OTHOGONALITY AND FIXED POINTS OF NONEXPANSIVE MAPS

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Abstract: *The concept of weak- orthogonality for a Banach lattice is examined. A proof that in such a lattice nonexpansive self maps of a nonempty weakly compact convex set have fixed points is outlined. A geometric generalization of weak- orthogonality is introduced and related to the Opial condition.*

AMS subject classification: 47H10

1. Preliminaries

The purpose of this note is not so much to present new results, although there are some, but rather to present known results from a new perspective, which points naturally to a number of open questions.

We will say a Banach space has the w–FPP (w^*–FPP) if every nonexpansive self mapping of a nonempty *weak* – (*weak**–) compact convex set has a fixed point. Conditions on a Banach space which ensure the w–FPP (w^*–FPP) have largely been of two types: Geometrical and Lattice theoretical.
Our aim is to identify and investigate possible common features underlying these two approaches.

Proofs in both cases proceed through the following fundamental observation, due to Brodskii and Mil'man [1948]. Let $T : C \longrightarrow C$ be a nonexpansive mapping on the nonempty *weak* – (*weak**–) compact convex set C, and let D be a minimal nonempty closed convex subset of C invariant under T (such *minimal invariant subsets* exist courtesy of Zorn's lemma). Then D is *diametral* in the sense that, for each $x \in D$,

$$\sup_{y \in D} \|x - y\| = \operatorname{diam} D.$$

If T is fixed point free then necessarily diam $D > 0$ and by a dilation we will assume that diam $D = 1$. This leads [Kirk, 1965] to the w– FPP for spaces containing no diametral sets with more than one point (spaces with *normal structure*).

More recent results have exploited the existence in D of *approximate fixed point sequences* (a_n) (that is, sequences with $\|a_n - Ta_n\| \longrightarrow 0$, which may be constructed by choosing $x_0 \in D$ and applying Banach's contraction mapping principle to $T_n x := (1 - \frac{1}{n})Tx + \frac{1}{n}x_0$ to obtain a fixed point a_n, for each n) and the key result [Karlovitz, 1976(a)] that (a_n) is *diameterizing* for D in that

$$\lim_n \|x - a_n\| = \text{diam } D,$$

for all $x \in D$. Clearly any subsequence of an approximate fixed point sequence is itself such a sequence and so when C is *weak* – (*sequentially weak**–) compact we may assume that (a_n) is *weak* – (*weak**–) convergent, and by a translation, if necessary, we may take the limit to be 0.

As we shall see these observations can be usefully combined with the following Banach space ultra-product construction.

Let U be a nontrivial ultrafilter over $\mathbb{N}$ and denote by $(X)_U$ the *Banach space ultra-power* of X defined by

$$(X)_U := \ell_\infty(X)/\{(x_n) \; : \; x_n \in X, \; \lim_U \|x_n\| = 0\},$$

with elements denoted by $\tilde{x} = [x_n]_U$ and the quotient norm given canonically by

$$\|[x_n]_U\| = \lim_U \|x_n\|.$$

The mapping $J : X \longrightarrow (X)_U : x \longmapsto [x_n]_U$, where $x_n := x$ for all n, is the *natural isometric embedding* of X into $(X)_U$.

Let

$$\tilde{D} := \{[x_n]_U : x_n \in D\},$$

and define

$$\tilde{T} : \tilde{D} \longrightarrow \tilde{D} : [x_n]_U \longmapsto [Tx_n]_U.$$

