BANACH ALGEBRAS OCCURRING IN SPECTRAL THEORY

Igor Kluvánek

1.   THE STONE THEOREM

The algebras referred to in the title will be specified at the end of the following section. This section is devoted to a few fairly simple observations about the Fourier L^p-multipliers. I hope that in the light of these observations the notions introduced in the following section will seem natural. At any rate, it will be apparent that the realm of mathematical objects to which these notions pertain is sufficiently rich to warrant our attention.

The algebra of all bounded linear operators on a complex Banach space, E, is denoted by $\mathrm{BL}(E)$. The identity operator is denoted by I. The norm on E is denoted as modulus and the operator (uniform) norm of an element, T, of $\mathrm{BL}(E)$ by $\|T\| = \sup\{\,|Tx| : |x| \leq 1, x \in E\}$.

To avoid complicated notation and circumlocution, we shall identify subsets of a given basic space with their characteristic functions.

Let λ be Lebesgue measure on $\mathbb{R}$. Let $1 < p < \infty$. Let $\mathcal{M}^p$ be the family of all (individual) functions on $\mathbb{R}$ which determine Fourier multiplier operators on the space $E = L^p(\mathbb{R})$. That is, $f \in \mathcal{M}^p$ if and only if there exists an operator $T_f \in \mathrm{BL}(E)$ such that $(T_f \varphi)\hat{} = f\hat{\varphi}$, for every $\varphi \in L^p \cap L^2(\mathbb{R})$. Here, of course, $\hat{\varphi}$ denotes the Fourier–Plancherel transform of an element, φ, of $L^2(\mathbb{R})$ and $f\hat{\varphi}$ is the point-wise product of f and $\hat{\varphi}$.

Let $\mathcal{R}^p$ be the family of all sets $X \subset \mathbb{R}$ such that $X \in \mathcal{M}^p$. Let $P^p(X) = T_X$, for every $X \in \mathcal{R}^p$.

PROPOSITION 1.   *Let f be an absolutely continuous function on $\mathbb{R}$, or else, let $w > 0$ and let f be a w-periodic function on $\mathbb{R}$ which is absolutely continuous in an interval $[s,t)$ with $t - s = w$.*

Then there exist numbers c_j and Borel sets X_j, $j = 1, 2, \ldots,$ such that

$$(1) \qquad \sum_{j=1}^{\infty} |c_j| < \infty ,$$

the sets X_j belong to $\mathcal{R}^p$ for every $p \in (1,\infty)$, and there is a constant $C_p \geq 1$ such that

$$(2) \qquad \| P^p(X_j) \| \leq C_p ,$$

for every $j = 1,2,\dots$, and

$$(3) \qquad f(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega) ,$$

for every $\omega \in \mathbb{R}$.

Proof. If K is an interval with end-points α and β, $-\infty < \alpha < \beta < \infty$, and a is a complex number let $a\lambda(K) = b$ and

$$\varphi(\omega) = \int_{-\infty}^{\omega} K \, d\lambda ,$$

for every $\omega \in \mathbb{R}$. Then $\varphi(\omega) = 0$, for every $\omega \in (-\infty, \alpha)$, $\varphi(\omega) = b(\omega-\alpha)(\beta-\alpha)^{-1}$, for $\omega \in [\alpha, \beta)$, and $\varphi(\omega) = b$, for $\omega \in [\beta, \infty)$. Let, further, $Y_\beta = [\beta, \infty)$ and, given an integer $n \geq 1$,

$$Y_{\alpha,\beta,n} = \bigcup_{j=1}^{2^{n-1}} [\alpha + 2^{-n}(2j-1)(\beta-\alpha), \alpha + 2^{1-n}j(\beta-\alpha)) .$$

Then

$$\varphi(\omega) = b Y_\beta(\omega) + \sum_{n=1}^{\infty} 2^{-n} b Y_{\alpha,\beta,n}(\omega) ,$$

for every $\omega \in \mathbb{R}$.

