OPEN PROBLEMS

The following problems were presented during the conference by the indicated participants. In most, but not all, cases, further details and references can be found in the author(s)' conference paper(s).

G. R. ALLAN

Let R be a commutative radical Banach algebra and let R_+ be its unitization. Say that $x \in R$ is *quasi-power-bounded* (qpb) if and only if $1 + x$ is power bounded (pb) in R_+. Clearly 0 is always qpb, and, by Gelfand's theorem, it is the *only* element of R such that both $1 + x$ and $(1 + x)^{-1}$ are pb. The result of Katznelson-Tzafriri-Esterle shows that if x is qpb then $x(1 + x)^n \to 0$. In particular $\{1 - (1 + x)^n\}$ is a bounded approximate identity for the closed ideal $(Rx)^-$. Using a result of Sinclair and Allan it follows, for example, that :

(i) if R is uniformly radical (that is, $\| x^n \|^{1/n} \to 0$ uniformly on the unit sphere of R) then R contains no (non-zero) qpb elements; this applies in particular to $C_*[0, 1]$ (the algebra of continuous functions on $[0, 1]$ with convolution product and the uniform norm).

Another elementary result is :

(ii) if x is a proper nilpotent in R, then x is *not* qpb.

Problem : Which radical algebras contain non-zero qpb elements (an example to show that the class is non-vacuous is described in my paper on pb elements). In particular, are there any qpb elements in $L^1[0, 1]$ or $L^1(\mathbb{R}^+, w)$? What are they ?

Remark. If $x \neq 0$ is qpb in R, then $I = \{ y \in R : y(1 + x)^n \to 0 \ (n \to \infty) \}$ is a non-zero closed ideal of R, of course it may happen that $I = R$.

W. G. BADE

Let A be a Silov algebra with structure space Φ_A. We say that A is *strongly regular* if $J(\varphi)^- = M(\varphi)$ for each $\varphi \in \Phi_A$. A has *bounded relative units* if for each $\varphi \in \Phi_A$, there exists a constant $K_\varphi > 0$ such that for $g \in J(\varphi)$, an element $h \in J(\varphi)$ can be found such that $gh = g$ and $\| h \| \leq K_\varphi$.

Does a strongly regular Silov algebra have bounded relative units if every maximal ideal $M(\varphi)$ has a bounded approximate identity ?

W. G. BADE and H. G. DALES

Let w be a radical weight on $\mathbb{R}^+$. Are all derivations from $L^1(\mathbb{R}^+, w)$ into a Banach $L^1(\mathbb{R}^+, w)$-module automatically continuous ? We fail to solve this in a paper Continuity of derivations from radical convolution algebras, *Studia Math.*, to appear.

P. C. CURTIS JR.

1. Can every amenable Banach algebra (or some closed cofinite ideal thereof) be represented in the form $\nu(L^1(G))^-$ where ν is a continuous homomorphism and G is locally compact and amenable ?

2. Do there exist commutative radical Banach algebras which are either amenable or weakly amenable ?

3. If G is locally compact abelian and I is a closed ideal in $A(G)$ such that $A(G)/I$ is not semisimple, can the radical of $A(G)/I$ be amenable, that is, can the radical have a bounded approximate identity ?

4. If T is a quasinilpotent operator on a Banach space X and A is the closed algebra generated by T, must A admit a non-zero bounded derivation into A^* ? This is the case (N. Grønbæk) if $A \cong l^1(w)$ where $w(n) = \|T^n\|$.

5. Which locally compact non-discrete abelian groups G have either of the properties :

(i) $A(G)_+ = \mathrm{sp}\{ x \in A(G)_+ : x$ is invertible and $\| x^n \| . \| x^{-n} \| = o(n) \}^-$,

(ii) $A(G) = \mathrm{sp}\{ x \in A(G) : \| e^{nx} \| . \| e^{-nx} \| = o(n) \}^-$,

where the norm $\| e^{nx} \|$ is taken in $A(G)_+$? This is true for $G = \mathbb{R}$.

6. Let $f \in A(\mathbb{T})$, $\mathbb{T}$ the unit circle. If $(fA(\mathbb{T}))^- = (f^2 A(\mathbb{T}))^-$, is the zero set of f a set of spectral synthesis for $A(\mathbb{T})$?

H.G.DALES

1. **Conjecture.** If A is a commutative Banach algebra and an integral domain, then A has a unique norm topology.