Then $\tilde{D}$ is a closed convex subset of $(X)_U$ containing JD, and $\tilde{T}$ is a well defined nonexpansive self mapping of $\tilde{D}$. Further for the approximate fixed point sequence (a_n) for T in D we have $[a_n]_U$ is a fixed point of $\tilde{T}$ with

$$\|[a_n]_U - Jx\| \; = \; \text{diam } \tilde{D} \; = \; \text{diam } D \; = \; 1,$$

for all $x \in D$. Indeed, since any nonempty closed convex subset K of $\tilde{D}$ which is invariant under $\tilde{T}$ contains approximate fixed point sequences for $\tilde{T}$, Karlovitz' result combined with a simple diagonalization argument shows that

$$\sup_{\tilde{y} \in K} \|\tilde{y} - Jx\| \; = \; \text{diam } \tilde{D} \; = \; 1,$$

for all $x \in D$. In particular K contains elements of norm arbitrarily near to 1, and for the approximate fixed point sequence (a_n) of T in D we have $\|[a_n]_U\| = 1$, since without loss of generality we have taken $a_n \rightharpoonup 0 \in D$.

2. Orthogonality in Banach lattices

In order to generalize Maurey's [1980] proof of the $w-$ FPP for c_0 to a larger class of Banach lattices Borwein and Sims [1984] introduced the notion of a *weakly*$-$ orthogonal Banach lattice. We will say a Banach lattice is *weakly*$-$ **orthogonal** if whenever (x_n) converges *weakly* to 0 we have

$$\lim_n \| \, |x_n| \wedge |x| \, \| = 0, \quad \text{for all } x \in X.$$

In Borwein and Sims it is shown that any Banach lattice X for which there exists a family of linear projections $\mathbf{P}$ satisfying:

(i) $P|x| = |Px|$, for all $x \in X$ and $P \in \mathbf{P}$

(ii) $P(X)$ is a finite dimensional ideal, for all $P \in \mathbf{P}$

(iii) $\inf\{\|x - Px\| : P \in \mathbf{P}\} = 0$, for all $x \in X$

is *weakly*$-$ orthogonal. In particular a wide class of *sequential* lattices are weakly orthogonal; including $c_0(\Gamma)$ and $\ell_p(\Gamma)$, $1 \le p < \infty$, but not the spaces $L_p[0,1]$ for $p \ne 2$. It is also shown that any *weakly*$-$ orthogonal Banach lattice with a *Riesz angle*

$$\alpha(X) := \sup\{\| \, |x| \vee |y| \, \| : \|x\| \le 1 \text{ and } \|y\| \le 1\} < 2$$

has the $w-$ FPP.

By adapting an argument of Lin [1985], it follows [Sims, 1986] that the assumption on Riesz angle is unnecessary; **all** ***weakly*****$-$ orthogonal Banach lattices have the** ***weak*****$-$ FPP**. When the unit ball is sequentially *weak*$^*-$ compact similar arguments establish the *weak*$^*-$ FPP for any *weak*$^*-$ orthogonal dual lattice, thereby providing a generalization of results by Soardi [1979]. Before sketching a proof for these last results, we examine in more detail the meaning of *weak*$-$ (sequential *weak*$^*-$) orthogonality for a Banach lattice.

Let X be a *weakly*$-$ orthogonal Banach lattice, let (x_n) be a weak null sequence in X, and let U be a nontrivial ultrafilter over $\mathbb{N}$, then:

1) In the Banach lattice $(X)_U$ [Sims, 1982] we have

$$|[x_n]_U| \wedge |Jx| = 0, \quad \text{for all } x \in X.$$

This is a direct consequence of the definitions of *weak*$-$ orthogonality and of the lattice opertations in $(X)_U$.

2) From the definition of *weak*$-$ orthogonality it follows that we can extract a subsequence (x_{n_k}) such that

$$\| \, |x_{n_m}| \wedge |x_{n_k}| \, \| < 1/k, \quad \text{for all } m < k.$$

In particular we have

$$\lim_k |x_{n_k}| \wedge |x_{n_{k+1}}| = 0,$$

or, in terms of the ultrapower the elements $\tilde{x} := [x_{n_k}]_U$ and $\tilde{y} := [x_{n_{k+1}}]_U$ are lattice orthogonal.

