of $\mathfrak{g}$.

3. Fundamental Analytic Solutions and Scattering Data for Gauge Equivalent Systems

The direct scattering problem for the Lax operator (1) is based on the Jost solutions:

$$\lim_{x \to \infty} \psi(x, \lambda) \, e^{i\lambda J x} = \mathbb{1}, \qquad \lim_{x \to -\infty} \phi(x, \lambda) \, e^{i\lambda J x} = \mathbb{1} \,, \tag{15}$$

and the scattering matrix:

$$T(\lambda) = (\psi(x, \lambda))^{-1} \phi(x, \lambda) \,. \tag{16}$$

The fundamental analytic solutions (FAS) $\xi^{\pm}(x, \lambda)$ of $L(\lambda)$ are analytic functions of λ for $\lambda \gtrless 0$ and are related to the Jost solutions by [5]

$$\xi^{\pm}(x, \lambda) = \phi(x, \lambda) S^{\pm}(\lambda) = \psi(x, \lambda) T^{\mp}(\lambda) D^{\pm}(\lambda) \tag{17}$$

where $T^{\pm}(\lambda)$, $S^{\pm}(\lambda)$ and $D^{\pm}(\lambda)$ are the factors of the Gauss decomposition of the scattering matrix:

$$T(\lambda) = T^{-}(\lambda) D^{+}(\lambda) \hat{S}^{+}(\lambda) = T^{+}(\lambda) D^{-}(\lambda) \hat{S}^{-}(\lambda) \,, \tag{18}$$

where $\hat{S} \equiv S^{-1}$, the superscripts + (resp. −) in $T^{\pm}(\lambda)$ and $S^{\pm}(\lambda)$ mean upper- (resp. lower-)triangularity; for the diagonal factors $D^{\pm}(\lambda)$ these superscripts mean that $D^{\pm}$ are analytic functions of λ for $\mathrm{Im}\,\lambda > 0$ and $\mathrm{Im}\,\lambda < 0$ respectively.

On the real axis $\xi^+(x, \lambda)$ and $\xi^-(x, \lambda)$ are related by

$$\xi^+(x, \lambda) = \xi^-(x, \lambda)G_0(\lambda), \qquad G_0(\lambda) = S^+(\lambda)\hat{S}^-(\lambda), \qquad (19)$$

and the function $G_0(\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues of (1) [10, 5].

If the potential $Q(x, t)$ of $L(\lambda)$ (1) satisfies equation (5) then $S^\pm(\lambda)$ and $T^\pm(\lambda)$ satisfy the linear equation:

$$\mathrm{i}\frac{\mathrm{d}S^\pm}{\mathrm{d}t} - \lambda[I, S^\pm] = 0, \qquad \mathrm{i}\frac{\mathrm{d}T^\pm}{\mathrm{d}t} - \lambda[I, T^\pm] = 0, \qquad (20)$$

while the functions $D^\pm(\lambda)$ are time-independent. In other words $D^\pm(\lambda)$ can be considered as the generating functions of the integrals of motion of (5).

In order to determine the scattering data for the gauge equivalent equations we need to start with the FAS for these systems:

$$\tilde{\xi}^\pm(x, \lambda) = g^{-1}(x, t)\xi^\pm(x, \lambda)g_- \qquad (21)$$

where $g_- = \lim_{x \to -\infty} g(x, t)$ and due to (16) and $g_- = \hat{T}(0)$. In order to ensure that the functions $\tilde{\xi}^\pm(x, \lambda)$ are analytic with respect to λ the scattering matrix $T(0)$ at $\lambda = 0$ must belong to the corresponding Cartan subgroup $\mathfrak{H}$. Then Equation (21) provide the fundamental analytic solutions of $\tilde{L}$. We can calculate their asymptotics for $x \to \pm\infty$ and thus establish the relations between the scattering matrices of the two systems:

$$\lim_{x \to -\infty} \tilde{\xi}^+(x, \lambda) = \mathrm{e}^{-\mathrm{i}\lambda Jx}T(0)S^+(\lambda)\hat{T}(0) \qquad (22)$$