(Recall that Cusack (*J. London Math. Soc.*, (2) 16 (1977), 493 – 500) has shown that if there is an integral domain which is a Banach algebra with respect to two inequivalent norms then there exists a topologically simple Banach algebra.)

Let A be an integral domain and suppose that A is a topologically simple Banach algebra with respect to two norms. Are these two norms equivalent ? A harder version of this is to relax the assumption to "topologically simple with respect to one of the norms".

2. Let $A(\mathbb{Z})$ be the Banach algebra of Fourier transforms of functions in $L^1(\mathbb{T})$, so that $l^1(\mathbb{Z}) \subset A(\mathbb{Z}) \subset c_0(\mathbb{Z})$. An ideal I in $A(\mathbb{Z})$ is *absolutely convex* if, whenever $\alpha = (\alpha_n) \in I$ and $\beta = (\beta_n) \in A(\mathbb{Z})$ with $|\beta_n| \leq |\alpha_n|$ $(n \in \mathbb{Z})$, it follows that $\beta \in I$. If P is a prime ideal in $A(\mathbb{Z})$, is P absolutely convex ?

It is easily seen that a prime ideal in $c_0(\mathbb{Z})$ itself is absolutely convex, cf. §4 of my paper in these proceedings.

3. For a Banach space E, $\mathcal{K}(E)$ denotes the Banach algebra of compact operators on E. The problem is to characterize the Banach spaces E such that $\mathcal{K}(E)$ is amenable. This is known for example, if $E = l^2$, or $E = C(\mathbb{T})$. Can one use the criterion that a Banach algebra is amenable if and only if A has a bounded approximate identity and the closed left ideal I^Δ in $A \hat{\otimes} A^{op}$ has a bounded right approximate identity ?

We have the following implications. If $\mathcal{K}(E)$ is amenable, then $\mathcal{K}(E)$ has a bounded approximate identity. By a result of P. G. Dixon, (Left approximate identities in algebras of compact operators on Banach spaces, *Proc. Royal Soc. Edinburgh*, 104A (1986), 169 – 175) the latter holds if and only if E has the bounded compact approximation property (BCAP). In particular, if E does not have BCAP, then $\mathcal{K}(E)$ is not amenable.

4. (See KBL) Let E and F be Banach spaces, and $R \in B(E)$, $S \in B(F)$. A linear map $T: E \rightarrow F$ *intertwines* R and S if $ST = TR$. Consider the following two statements :

(a) each linear map which intertwines R and S is automatically continuous;

(b) the pair (R, S) has no critical eigenvalues and either R is algebraic or $E_S = \emptyset$.

(A number $z \in \mathbb{C}$ is a *critical eigenvalue* of (R, S) if z is an eigenvalue of S and $(zI_E - R)(E)$ has infinite codimension in E. The operator R is *algebraic* if there is a non-zero polynomial $p \in \mathbb{C}[x]$ with $p(T) = 0$. The space E_S is the algebraic spectral space : it is the maximal linear subspace G of F such that $(zI_F - R)(G) = G$ for each $z \in \mathbb{C}$.)

It is a well known and fairly easy result that (a) always implies (b). The converse implication has been proved in a variety of cases. See, for example, A. M. Sinclair, *Automatic continuity of linear operators*, §4. That (b) implies

(a) in the case where R is decomposable and S is super-decomposable is a result of K. B. Laursen and M. M. Neumann, Decomposable operators and automatic continuity, *J. Operator Theory*, 15 (1986), 33 – 51; and this result is proved in §4.3 of my forthcoming monograph.

Are there any counterexamples to the conjecture that (b) *always* implies (a) ? Can one show that (b) implies (a) under the hypothesis that R and S are decomposable (rather than S is super-decomposable) ?

J. E. GALÉ

1. Let A be a regular semisimple tauberian Banach algebra. Does A necesarily contain a non-zero analytic semigroup $(a^z)_{\mathcal{Re}\, z > 0}$ such that

$$\int_{-\infty}^{+\infty} \frac{\log^+ \| a^{1+iy} \|}{1+y^2}\, dy < \infty \, ?$$

2. Let $(a^z)_{\mathcal{Re}\, z > 0}$ be the Gaussian or Poisson semigroup in $L^1(\mathbb{R}^n)$, and let A be the Banach subalgebra of $L^1(\mathbb{R}^n)$ generated by this semigroup. Does spectral synthesis hold in A ?