The observation in 2) is sharpened, and better understood, by means of the following.

Proposition: Let (x_n) be any sequence in a Banach lattice X for which

$$\lim_n \| \, |x_m| \wedge |x_n| \, \| = 0, \quad \text{for all } m,$$

then there exists a subsequence (x_{n_k}) and an orthogonal sequence of elements (y_k) with

$$\lim_k \|x_{n_k} - y_k\| = 0.$$

In order to prove this we will need the following easily verified lattice facts.

i) Given any two elements x and y in a Banach lattice the elements

$$x' := x \quad - \quad x^+ \wedge |y| \quad + \quad x^- \wedge |y|$$

and

$$y' := y \quad - \quad y^+ \wedge |x| \quad + \quad y^- \wedge |x|$$

are lattice orthogonal with $\|x - x'\|, \ \|y - y'\| \ \leq \ 2\| \, |x| \wedge |y| \, \|$.

ii) If x and z are lattice orthogonal and y is any element, then if we form x' as above we have $|x'| \wedge |z| \leq |x| \wedge |z| = 0$ (as, $|x'| \leq |x|$). That is, orthogonalities with x are preserved by the operation $x \longmapsto x'$.

Proof of the proposition. Let $x_{n_1} = x_1$ and, using 2) above, inductively select a subsequence (x_{n_k}) so that

$$\| \, |x_{n_j}| \wedge |x_{n_k}| \, \| \ < \ 1/2^{j+k} \qquad (j \ < \ k).$$

Let $x_k(1) := x_{n_k}$ for all $k \in \mathbb{N}$, and apply facts i) and ii) above to obtain a sequence $(x_k(2))$ with

$$|x_1(2)| \wedge |x_k(2)| = 0, \quad \text{for } k = 2, 3, \ldots$$

where

$$\|x_1(1) - x_1(2)\| \leq 2 \sum_{k=2}^{\infty} \| \, |x_1(1)| \wedge |x_k(1)| \, \|$$

and for $k \geq 2$

$$\|x_k(1) - x_k(2)\| \leq 2\| \, |x_1(1)| \wedge |x_k(1)| \, \|.$$

Repeat the procedure for $k \geq 2$ to obtain $(x_k(3))$ with

$$|x_2(3)| \wedge |x_k(3)| = 0, \quad \text{for } k = 3, 4, \ldots \quad \text{and by fact 2) for } k = 1 \text{ also,}$$

where

$$\|x_2(2) - x_2(3)\| \le 2\sum_{k=3}^{\infty} \| \, |x_2(2)| \wedge |x_k(2)| \, \|$$
$$\le 2\sum_{k=3}^{\infty} \| \, |x_2(1)| \wedge |x_k(1)| \, \|$$

and for $k \ge 3$

$$\|x_k(3) - x_k(2)\| \le 2\| \, |x_2(2)| \wedge |x_k(2)| \, \| \le 2\| \, |x_2(1)| \wedge |x_k(1)| \, \|.$$

Continuing in this way the sequence (y_k) defined by

$$y_k := x_k(k+1)$$

is orthogonal and satisfies

$$\begin{aligned} \|x_{n_k} - y_k\| &\le \|x_k(1) - x_k(k+1)\| \\ &\le 2 \sum_{\substack{j=1 \\ j \ne k}}^{\infty} \| \, |x_j(1)| \wedge |x_k(1)| \, \| \\ &\le \sum_{j=1}^{\infty} 2^{1-k-j} \\ &= 1/s^{k-1}. \end{aligned}$$

Thus $\|x_{n_k} - y_k\| \longrightarrow 0$ as $k \longrightarrow \infty$.

Corollary: Let X be a *weakly–* orthogonal, order continuous Banach lattice. Then every order bounded weakly compact set is norm compact. In particular every order interval is norm compact.

Proof. This follows from known equivalents to an order continuous norm. See for example Schaefer [1980], Theorem 4.4.