$$\lim_{x \to \infty} \tilde{\xi}^+(x, \lambda) = \mathrm{e}^{-\mathrm{i}\lambda Jx}T^-(\lambda)D^+(\lambda)\hat{T}(0) \qquad (23)$$

with the result:

$$\tilde{T}(\lambda) = T(\lambda)\hat{T}(0). \qquad (24)$$

Obviously $\tilde{T}(0) = \mathbb{1}$. The factors in the corresponding Gauss decompositions are related by:

$$\tilde{S}^\pm(\lambda) = T(0)S^\pm(\lambda)\hat{T}(0), \qquad \tilde{T}^\pm(\lambda) = T^\pm(\lambda)$$
$$\tilde{D}^\pm(\lambda) = D^\pm(\lambda)\hat{T}(0). \qquad (25)$$

On the real axis again the FAS $\tilde{\xi}^+(x, \lambda)$ and $\tilde{\xi}^-(x, \lambda)$ are related by:

$$\tilde{\xi}^+(x, \lambda) = \tilde{\xi}^-(x, \lambda)\tilde{G}_0(\lambda) \qquad (26)$$

with the normalization condition $\tilde{\xi}(x, \lambda = 0) = \mathbb{1}$ and $\tilde{G}_0(\lambda) = \tilde{S}^+(\lambda)\hat{\tilde{S}}^-(\lambda)$ again can be considered as a minimal set of scattering data.

4. $\mathbb{Z}_2$-reductions

The numerous $\mathbb{Z}_2$-reductions have been recently classified for the N-wave equations [3,4] using the reduction group introduced by Mikhailov [9]. They can easily be reformulated for the gauge equivalent systems. Here we briefly outline the main steps in this. In [4] we studied four type of reductions:

$$\text{a)} \quad C_1\Big(U^\dagger(x, t, \eta\lambda^*)\Big) = U(x, t, \lambda)\,, \qquad \eta = \pm 1\,, \qquad (27)$$

$$\text{b)} \quad C_2\Big(U^T(x, t, -\lambda)\Big) = -U(x, t, \lambda)\,, \qquad\qquad\qquad (28)$$

$$\text{c)} \quad C_3\Big(U^*(x, t, \eta\lambda^*)\Big) = -U(x, t, \lambda)\,, \qquad \eta = \pm 1\,, \qquad (29)$$

$$\text{d)} \quad C_4\Big(U(x, t, \eta\lambda)\Big) = U(x, t, \lambda)\,, \qquad \eta = \pm 1 \qquad\quad (30)$$

where $U(x, t, \lambda) = [J, Q(x, t)] - \lambda J$ is the potential part of the Lax operator (1) and C_k, $k = 1, \ldots, 4$ are involutive automorphisms of the Lie algebra $\mathfrak{g}$. The reductions for the gauge equivalent systems are obtained from (27-30) by replacing $U(x, t, \lambda)$ by $\lambda S(x, t)$. In order to describe their effect on the coefficients of $S(x, t)$ we parametrize it by:

$$S(x, t) = \sum_{k=1}^{r} s_k(x, t)H_{e_k} + \sum_{\alpha \in \Delta} S_\alpha(x, t)E_\alpha\,. \qquad (31)$$

These coefficients are subject to the constraints (14). For example, we have:

$$\frac{1}{2}\,\text{trace}\,S^2(x, t) = (\vec{s}, \vec{s}) + \sum_{\alpha \in \Delta^+} \frac{4}{(\alpha, \alpha)}\,S_\alpha(x, t)S_{-\alpha}(x, t) = \sum_{k=1}^{r} J_k^2 \qquad (32)$$

where by $\vec{s}(x, t)$ we have denoted the r-component vector $\vec{s}(x, t) = (s_1, \ldots, s_r)(x, t)$ and $(\vec{s}, \vec{s}) = \sum_{k=1}^{r} s_k^2$.