Assuming that $s \leq \alpha < \beta \leq t$, let Z_β be the w-periodic function such that $Z_\beta(\omega) = 0$, for $\omega \in [s,\beta)$, and $Z_\beta(\omega) = 1$, for $\omega \in [\beta, t)$. Furthermore, for any integer $n \geq 1$, let $Z_{\alpha,\beta,n}$ be the w-periodic function such that $Z_{\alpha,\beta,n}(\omega) = Y_{\alpha,\beta,n}(\omega)$, for every $\omega \in [s,t)$. Let ψ be the w-periodic function such that

$$\psi(\omega) = \int_s^{\omega} K \, d\lambda ,$$

for every $\omega \in [s,t)$, where, again, K is an interval with end-points α and β. Then

$$\psi(\omega) = bZ_\beta(\omega) + \sum_{n=1}^{\infty} 2^{-n} bZ_{\alpha,\beta,n}(\omega) \,,$$

for every $\omega \in \mathbb{R}$.

By a classical theorem of M. Riesz (see, for example, [2], Theorem 6.3.3) all intervals of all kinds belong to $\mathcal{R}^p$, for every $p \in (1,\infty)$, and there exists a constant $A_p \geq 1$ such that $\|P^p(X)\| \leq A_p$, for every interval $X \subset \mathbb{R}$. Furthermore, let us call a set $X \subset \mathbb{R}$ a periodic interval, if there exists a real number x and a connected subset, J, of the unit circle, $\mathbb{T} = \{\exp it : t \in \mathbb{R}\}$, such that $X = \{\omega : \exp i\omega x \in J\}$. By a lemma of A. Gillespie (see [6], Lemma 6, or [3], Lemma 20.15), every periodic interval belongs to $\mathcal{R}^p$, for every $p \in (1,\infty)$, and there exists a constant $B_p \geq 1$, such that $\|\mathrm{P}^p(X)\| \leq B_p$, for every periodic interval $X \subset \mathbb{R}$.

Let us now note that every set of the form Y_β is an interval, every set Z_β is a periodic interval, every set $Y_{\alpha,\beta,n}$ is the intersection of an interval and a periodic interval and every set $Z_{\alpha,\beta,n}$ is the intersection of two periodic intervals. Consequently, every set, X, which is of one of these four kinds belongs to $\mathcal{R}^p$, for every $p \in (1,\infty)$, and $\|P^p(X)\| \leq C_p$, for some constant C_p depending on p alone.

Now, let g be a λ-integrable function on $\mathbb{R}$, and let

$$f(\omega) = \int_{-\infty}^{\omega} g d\lambda \,,$$

for every $\omega \in \mathbb{R}$. Then there exist numbers a_k and bounded intervals K_k, $k = 1,2,...,$ such that

(4)
$$\sum_{k=1}^{\infty} |a_k| \lambda(K_k) < \infty$$

and

(5)
$$g(\omega) = \sum_{j=1}^{\infty} a_k K_k(\omega)$$

for λ-almost every $\omega \in \mathbb{R}$, so that

$$f(\omega) = \sum_{k=1}^{\infty} a_k \int_{-\infty}^{\omega} K_k d\lambda \,,$$

for every $\omega \in \mathbb{R}$. Let $b_k = a_k \lambda(K_k)$ and let α_k and β_k, $\alpha_k < \beta_k$, be the end-points of the interval K_k, for every $k = 1,2,\dots$. Then

$$f(\omega) = \sum_{k=1}^{\infty} \left[b_k Y_{\beta_k}(\omega) + \sum_{n=1}^{\infty} 2^{-n} b_k Y_{\alpha_k,\beta_k,n}(\omega) \right],$$

for every $\omega \in \mathbb{R}$. The desired representation (3) of f is obtained by rearrangement, because, by (4),

$$\sum_{k=1}^{\infty} \left[|b_k| + \sum_{n=1}^{\infty} 2^{-n} |b_k| \right] = 2 \sum_{k=1}^{\infty} |b_k| < \infty.$$

Similarly, let f be a w-periodic function on $\mathbb{R}$ and g a λ-integrable function in the interval $[s,t)$ such that

$$f(\omega) = \int_s^{\omega} g d\lambda,$$

for every $\omega \in [s,t)$. Then there exist numbers a_k and intervals $K_k \subset [s,t)$, $k = 1,2,\dots$, such that (4) holds and (5) holds for λ-almost every $\omega \in [s,t)$, so that

$$f(\omega) = \sum_{k=1}^{\infty} a_k \int_s^{\omega} K_k d\lambda,$$

for every $\omega \in [s,t)$. If $b_k = a_k \lambda(K_k)$ and α_k and β_k, $\alpha_k < \beta_k$, are the end-points of the interval K_k, for every $k = 1,2,\dots$, then

$$f(\omega) = \sum_{k=1}^{\infty} \left[b_k Z_{\beta_k}(\omega) + \sum_{n=1}^{\infty} 2^{-n} b_k Z_{\alpha_k,\beta_k,n}(\omega) \right],$$

for every $\omega \in \mathbb{R}$.