We conclude this section by outlining a proof that every *weakly–* orthogonal Banach lattice has the *weak–* FPP. For more details and extensions to spaces which are *near* to such a lattice see Sims [1986].

Suppose D is a minimal invariant set for the nonexpansive mapping T with diam $D = 1$. Let (a_n) be an approximate fixed point sequence for T in D. From the discussion above we may, by passing to a subsequence if necessary, assume that

$$a_n \rightharpoonup 0,$$

$$\lim_n \|a_{n+1} - a_n\| = 1$$

and

$$\lim_n \| \; |a_n| \wedge |a_{n+1}| \; \| = 0.$$

Thus defining $\tilde{a}_1 := [a_n]_U$ and $\tilde{a}_2 := [a_{n+1}]_U$ we have:

i) $\|\tilde{a}_i - Jx\| = 1$ for all $x \in D$, and $i = 1, 2$. In particular $\|\tilde{a}_i\| = 1$, since $0 \in D$.

ii) $\|\tilde{a}_1 - \tilde{a}_2\| = 1$.

iii) For $i = 1, 2$ and $x \in D$ $|\tilde{a}_i| \wedge |Jx| = 0$, and $|\tilde{a}_1| \wedge |\tilde{a}_2| = 0$.

Let

$$W := \{\tilde{w} \in \tilde{D} : \text{ for } i = 1, 2 \;\; \|\tilde{w} - \tilde{a}_i\| = 1/2 \text{ and } \operatorname{dist}(\tilde{w}, \; JD) \leq 1/2\}.$$

Then W is a closed convex subset of $\tilde{D}$ which is invariant under $\tilde{T}$ and nonempty, since

$$\begin{aligned} \|\frac{\tilde{a}_1 + \tilde{a}_2}{2} - 0\| &= \frac{1}{2}\|\tilde{a}_1 + \tilde{a}_2\| \\ &= \frac{1}{2}\|\tilde{a}_1 - \tilde{a}_2\|, \quad \text{as } |\tilde{a}_1| \wedge |\tilde{a}_2| = 0 \\ &= \frac{1}{2}, \end{aligned}$$

so $\dfrac{\tilde{a}_1 + \tilde{a}_2}{2} \in W$.

Hence W contains elements of norm arbitrarily near to 1.

In order to derive a contradiction, and so establish the *weak*$-$ FPP, we imbed the construction in the countably order complete Banach lattice $(X)_U^{**}$ where we may construct the principle band projections

$$P_{\tilde{a}_i} := \bigvee_{n=1}^{\infty} \left[(n|\tilde{a}_i|) \wedge \tilde{y}^+ \right] - \bigvee_{n=1}^{\infty} \left[(n|\tilde{a}_i|) \wedge \tilde{y}^- \right],$$

for $i = 1, 2$.

Then $P_{\tilde{a}_i}$, $I - P_{\tilde{a}_i}$ and $P_{\tilde{a}_1} + P_{\tilde{a}_2}$ are norm one projections with $P_{\tilde{a}_i}\tilde{a}_i = \tilde{a}_i$ and $P_{\tilde{a}_i}J = 0$. Thus for each $\tilde{w} \in W$, if $x \in D$ is such that $\|\tilde{w} - Jx\| \leq 1/2$, we have

$$\begin{aligned} \|\tilde{w}\| &= \frac{1}{2}\|(P_{\tilde{a}_1} + P_{\tilde{a}_2})(\tilde{w} - Jx) + (I - P_{\tilde{a}_1})(\tilde{w} - \tilde{a}_1) + (I - P_{\tilde{a}_2})(\tilde{w} - \tilde{a}_2)\| \\ &\leq \frac{1}{2}(\|\tilde{w} - Jx\| + \|\tilde{w} - \tilde{a}_1\| + \|\tilde{w} - \tilde{a}_2\|) \\ &\leq \frac{3}{4}. \end{aligned}$$