The automorphisms C_k that we will use below will be of two types: elements of the Cartan subgroup (type 1) or of the Weyl group $W_\mathfrak{g}$ of $\mathfrak{g}$ (type 2). For the type 1 reductions we will use

$$K = \exp H_{\vec{k}}, \qquad K^{-1}E_\alpha K = e^{-(\vec{k}, \alpha)}E_\alpha \qquad (33)$$

where $\vec{k}$ is the vector in the root space dual to $H_{\vec{k}} \in \mathfrak{h}$. Such reductions impose on $S(x, t)$ the following constraints:

$$\text{1a)} \quad s_k^* = \eta s_k\,, \qquad S_{-\alpha} = \eta\,e^{(\vec{k}, \alpha)}S_\alpha^*\,, \qquad (34)$$

1b) $\quad S_{-\alpha} = \mathrm{e}^{(\vec{k},\alpha)} S_\alpha \,,$ $\hfill$ (35)

1c) $\quad s_k^* = -\eta s_k \,,$ $\qquad$ $S_\alpha = -\eta\, \mathrm{e}^{-(\vec{k},\alpha)} S_\alpha^* \,,$ $\hfill$ (36)

1d) $\quad S_\alpha = \mathrm{e}^{-(\vec{k},\alpha)} S_\alpha \,.$ $\hfill$ (37)

In the cases 1b) and 1d) the reduction does not affect $s_k(x,t)$.

In applying the type 2 reductions we have to keep in mind that the Weyl reflection w_α with respect to the root α acts on the Cartan–Weyl basis as follows:

$$w_\alpha(H_{\vec{k}}) = H_{w_\alpha(\vec{k})} \,, \qquad w_\alpha(E_\beta) = n_{\alpha,\beta} E_{\beta'} \qquad (38)$$

where $\beta' = w_\alpha(\beta) = \beta - \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ and $n_{\alpha,\beta}$ take values ± 1. The restrictions on $S(x,t)$ imposed by the type 2 reductions are as follows:

2a) $\quad w_\alpha(\vec{s}^*) = \eta\vec{s},$ $\qquad$ $S_{\beta'} = \eta n_{\alpha,\beta} S_{-\beta}^*,$ $\hfill$ (39)

2b) $\quad w_\alpha(\vec{s}) = \vec{s},$ $\qquad$ $S_{\beta'} = n_{\alpha,\beta} S_{-\beta} \,,$ $\hfill$ (40)

2c) $\quad w_\alpha(\vec{s}^*) = -\eta\vec{s},$ $\qquad$ $S_{\beta'} = -\eta n_{\alpha,\beta} S_\beta^*,$ $\hfill$ (41)

2d) $\quad w_\alpha(\vec{s}) = \eta\vec{s},$ $\qquad$ $S_{\beta'} = \eta n_{\alpha,\beta} S_\beta \,.$ $\hfill$ (42)

5. Dressing Factors and 1-Soliton Solutions

The main idea of the dressing method is starting from a FAS $\tilde{\xi}_{(0)}^\pm(x,\lambda)$ of $\tilde{L}$ with potential $\mathcal{S}_{(0)}$ to construct a new singular solution $\tilde{\xi}_{(1)}^\pm(x,\lambda)$ of the RHP (26) with singularities located at prescribed positions $\lambda_1^\pm$. Then the new solutions $\tilde{\xi}_{(1)}^\pm(x,\lambda)$ will correspond to a potential $\mathcal{S}_{(1)}$ of $\tilde{L}$ with two discrete eigenvalues $\lambda_1^\pm$. It is related to the regular one by the dressing factors $\tilde{u}(x,\lambda)$:

$$\begin{aligned} \tilde{\xi}_{(1)}^\pm(x,\lambda) &= \tilde{u}(x,\lambda)\tilde{\xi}_{(0)}^\pm(x,\lambda)\tilde{u}_-^{-1}(\lambda)\,, \\ \tilde{u}_-(\lambda) &= \lim_{x\to-\infty} \tilde{u}(x,\lambda)\,, \end{aligned} \qquad (43)$$

and the dressing factors for the gauge equivalent equations $\tilde{u}(x,\lambda)$ are related to $u(x,\lambda)$ by

$$\tilde{u}(x,\lambda) = g_{(0)}^{-1}(x,t)u^{-1}(x,\lambda=0)u(x,\lambda)g_{(0)} \,. \qquad (44)$$