To draw some consequences of this proposition, let us recall that $\|f\|_{\infty} \leq \|T_f\|$, for every function $f \in \mathcal{M}^p$ and every $p \in (1,\infty)$, where $\|f\|_{\infty}$ denotes the λ-essential supremum of the function $|f|$. Moreover, the map $f \mapsto T_f$, $f \in \mathcal{M}^p$, is λ-essentially injective, that is, the equality $T_f = T_g$ implies that $\|f-g\|_{\infty} = 0$.

So, if f is a function satisfying the assumptions of Proposition 1, then $f \in \mathcal{M}^p$, for every $p \in (1,\infty)$, and

$$T_f = \sum_{j=1}^{\infty} c_j P^p(X_j) \ ,$$

where the c_j are numbers and the X_j sets, $j = 1,2,...,$ whose existence is guaranteed by Proposition 1. Because the operator T_f does not depend on the choice of the numbers c_j and the sets X_j, $j = 1,2,...,$ we may even write

$$\int_{\mathbb{R}} f(\omega) P^p(d\omega) = \int_{\mathbb{R}} f dP^p = \sum_{j=1}^{\infty} c_j P^p(X_j) \ ,$$

so that

$$T_f = \int_{\mathbb{R}} f dP^p \ .$$

In particular, if $x \in \mathbb{R}$ and $f(\omega) = \exp i x \omega$, for every $\omega \in \lambda$, then T_f is the operator of translation by x and is denoted simply by T_x. Thus, given an $x \in \mathbb{R}$, we have

$$T_x = \int_{\mathbb{R}} (\exp i x \omega) P^p(d\omega) \ ,$$

for every $p \in (1,\infty)$. For $p = 2$, this is of course a case of the Stone theorem.

By way of concluding these introductory remarks, we re-iterate that, for every $x \in \mathbb{R}$, there exist numbers c_j and sets X_j, $j = 1,2,...,$ such that the inequality (1) holds and for every $p \in (1,\infty)$, the sets X_j belong to $\mathcal{R}^p$, and there exists a constant C_p such that the inequality (2) holds for every $j = 1,2,...,$ and the operator, T_x, of translation by x in the space $L^p(\mathbb{R})$ is given by

$$T_x = \sum_{j=1}^{\infty} c_j P^p(X_j) \ .$$

2. CLOSABLE SPECTRAL SET FUNCTIONS

In this section, we summarize some definitions motivated by the considerations of the previous section. More detail can be found in [8]. Also the statements, which are presented here without proofs, are proved in [8].

Let $\mathcal{Q}$ be a quasialgebra of sets in a space Ω. That is, $\mathcal{Q}$ is a family of subsets of Ω such that $\emptyset \in \mathcal{Q}$, $\Omega \in \mathcal{Q}$ and, for any $X \in \mathcal{Q}$ and $Y \in \mathcal{Q}$, the sets $X \cap Y$ and

$X \setminus Y$ can be expressed as the unions of finite families of pair-wise disjoint elements of $\mathcal{Q}$. The family of all $\mathcal{Q}$-simple functions is denoted by $\text{sim}(\mathcal{Q})$. That is, $f \in \text{sim}(\mathcal{Q})$, if and only if, f is a complex linear combination of a finite collection of elements of $\mathcal{Q}$.

A map $P : \mathcal{Q} \to \text{BL}(E)$ is called a *spectral set function* if it is additive, multiplicative and $P(\Omega) = I$. A spectral set function $P : \mathcal{Q} \to \text{BL}(E)$ has a unique linear extension, also denoted P, to the whole of $\text{sim}(\mathcal{Q})$ with values in $\text{BL}(E)$ which is an algebra homomorphism. Let $A(P)$ denote the closure in $\text{BL}(E)$, with respect to the operator-norm, of the algebra of operators $\{P(f) : f \in \text{sim}(\mathcal{Q})\}$.