3. Geometric orthogonality

Given the results of the last section it seems reasonable to consider *geometric* analogues of the *weak*$-$ orthogonality condition for lattices. In particular we will investigate the following natural generalization of *weak*$-$ (*weak*$^*-$) orthogonality:

Whenever (x_n) is a *weak* (*weak**) null sequence we have

$$\lim_n \Big| \, \|x + x_n\| \; - \; \|x - x_n\| \, \Big| \; = \; 0, \quad \text{for all } x \in X.$$

We will say that a Banach space with this property has **WORTH** (**W*ORTH**).

In what follows we will only consider spaces with WORTH, however analogous conclusions are valid for dual spaces with W*ORTH.

For a Banach space X with WORTH and a nontrivial ultrafilter U over $\mathbb{N}$ we have

$$\|[x_n]_U \; + \; Jx\| \; = \; \|[x_n]_U \; - \; Jx\|,$$

for all $x \in X$, whenever $x_n \rightharpoonup 0$. That is, in the ultrapower $(X)_U$ the element $[x_n]_U$ is orthogonal in the sense of James [1947] to each element of the subspace JX, which we will denote by writing $[x_n]_U \sqcup JX$. Indeed, since (λx_n) is also a weak null sequence, we have $\lambda[x_n]_U \sqcup Jx$, for all scalars λ and all $x \in X$.

While we do not know whether spaces with WORTH enjoy the $w-$ FPP, we can establish a suggestive connection between WORTH and a *geometric* condition known to be sufficient for the $w-$ FPP.

Let X be a Banach space with WORTH, and let (x_n) be a *weak* null sequence. Then for all $x \in X$ we have $\lambda[x_n]_U \sqcup Jx$, and so in particular

$$\begin{aligned} \|[x_n]_U\| \; &= \; \frac{1}{2}\|([x_n]_U + Jx) \; + \; ([x_n]_U - Jx)\| \\ &\leq \frac{1}{2}\Big(\|[x_n]_U + Jx\| \; + \; \|[x_n]_U - Jx\|\Big) \\ &= \; \|[x_n]_U + Jx\|. \end{aligned}$$

A similar calculation shows that $\|Jx\| \leq \|Jx + \lambda[x_n]_U\|$, for all λ. That is, for all $x \in X$ we have $[x_n]_U \perp Jx$ and $Jx \perp [x_n]_U$, where "$\perp$" denotes orthogonality in the sense of James-Birkhoff [see, James 1947]. The second of these simply reflects the *weak*$-$ lower semi-continuity of the norm, however the first gives

$$\lim_U \|x_n\| \; \leq \; \lim_U \|x_n + x\|$$

and so

$$\liminf_n \|x_n\| \; \leq \; \limsup_n \|x_n + x\|,$$

for all $x \in X$. By extraction of appropriate subsequences, this last condition is readily seen to be equivalent to the **non-strict Opial condition**:

whenever (x_n) is a *weak* null sequence we have

$$\liminf_n \|x_n\| \leq \liminf_n \|x_n + x\|, \quad \text{for all } x \in X.$$

Thus, **spaces with WORTH satisfy the non-strict Opial condition.**

The (strict) **Opial condition**:

whenever (x_n) is a *weak* null sequence we have

$$\liminf_n \|x_n\| < \liminf_n \|x_n + x\|, \quad \text{for all } x \neq 0,$$

is easily seen to imply the $w-$ FPP [see, van Dulst 1982]. Indeed it is known to imply *weak*$-$ normal structure [Gossez and Lami Dozo, 1972] and so is more than sufficient for the $w-$ FPP. This leads naturally to the following.

Question: Does the non-strict Opial condition imply the $w-$ FPP?

A close connection between the non-strict (strict) Opial condition and (*uniform*) *approximate symmetry* for James-Birkhoff orthogonality has been noted previously by Karlovitz [1976(b)].

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