3. (i) (J. Esterle) Is there some non-zero analytic semigroup $(a^z)_{\mathcal{Re}\, z > 0}$ in $L^1(\mathbb{R})$ such that $\sup \{ \| a^{1+iy} \| : y \in \mathbb{R} \} < \infty$?

(ii) Suppose there exists $(a^z)_{\mathcal{Re}\, z > 0}$ in $L^1(\mathbb{R})$ as in (i). Does there exist a non-zero semigroup $(b^z)_{\mathcal{Re}\, z > 0}$ in $L^1(\mathbb{R})$ such that $\{ \| b^{1+iy} \| : y \in \mathbb{R} \}$ is relatively weakly compact ? Is the map $y \to \varphi(a^{1+iy})$ weakly almost periodic for every character $\varphi \in \mathbb{R}^\wedge = \mathbb{R}$? (See my paper, and reference [8] there.) And what about the same question for $L^1(G)$, G a locally compact group ?

4. Let G be a locally compact group. Suppose there exists $f \in L^1(G) \setminus \{0\}$ with countable spectrum. Does this imply some kind of structural properties for G ? This is the case if G is central. If, besides, G is connected, must G be compact ? For central G the answer is yes. (See reference [8] of my paper.)

5. Let G be a metrizable compact abelian group. Can $L^1(G)$ be characterized in terms of analytic semigroups ? (If G is such a group with dual group

$\hat{G} = \{X_n\}_{n=1}^{\infty}$, and we define $a^z = \sum_{n=1}^{\infty} e^{-nz} X_n(t)$ $(t \in G, \mathcal{Re}\, z > 0\}$, then $(a^z)_{\mathcal{Re}\, z > 0}$ is an analytic semigroup such that $L^1(G)$ is polynomially generated by a^1, and $\{a^{1+iy} : y \in \mathbb{R}\}$ is norm compact. A partial converse to this is given in my paper.)

S. GRABINER

1. For which radical $L^1(w)$ is $L^1(w) * f$ always dense in $L^1(w)$ whenever $\alpha(f) = 0$? When is such $L^1(w) * f$ dense in the (relative) weak*-topology of $L^1(w) \subseteq M(w) = c_0(1/w)^*$? In particular, is this true for regulated weights ?

2. We call a semigroup $\{\mu_t\}_{t \geq 0} \subseteq M(w)$, viewed as an algebra of operators on $L^1(w)$, *almost continuous*, if $\mu_t * f$ is continuous for $t > 0$ for each $f \in L^1(w)$ and $\|\mu_t\|$ is bounded as $t \to 0+$. For which $L^1(w)$ are almost continuous semigroups $\{\mu_t\}_{t \geq 0}$ actually strongly continuous semigroups ? Equivalently, for which $L^1(w)$ does every continuous non-zero endomorphism φ have the property that $L^1(w) * f$ dense implies $L^1(w) * \varphi(f)$ is dense ? For other formulations, see Theorems (1.2) and (2.8) of my paper on convolution algebras. In particular, do these conditions hold for $w(x) \equiv 1$, that is, for $L^1(\mathbb{R})$?

3. Can an ideal in $L^1(w)$ be closed but not weak*-closed ? Are there any proper non-zero closed prime ideals ?

4. Must every continuous non-zero homomorphism $\varphi : L^1(w_1) \to L^1(w_2)$ be on-to-one ? The answer is yes for $w_1 = w_2$ a radical weight, and also for w_1, w_2 semisimple weights.

5. Can the umbral calculus be further extended to give interesting information about analytic functions which are not entire ?

6. We know that if $\{w_n\}_{n=0}^{\infty}$ is a regulated weight then $f(t) = \sum_{n=1}^{\infty} a_n t^n$ in $l^1(w_n)$ generates a nonstandard ideal of $l^1(w_n)$ if and only if the infinite order differential equation $\sum_{n=1}^{\infty} a_n y^{(n)}(x) = 0$ has a non-polynomial solution in the predual $c_0(n!/w_n)$. Can this fact be used to give new examples of nonstandard elements ?

N. GRØNBÆK

1. Let I be a closed ideal in a Banach algebra A. Develop methods to determine whether I has the ET property, that is, when the canonical map $Z^0(A, A^*) \to Z^0(A, I^*)$ is surjective.