Given a spectral set function $P : \mathcal{Q} \to \text{BL}(E)$, a set $Y \subset \Omega$ will be called P-null if it can be covered by countably many sets $X \in \mathcal{Q}$ such that $P(X) = 0$. Let $\mathcal{N}$ be the family of all P-null sets. For the sake of simplicity, we have not indicated the spectral set function, P, in this notation, even though $\mathcal{N}$ depends on it. The same licence is used in denoting other objects, such as the following one.

For a complex valued function, f, on Ω, we define

$$\|f\|_{\infty} = \inf\{\sup\{|f(\omega)| : \omega \in \Omega \setminus Y\} : Y \in \mathcal{N}\} .$$

By $[f]_P$ is then denoted the class of all functions, g, on Ω such that $\|f - g\|_{\infty} = 0$. By the P-essential range of the function f is understood the set

$$\bigcap_{Y \in \mathcal{N}} \{f(\omega) : \omega \in \Omega \setminus Y\}^{-} ,$$

where the bar indicates closure in the complex plane.

The family of all functions, f, on Ω such that, for every $\epsilon > 0$, there exists a function $g \in \text{sim}(\mathcal{Q})$ with $\|f - g\|_{\infty} < \epsilon$, will be denoted by $\mathcal{L}^{\infty}(P)$. Then we define $L^{\infty}(P) = \{[f]_P : f \in \mathcal{L}^{\infty}(P)\}$. It turns out that $L^{\infty}(P)$ is a Banach algebra with respect to the operations induced by the natural operations in $\mathcal{L}^{\infty}(P)$ and with respect to the norm induced by the seminorm $f \mapsto \|f\|_{\infty}$, $f \in \mathcal{L}^{\infty}(P)$.

The spectral set function $P : \mathcal{Q} \to \text{BL}(E)$ will be called *closable* if, for any functions $f_j \in \text{sim}(\mathcal{Q})$, $j = 1, 2, \ldots$, such that

$$(6) \qquad \sum_{j=1}^{\infty} \|P(f_j)\| < \infty$$

and

$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ for which

(7)
$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty \,,$$

it follows that

$$\sum_{j=1}^{\infty} P(f_j) = 0 \,.$$

Let $P : \mathcal{Q} \to \mathrm{BL}(E)$ be a closable spectral set function. We say that a function f on Ω is P-integrable if there exist functions $f_j \in \mathrm{sim}(\mathcal{Q})$, $j = 1,2,...$, satisfying condition (6) such that

(8)
$$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega) \,,$$

for every $\omega \in \Omega$ for which the inequality (7) holds. We then define

$$P(f) = \int_{\Omega} f(\omega) P(d\omega) = \int_{\Omega} f dP = \sum_{j=1}^{\infty} P(f_j) \,,$$

where f_j, $j = 1,2,...$, are $\mathcal{Q}$-simple functions, satisfying condition (6), such that the equality (8) holds for every $\omega \in \Omega$ for which the inequality (7) does. By the assumption that the spectral set function P is closable, this definition is unambiguous. It then follows that

$$\|P(f)\| = \inf \sum_{j=1}^{\infty} \|P(f_j)\| \,,$$

where the infimum is taken over all choices of such $\mathcal{Q}$-simple functions f_j, $j = 1,2,...$.

LEMMA. *If f is a complex valued function on Ω, then $\|f\|_{\infty} = 0$ if and only if f is P-integrable and $P(f) = 0$.*

The family of all P-integrable functions is denoted by $\mathcal{L}(P)$. We define $L(P) = \{[f]_P : f \in \mathcal{L}(P)\}$. By the lemma, the seminorm $f \mapsto \|P(f)\|$, $f \in \mathcal{L}(P)$, induces a norm in the space $L(P)$. The space $L(P)$ is complete in this norm.

PROPOSITION 2. *Let* $P : Q \rightarrow \mathrm{BL}(E)$ *be a closable spectral set function. Then* $\mathcal{L}(P) \subset \mathcal{L}^\infty(P)$ *and* $\|f\|_\infty \leq \|P(f)\|$, *for every* $f \in \mathcal{L}(P)$.