If $\mathfrak{g} \simeq \mathbf{A}_r$ then the gauge equivalent dressing factors are

$$\tilde{u}(x, \lambda) = \mathbb{1} + \left(\frac{c_1(\lambda)}{c_1(0)} - 1 \right) P_1, \qquad c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}$$

$$P_1(x) = \frac{|n(x)\rangle\langle m(x)|}{\langle m(x)|n(x)\rangle}, \tag{45}$$

$$|n(x)\rangle = \xi_0^+(\lambda_1^+)|n_0\rangle, \qquad \langle m(x)| = \langle m_0|\hat{\xi}_0^-(\lambda_1^-)$$

where $|n_0\rangle$ and $\langle m_0|$ are constant vectors and these dressing factors satisfy the equation:

$$i\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} - \lambda S_{(1)}\tilde{u} + \lambda \tilde{u} S_{(0)} = 0. \tag{46}$$

If $\mathfrak{g} \simeq \mathbf{B}_r, \mathbf{D}_r$ the dressing factors take the form [3]:

$$u(x, \lambda) = \mathbb{1} + (c_1(\lambda) - 1)P_1 + (c_1^{-1}(\lambda) - 1)P_{-1} \tag{47}$$

$$\tilde{u}(x, \lambda) = \mathbb{1} + \left(\frac{c_1(\lambda)}{c_1(0)} - 1 \right) \tilde{P}_1 + \left(\frac{c_1(0)}{c_1(\lambda)} - 1 \right) \tilde{P}_{-1} \tag{48}$$

where $P_{-1}(x) = \mathbf{S}_0 P_1^{\mathsf{T}}(x)\mathbf{S}_0^{-1}$, $P_1(x)$ is the rank one projector (45), $\tilde{P}_{\pm 1} = g_{(0)}^{-1} P_{\pm 1} g_{(0)}(x, t)$. If $\mathfrak{g} \simeq \mathbf{B}_r$ then $N = 2r + 1$,

$$\mathbf{S}_0 = \sum_{k=1}^{r} (-1)^{k+1}(E_{k\bar{k}} + E_{\bar{k}k}) + (-1)^r E_{r+1,r+1}; \tag{49}$$

$\bar{k} = N - k + 1$, $(E_{km})_{il} = \delta_{ik}\delta_{ml}$; if $\mathfrak{g} \simeq \mathbf{D}_r$ then $N = 2r$ and

$$\mathbf{S}_0 = \sum_{k=1}^{r} (-1)^{k+1}(E_{k\bar{k}} + E_{\bar{k}k}). \tag{50}$$

If the dressing factors of the gauge equivalent equations satisfy (46) then the projectors $\tilde{P}_{\pm 1}$ satisfy the equations:

$$i\frac{\mathrm{d}\tilde{P}_1}{\mathrm{d}x} + \lambda_1^- \tilde{P}_1 S_{(0)} - \lambda_1^- S_{(1)}\tilde{P}_1 = 0,$$

$$i\frac{\mathrm{d}\tilde{P}_{-1}}{\mathrm{d}x} + \lambda_1^+ \tilde{P}_{-1} S_{(0)} - \lambda_1^+ S_{(1)}\tilde{P}_{-1} = 0, \tag{51}$$

and the **"dressed" potential** can be obtained by:

$$S_{(1)} = S_{(0)} + i\frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ \lambda_1^-} \frac{\mathrm{d}}{\mathrm{d}x}(\tilde{P}_1(x) - \tilde{P}_{-1}(x)). \tag{52}$$

The dressing factors can be written in the form:

$$\tilde{u}(x,\lambda) = \exp\left[\ln\frac{c_1(\lambda)}{c_1(0)}\,\tilde{p}(x)\right], \tag{53}$$

where $\tilde{p}(x) = \tilde{P}_1 - \tilde{P}_{-1} \in \mathfrak{g}$ and consequently $\tilde{u}(x,\lambda)$ belongs to the corresponding orthogonal group.