2. Let $\theta : A \to B$ be a Banach algebra homomorphism with dense range. Assume that A is weakly amenable and that $\ker \theta$ has the ET property. Does it follow that B is weakly amenable ? Yes, if θ is surjective or if the map $d : B \hat{\otimes} B \to B$, given by $d(b_1 \otimes b_2) = b_1 b_2 - b_2 b_1$, has closed range. Equivalently, with θ, A and B satisfying the above condition, does it follow that d has closed range ? If the answer is yes, then one can prove that C*-algebras are weakly amenable without using Grothendieck's inequality for C*-algebras.

3. To what extent does $H^1(A, A^*) = 0$, A a C*-algebra, depend on the Grothendieck inequality. It seems plausible, at least, that one can make do with the Grothendieck inequality for commutative C*-algebras. It is true that for a general Banach algebra we have $H^1(A, A^*) = 0 \Rightarrow d$ has closed range, that is, there is $K > 0$ such that for all $x \in$ range(d),

$$\inf \{\| \sum (a_i \otimes b_i - b_i \otimes a_i) \|_\pi : \sum (a_i b_i - b_i a_i) = x \} \leq K \| x\| ?$$

So some means of reversing the inequality

$$\| \sum (a_i \otimes b_i - b_i \otimes a_i) \|_\pi \geq \| \sum (a_i b_i - b_i a_i) \|$$

on the subspace $\{ \sum (a_i \otimes b_i - b_i \otimes a_i) \}$ of $A \hat{\otimes} A$ is needed.

4. What is the relationship between $H^n_\lambda (A \hat{} B)$ and $H^n_\lambda (A) \times H^n_\lambda (B)$ for $n \geq 2$? Here $A \hat{} B$ is the Banach algebraic free product of A and B.

K. B. LAURSEN

Let X, Y be Banach spaces. Suppose $S \in B(Y)$ is an isometry for which $\bigcap_{n \geq 1} S^n Y = \{0\}$ and suppose $T \in B(X)$ is decomposable. It is known that if $\theta : X \to Y$ is linear and $S\theta = \theta T$, then $\theta = 0$. Moreover, if S is any isometry, then every linear $\theta : X \to Y$ for which $S\theta = \theta T$ is necessarily continuous if and only if (S, T) has no critical eigenvalues. (See HGD, #4)

Can decomposability of T be relaxed to property (β), say ? Can the isometric hypothesis on S be relaxed ? What if the roles of S and T are interchanged, that is, if we look at maps $\psi : Y \to X$ for which $\psi S = T\psi$? Are any continuity conclusions available ? A specific example would be S the unilateral right shift on $l^2(\mathbb{N})$, T the bilateral shift on $l^2(\mathbb{Z})$.

In my paper the assumption of intertwining is not just $C(S, T)\theta = 0$, but $C(S, T)^n\theta = 0$ for some $n \in \mathbb{N}$. Can this be relaxed to the condition that $C(S, T)^n\theta$ be continuous, for some $n \in \mathbb{N}$? M. M. Neumann has some work on this.

R. J. LOY

1. Let A be a commutative Banach algebra such that all of the closed subalgebras of A are amenable. It follows that the spectrum of every element of A has empty interior and connected complement. Does it follow that Φ_A is scattered ? What else can be said about A ?

2. (B. E. Johnson) Let A be a superamenable (=contractible) Banach algebra with diagonal $\sum a_i \otimes b_i$, so that $\sum aa_i \otimes b_i = \sum a_i \otimes b_i a$ $(a \in A)$ and $\sum a_i b_i = 1$. If X is an irreducible A-module, must X be finite dimensional ? If so, then finite dimensionality of A follows by a result of Selivanov. (See Helemskii article.)

Remark. For $T \in B(X)$ define $\Phi(T) = \sum a_i T b_i$. By irreducibility of X and the diagonal property it follows that $\Phi(T) = \varphi(T)\, \mathrm{Id}_X$ for some linear functional φ. Since $\Phi(T)$ lies in the closed ideal generated by T we have φ vanishing on every proper closed ideal in $B(X)$, yet $\varphi(\mathrm{Id}_X) = 1$. Further, φ is continuous in the topology of uniform convergence on compact subsets, and so has the form $\sum \xi_i \otimes \eta_i \in X \hat{\otimes} X^*$. Thus X cannot be infinite dimensional and have the compact approximation property. Are there Banach spaces where, for example, the weakly compact operators are not $\sigma(B(X), X \hat{\otimes} X^*)$-dense ?