If $f \in \mathcal{L}(P)$ *and* $g \in \mathcal{L}(P)$, *then* $fg \in \mathcal{L}(P)$ *and* $P(fg) = P(f)P(g)$. *So,* $\mathcal{L}(P)$ *is an algebra of functions and* $L(P)$ *is a Banach algebra with respect to the operations induced by those of* $\mathcal{L}(P)$.

The range of the integration map $P : \mathcal{L}(P) \rightarrow B(E)$ *is equal to* $A(P)$. *The Banach algebra* $A(P)$ *is semisimple. The integration map* $P : L(P) \rightarrow A(P)$ *is an isomorphism of the algebra* $L(P)$ *onto the algebra* $A(P)$.

If $f \in \mathcal{L}(P)$, *then the spectrum of the operator* $T = P(f)$ *is equal to the* P-*essential range of the function* f .

PROPOSITION 3. *A spectral set function* $P : Q \rightarrow B(E)$ *is closable if and only if there exists an injective map* $\Phi : A(P) \rightarrow L^\infty(P)$ *such that* $\|\Phi(T)\|_\infty \leq \|T\|$, *for every* $T \in A(P)$, *and* $\Phi(P(f)) = [f]_P$ *for every* $f \in sim(Q)$.

If the spectral set function $P : Q \rightarrow B(E)$ *is indeed closable, then such a map* Φ *is unique, its range is equal to* $L(P)$ *and the map* Φ *is equal to the inverse of the integration map.*

Recall that an algebra of sets is a quasialgebra of sets containing the union of any two of its members.

PROPOSITION 4. *Let* Q *be an algebra of sets in a space* Ω . *Let* $P : Q \rightarrow \mathrm{BL}(E)$ *be a bounded spectral set function such that* $P(Y) = 0$ *for every* P-*null set* $Y \in Q$. *Then* P *is a closable spectral set function.*

This proposition implies, in particular, that, if Q is a σ-algebra and $P : Q \rightarrow \mathrm{BL}(E)$ is a σ-additive spectral measure (see [4], Definition XV.2.3), then P is a closable spectral set function. It is straightforward that, in this case, $\mathcal{L}(P) = \mathcal{L}^\infty(P)$ and that the integral in the sense of our definition coincides with that discussed in [4].

Let us call a Boolean algebra of projections, $W \subset \mathrm{BL}(E)$, semisimple, if the smallest Banach algebra, $A(W)$, such that $W \subset A(W) \subset \mathrm{BL}(E)$, is semisimple.

PROPOSITION 5. *A Boolean algebra of projections,* $W \subset \mathrm{BL}(E)$ *, is semisimple if and only if there exists a semialgebra of sets,* Q *, in a space* Ω *and a closable spectral set function* $P : Q \to \mathrm{BL}(E)$ *such that* $A(W) = A(P)$ *.*

Now we can easily describe the type of Banach algebras referred to in the title of this note. They are Banach algebras of the form $A(W)$, where $W \subset \mathrm{BL}(E)$ is a semisimple Boolean algebra of projections. By Proposition 5 and Proposition 3, such an algebra is equal to the range of the integration map with respect to a closable spectral set function. So, every element of it can be approximated in the operator norm by linear combinations of commuting projections. Elements of such algebras can be viewed as generalizations of diagonalizable matrices. Indeed, by Proposition 4, and remarks after it, they include all scalar operators in the sense of N. Dunford. What is more, all elements of such algebras are decomposable operators in the sense of C. Foiaş (see [2], Definition 2.1.1). In fact, if W is a semisimple Boolean algebra of projections, then the structure space, Δ, of the Banach algebra $A(W)$, is totally disconnected, because it can be identified with the Stone space of W, so that W itself is (lattice) isomorphic with the Boolean algebra of all open and closed sets in Δ. Then, by a result of E. Albrecht ([1], Corollary 4.7), all elements of such an algebra are decomposable operators.

There is another step to be made in order to obtain a generalization of the whole Dunford's theory of spectral operators. Namely, to introduce commutative Banach algebras, $B \subset \mathrm{BL}(E)$, such that B is the direct sum of its radical and an algebra $A(W)$, for some semisimple Boolean algebra of projections, W. However, such algebras will not be considered here.