Making use of the explicit form of the projectors $P_{\pm 1}(x)$ valid for the typical representation of $\mathbf{B}_r$ we have [3]

$$\tilde{p}(x) = \frac{2}{\langle m|n\rangle}\sum_{k=1}^{r}\tilde{h}_k(x)H_{e_k} + \frac{2}{\langle m|n\rangle}\sum_{\alpha\in\Delta_+}(\tilde{P}_\alpha(x)E_\alpha + \tilde{P}_{-\alpha}(x)E_{-\alpha}) \tag{54}$$

where we assumed $S_{(0)} = J$, $g_{(0)} = \mathbb{1}$. Thus

$$\tilde{h}_k(x,t) = n_{0,k}m_{0,k}\,\mathrm{e}^{2\nu_1 y_k} - n_{0,\bar{k}}m_{0,\bar{k}}\,\mathrm{e}^{-2\nu_1 y_k},$$

$$\langle m|n\rangle = \sum_{k=1}^{r}(n_{0,k}m_{0,k}\,\mathrm{e}^{2\nu_1 y_k} + n_{0,\bar{k}}m_{0,\bar{k}}\,\mathrm{e}^{-2\nu_1 y_k}) + n_{0,r+1}m_{0,r+1},$$

$$\tag{55}$$

$$\tilde{P}_\alpha = \begin{cases} \tilde{P}_{ks}, & \text{for } \alpha = e_k - e_s \\ \tilde{P}_{k\bar{s}}, & \text{for } \alpha = e_k + e_s \\ \tilde{P}_{k,r+1}, & \text{for } \alpha = e_k. \end{cases}$$

Here $1 \leq k,s \leq r$, $\mu_1 = \mathrm{Re}\,\lambda_1^+$, $\nu_1 = \mathrm{Im}\,\lambda_1^+$ and

$$\tilde{P}_{ks} = \mathrm{e}^{\mathrm{i}\mu_1(y_s-y_k)}\left(n_{0,k}m_{0,s}\,\mathrm{e}^{\nu_1(y_s+y_k)} - (-1)^{k+s}n_{0,\bar{s}}m_{0,\bar{k}}\,\mathrm{e}^{-\nu_1(y_s+y_k)}\right)$$

$$y_k = J_k x + I_k t, \qquad y_{\bar{k}} = -y_k, \qquad y_{r+1} = 0. \tag{56}$$

The corresponding result for the $\mathbf{D}_r$ series is obtained formally if in the above expressions (55) and (56) we put $n_{0,r+1} = m_{0,r+1} = 0$. Thus $\tilde{P}_{k,r+1} = \tilde{P}_{r+1,k} = 0$ and the last term in the right hand side of $\langle m|n\rangle$ (55) is missing.

The N-soliton solutions can be obtained by applying successively N times the dressing procedure.

It is easy to determine the effect of each of the reductions (27–30) on the fundamental analytic solutions and on the scattering matrix of both $L(\lambda)$ and $\tilde{L}(\lambda)$. Here we will only formulate the properties of the dressing factors (32) and (33):

$$\text{a)}\quad C_1\Big(u^\dagger(x,\eta\lambda^*)\Big) = u^{-1}(x,\lambda), \tag{57}$$

$$\text{b)}\quad C_2\Big(u^T(x,-\lambda)\Big) = u^{-1}(x,\lambda), \tag{58}$$

$$\text{c)}\quad C_3\Big(u^*(x,\eta\lambda^*)\Big) = u(x,\lambda), \tag{59}$$

$$\text{d)} \quad C_4\Big(u(x,\eta\lambda)\Big) = u(x,\lambda)\,. \tag{60}$$

Obviously the concrete restrictions on the corresponding eigenvalues λ_1^+ and λ_1^- and on the projectors $P_{\pm 1}$ will depend on the choice of the automorphisms C_k. Skipping the details we formulate some of them:

$$\text{1a)} \quad \lambda_1^- = \eta(\lambda_1^+)^*, \qquad K^{-1}P_{\pm 1}^\dagger K = P_{\pm 1}\,, \qquad |m\rangle = K|n^*\rangle \tag{61}$$