M. MATHIEU

1. Is every ultraprime normed algebra semisimple ?

2. Is every primitive Banach algebra ultraprime ? Indeed, is every ultrapower of a primitive Banach algebra again primitive ?

3. Suppose that A is a semisimple Banach algebra, and that for all $a, b \in A$ the following identity for spectral radii holds : $\nu(L_a R_b) = \nu(a)\nu(b)$. Does it follow that A is ultraprime ?

Note that the answers to questions 2 and 3 are affirmative for C*-algebras, see my paper in these proceedings, and references 19 and 21 there.

J. P. McCLURE

Let w be a radical weight on $[0, \infty)$. Assume that w is continuous, with $w(0) = 1$. Consider the following conditions for a Radon measure μ on $[0, \infty)$:

(i) $\sup \{ \frac{x}{w(x)} \int e^{-\lambda(x+y)} w(x+y) d|\mu|(y) \} < \infty$ for all $\lambda > 0$;

(ii) $\int e^{-\lambda y} w(y) d|\mu|(y) < \infty$ for (a) for some $\lambda > 0$, or (b) for all $\lambda > 0$.

Does (i) imply (iia) ? Note that (iib) implies (i), and we know that (i) implies (iib) in certain cases, for example $w(x) = e^{-x^2}$.

J. P. McCLURE and F. GHAHRAMANI

Let $\mathcal{W}^+$ be the class of all weights such that :

(i) $p(a) = \sup \{ x \frac{w(x+a)}{w(a)} : x \in \mathbb{R}^+ \} < \infty$ for some $a > 0$;

(ii) $\inf \{ a : p(a) < \infty \} > 0$.

We have proved (*J.London Math. Soc.* to appear), that if $w \in \mathcal{W}^+$ and θ is an automorphism of $L^1(w)$, then there exist $\alpha \in \mathbb{R}, \lambda > 0$ and a derivation D on $L^1(w)$ such that $\theta = e^{i\alpha X} e^{\lambda X} e^{D} e^{-\lambda X}$.

1. Is an automorphism of the form $e^{\lambda X} e^{D} e^{-\lambda X}$ equal to $e^{D'}$ for some derivation D' ? An affirmative answer to this question would imply that the group of principal automorphisms of $L^1(w)$ is connected when $w \in \mathcal{W}^+$.

2. Suppose that w is a radical weight. Is there a simple representation for the automorphisms of $L^1(w)$ as in the case $w \in \mathcal{W}^+$?

3. Suppose that w is a radical weight and θ is a principal automorphism of $L^1(w)$. We have shown, *ibidem*, that for every $\lambda \geq 0$, $(e^{\lambda X}e^{D}e^{-\lambda X})^{-}$ defines an automorphism of $L^1(w)$ and the map $\lambda \to (e^{\lambda X}e^{D}e^{-\lambda X})^{-}$ is norm continuous on $(0, \infty)$. Is it norm continuous at $\lambda = 0$? An affirmative answer to this question would again imply that the group of principal automorphisms of $L^1(w)$ is connected when $w \in \mathcal{W}^+$.

G. A. WILLIS

1. Is there an amenable integral domain other than $\mathbb{C}$? The answer is "no" if the answer to question 1 of P. C. Curtis is "yes". If $\mathbb{C}$ is the only amenable integral domain, then all derivations from commutative amenable Banach algebras are continuous. See my second paper in these proceedings.

2. Let A be an infinite dimensional integral domain. Is there a non-zero continuous derivation from A to A^* ?

3. Is every (weakly) amenable prime algebra ultraprime ? This is a non-commutative version of questions 1 and 2.

4. Let G be a locally compact group, $L_0^1(G) = \{f \in L^1(G) : \int_G f(x)dm(x) = 0\}$, where m is (left) Haar measure on G. Is every left $L_0^1(G)$-module homomorphism from $L_0^1(G)$ to X, an arbitrary left $L^1(G)$-module, continuous ? If G is amenable, then the answer is "yes". If G is discrete, then any module homomorphism from $L_0^1(G)$ may be extended to be a derivation from $L^1(G)$ and so this question is a special case of the problem of continuity of derivations from $L^1(G)$.