3. THE EXTENT OF THE INTRODUCED NOTIONS

The title of this section is of course not to be taken too strictly. We merely make some comments indicating the extent of the notions introduced in the previous section. Already the remarks of Section 1 indicate that these notions are neither vacuous nor superfluous. They show, for example, that, for every $p \in (1, \infty)$, the

operator of translation, T_x, is equal to the integral of an integrable function with respect to the closable spectral set function P^p. On the other hand, if $p \neq 2$, then T_x is not spectral in the sense of N. Dunford and, if $p < 2$, not even in the extended sense of W. Ricker (see [5]). We may note also that those remarks can be extended so as to cover arbitrary locally compact Abelian groups instead of $\mathbb{R}$ (see [8]), since the Gillespie lemma used in the proof of Proposition 1 is valid (and originally stated) in such generality. Now, we present examples of closable spectral set functions of a different kind. We show also that the product of two closable spectral set functions is again closable.

Let $\mathcal{Q}$ be a quasialgebra of sets in a space Ω. Let $\mu : \mathcal{Q} \to [0,\infty)$ be a σ-additive set function. Let φ be a continuous, increasing and concave function on $[0,\infty)$ such that $\varphi(0) = 0$. Let

$$\rho(X) = \varphi(\mu(X)) \, ,$$

for every $X \in \mathcal{Q}$.

We denote by $\mathcal{L}(\rho,\mathcal{Q})$ the family of all (individual) functions, u, on Ω for which there exist numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,...$, such that

$$(9) \qquad \sum_{j=1}^{\infty} |c_j| \rho(X_j) < \infty \, ,$$

and the equality

$$(10) \qquad u(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

holds for every $\omega \in \Omega$ satisfying the condition

$$(11) \qquad \sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty \, .$$

For every such function, $u \in \mathcal{L}(\rho,\mathcal{Q})$, we define

$$q(u) = \inf \sum_{j=1}^{\infty} |c_j| \rho(X_j) \, ,$$

where the infimum is taken over all choices of the numbers c_j and sets $X_j \in \mathcal{Q}$, satisfying (9), such that (10) holds for every $\omega \in \Omega$ for which (11) does.

The family of functions $\mathcal{L}(\rho,\mathcal{Q})$ is a vector space and q a seminorm on it such that $q(X) = \rho(X)$, for every $X \in \mathcal{Q}$. The norm induced by this seminorm on the quotient-space $L(\rho,\mathcal{Q}) = \mathcal{L}(\rho,\mathcal{Q})/q^{-1}(\{0\})$ is still denoted by q. The space $L(\rho,\mathcal{Q})$ is complete in the norm q (see [9], Proposition 2.26 and Proposition 2.1). The element of the space $L(\rho,\mathcal{Q})$ determined by a function $u \in \mathcal{L}(\rho,\mathcal{Q})$ is denoted by $[u]$.

Let $E = L(\rho,\mathcal{Q})$. Given a set $X \in \mathcal{Q}$, let $P(X)[u] = [Xu]$, for every $u \in \mathcal{L}(\rho,\mathcal{Q})$, where Xu is the point-wise product of X and u. It is straightforward that this defines unambiguously an element, $P(X)$, of $\mathrm{BL}(E)$ and that $P(X)$ is a projection. Moreover, we have the following

PROPOSITION 6. *The resulting map* $P: \mathcal{Q} \to \mathrm{BL}(E)$ *is a closable spectral set function.*

Proof. It is obvious that P is a spectral set function. So, it suffices to prove that it is closable. Before doing that, let us note that $P(f)[u] = [fu]$, for any function $f \in \mathrm{sim}(\mathcal{Q})$ and any $u \in \mathcal{L}(\rho,\mathcal{Q})$. Now, let $f_j \in \mathrm{sim}(\mathcal{Q})$, $j = 1,2,...,$ be functions such that