where $K = K^*$;

$$\text{1b)} \quad \lambda_1^- = -\lambda_1^+\,, \qquad K^{-1}P_{\pm 1}^T K = P_{\pm 1}\,, \qquad |m\rangle = K|n\rangle \tag{62}$$

$$\text{1c)} \quad \lambda_1^- = \eta(\lambda_1^+)^*\,, \qquad K^{-1}P_{-1}^\dagger K = P_1\,, \qquad |m\rangle = \mathbf{S}_0 K|n^*\rangle \tag{63}$$

$$\text{2a)} \quad \lambda_1^- = \eta(\lambda_1^+)^*\,, \qquad w_\alpha(P_{\pm 1}^\dagger) = P_{\pm 1}\,, \qquad |m\rangle = \tilde{w}_\alpha|n^*\rangle\,, \tag{64}$$

$$\text{2b)} \quad \lambda_1^- = -\lambda_1^+\,, \qquad w_\alpha(P_{\pm 1}^T) = P_{\pm 1}\,, \qquad |m\rangle = \tilde{w}_\alpha|n\rangle\,, \tag{65}$$

$$\text{2c)} \quad \lambda_1^- = \eta(\lambda_1^+)^*\,, \qquad w_\alpha(P_1^*) = P_{-1}\,, \qquad |m\rangle = \mathbf{S}_0\tilde{w}_\alpha|n^*\rangle\,. \tag{66}$$

Here we have made use of the fact that to each element $w_\alpha \in W_\mathfrak{g}$ we can relate an inner automorphism of $\mathfrak{g}$, i. e., there exist a nondegenerate matrix $\tilde{w}_\alpha$ belonging to the group $\mathfrak{G}$ and such that:

$$w_\alpha(X) \equiv \tilde{w}_\alpha X \tilde{w}_\alpha^{-1} \tag{67}$$

for each element $X \in \mathfrak{g}$.

Applying the above restrictions to (32) and (33) we will get dressing factors satisfying automatically the corresponding reduction conditions. Finally the corresponding soliton solutions can be recovered from (37).

The dressing factors (32) and (33) are the simplest possible ones if we choose the algebra $\mathfrak{g}$ to an orthogonal one. More complicated dressing factors should contain at least four poles and zeroes in λ whose residues can again be reconstructed from the 'bare' solutions of $L(\lambda)$. These problems will be addressed in future publications.

6. Examples

Let us give some examples of the above constructions. As such we will use the 4-wave equations related to the $\mathbf{B}_2 \simeq so(5)$ algebra and their gauge equivalent. We also construct their one-soliton solutions.

The algebra $so(5)$ has four positive roots: $e_1 \pm e_2$, e_1 and e_2. The corresponding 4-wave system subject to the reduction (27) with $C_1(X) = K^{-1}XK$ and

$K = \text{diag}(K_1, K_2, 1, K_2, K_1)$, $K_i = \pm 1$ and $\eta = 1$ has the form:

$$
\begin{aligned}
\mathrm{i}(J_1 - J_2)q_{10,t} - \mathrm{i}(I_1 - I_2)q_{10,x} + 2\kappa K_1 K_2 q_{11}q_{01}^* &= 0, \\
\mathrm{i}J_2 q_{01,t} - \mathrm{i}I_2 q_{01,x} + \kappa(K_1 K_2 q_{11}^* q_{12} + q_{11}q_{10}^*) &= 0, \\
\mathrm{i}J_1 q_{11,t} - \mathrm{i}I_1 q_{11,x} + \kappa(K_1 K_2 q_{12}q_{01}^* - q_{10}q_{01}) &= 0, \\
\mathrm{i}(J_1 + J_2)q_{12,t} - \mathrm{i}(I_1 + I_2)q_{12,x} - 2\kappa q_{11}q_{01} &= 0
\end{aligned}
\tag{68}
$$

where $\kappa = J_1 I_2 - J_2 I_1$ and the subscripts 10, 01, 11 and 12 refer to the roots $e_1 - e_2$, e_1, e_2 and $e_1 + e_2$ respectively. This system with $K_1 = K_2 = 1$ is known to have applications in nonlinear optics, see [13, 3] and the references therein. Its gauge equivalent has the form:

$$
S_t - f_1 S_x - f_3(S^3)_x = 0,
$$

$$
f_1 = \frac{I_2 J_1^3 - I_1 J_2^3}{J_1 J_2(J_1^2 - J_2^2)}, \qquad f_3 = \frac{I_1 J_2 - I_2 J_1}{J_1 J_2(J_1^2 - J_2^2)}
\tag{69}
$$

where the 5×5 matrix S is constrained by $K^{-1}S(x,t)K = S(x,t)$ and:

$$
\text{trace}\, S^2 = 2(J_1^2 + J_2^2), \qquad \text{trace}\, S^4 = 2(J_1^4 + J_2^4),
$$

$$
S(S^2 - J_1^2)(S^2 - J_2^2) = 0.
\tag{70}
$$

Here we write down the 1-soliton solution for a special choice of the soliton parameters:

$$
\begin{aligned}
n_{0,1} &= 1, & n_{0,2} &= \rho, & n_{0,3} &= \sqrt{2(\rho^2 - 1)}, \\
n_{0,k} &= n_{0,\bar{k}}, & m_{0,k} &= K_k n_{0,k}, & K_3 &= 1, & K_{\bar{k}} &= K_k.
\end{aligned}
\tag{71}
$$

We also assume that $\rho \geq 1$ is real. The choice (71) obviously satisfies (61). Inserting this choice into the general formulae (54–55) after some rearrangements we get:

$$
\begin{aligned}
\langle m|n \rangle &= 2(K_1 \cosh(2\nu_1 y_1) + K_2 \rho^2 \cosh(2\nu_1 y_2) + \rho^2 - 1), \\
\tilde{h}_1 &= 2K_1 \sinh(2\nu_1 y_1), \qquad \tilde{h}_2 = 2K_2 \rho^2 \sinh(2\nu_1 y_2), \\
\tilde{P}_{e_1 \pm e_2} &= \rho\, \mathrm{e}^{-\mathrm{i}\mu_1(y_1 \pm y_2)} \left(K_2\, \mathrm{e}^{\nu_1(y_1 \mp y_2)} + K_1\, \mathrm{e}^{-\nu_1(y_1 \mp y_2)} \right), \\
\tilde{P}_{e_1} &= \sqrt{2(\rho^2 - 1)}\, \mathrm{e}^{-\mathrm{i}\mu_1 y_1} \left(\mathrm{e}^{\nu_1 y_1} - K_1\, \mathrm{e}^{-\nu_1 y_1} \right), \\
\tilde{P}_{e_2} &= \sqrt{2(\rho^2 - 1)}\, \mathrm{e}^{-\mathrm{i}\mu_1 y_2} \left(\mathrm{e}^{\nu_1 y_2} + K_2\, \mathrm{e}^{-\nu_1 y_2} \right),
\end{aligned}
\tag{72}
$$

and $\tilde{P}_{-\alpha} = \tilde{P}_{\alpha}^*$. If we let $\rho = 1$ we get a 1-soliton solution associated with the $D_2 \simeq A_1 \oplus A_1$ subalgebra; if we put $\rho = 0$ the result is a 1-soliton solution associated with the $so(3)$ subalgebra of B_2. In both subcases the subsets of

roots (resp. $\{\pm e_1 \pm e_2\}$ and $\{\pm e_1, \pm e_2\}$) for which $\tilde{P}_\alpha \neq 0$ contain only roots with the same length.

7. Discussion

We outlined the construction of the class of nonlinear evolution equations gauge equivalent to the N-wave equations. Although at some point we made explicit use of the typical representation of $\mathfrak{g}$ we believe that in fact these results may be extended to any irreducible representation of $\mathfrak{g}$.

It remains also to be studied the internal structure of the soliton solutions of both (5) and (8) and the N-soliton interactions. Another open problem is the study of the $\mathbb{Z}_2$-reductions of (8) along the ideas outlined in [3, 4].

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