(6)
$$\sum_{j=1}^{\infty} \|P(f_j)\| < \infty,$$

and

(12)
$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ for which

(7)
$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty.$$

Let u be an arbitrary element of $\mathcal{L}(\rho,\mathcal{Q})$. Then, by (6),

$$\sum_{j=1}^{\infty} q(P(f_j)[u]) = \sum_{j=1}^{\infty} q(f_j u) < \infty$$

and, by (12),

$$\sum_{j=1}^{\infty} (f_j u)(\omega) = u(\omega) \sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |(f_j u)(\omega)| = |u(\omega)| \sum_{j=1}^{\infty} |f_j(\omega)| < \infty .$$

Therefore, by Proposition 2.1 of [9],

$$\lim_{n \to \infty} q\left[\sum_{j=1}^{n} P(f_j)[u] \right] = \lim_{n \to \infty} q\left[\sum_{j=1}^{n} f_j u \right] = 0 .$$

It follows that

$$\sum_{j=1}^{\infty} P(f_j) = 0 ,$$

which means that P is closable.

If the quasialgebra Q is not actually an algebra and $t/\varphi(t) \to 0$, as $t \to 0+$, then, in most cases, the additive extension of P to the algebra of sets generated by Q is not bounded. In such case, P does not have a σ-additive extension on the σ-algebra generated by Q.

EXAMPLE. Let $\Omega = (0,1]$, $Q = \{(s,t] : 0 \le s \le t \le 1\}$. Let $p > 1$ and $\rho(X) = (\lambda(X))^{1/p}$, for every $X \in Q$, where λ is the one-dimensional Lebesgue measure. Let $E = L(\rho, Q)$.

Because $\mathcal{L}(\rho, Q) \ne \mathcal{L}^p(\lambda)$ (see Example 4.16(ii) in Section 4C of [9]), the spectral set function $P : Q \to \mathrm{BL}(E)$ is surely not σ-additive; indeed, its additive extension on the algebra of sets generated by Q is not bounded. Nevertheless, if $n \ge 1$ is an integer and a set X is equal to the union of n pair-wise disjoint sets, X_k, $k = 1,2,...,n$, belonging to Q, then $\|P(X)\| \le n^{(p-1)/p}$. In fact, let u be a function belonging to $\mathcal{L}(\rho, Q)$. Let c_j be numbers and $Y_j \in Q$ sets, $j = 1,2,...,$ such that

$$\sum_{j=1}^{\infty} |c_j| \rho(Y_j) < \infty$$

and

$$u(\omega) = \sum_{j=1}^{\infty} c_j Y_j(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j| Y_j(\omega) < \infty .$$

Then

$$q(Y_j \cap X) \leq \sum_{k=1}^{n} \rho(Y_j \cap X_k) = \sum_{k=1}^{n} (\lambda(Y_j \cap X_k))^{1/p} \leq$$

$$\leq n^{(p-1)/p} \left[\sum_{k=1}^{n} \lambda(Y_j \cap X_k)\right]^{1/p} \leq n^{(p-1)/p}(\lambda(Y_j))^{1/p} = n^{(p-1)/p}\rho(Y_j) ,$$

for every $j = 1, 2, \ldots$, and

$$(Xu)(\omega) = \sum_{j=1}^{\infty} c_j Y_j(\omega) X(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j| (Y_j \cap X)(\omega) < \infty .$$

Therefore, $Xu \in \mathcal{L}(\rho, \mathcal{Q})$ and $q(Xu) \leq n^{(p-1)/p} q(u)$.

Now, let Z be the function on $\mathbb{R}$ which is periodic with period 1 and its restriction to Ω is equal to the characteristic function of the interval $(\frac{1}{2}, 1]$. For every $j = 1, 2, \ldots$, let X_j be the function $\omega \mapsto Z(2^{j-1}\omega)$, $\omega \in \Omega$. Hence $X_j \in \text{sim}(\mathcal{Q})$ and $\|P(X_j)\| \leq 2^{(j-1)(p-1)/p}$, for every $j = 1, 2, \ldots$. Also, if $f(\omega) = \omega$, then

$$f(\omega) = \sum_{j=1}^{\infty} 2^{-j} X_j(\omega) ,$$

for every $\omega \in \Omega$. Therefore, $f \in \mathcal{L}(P)$ and

$$\|P(f)\| \leq \sum_{j=1}^{\infty} 2^{-j} 2^{(j-1)(p-1)/p} = \frac{1}{2} \frac{2^{1/p}}{2^{1/p} - 1} .$$

Let us now consider products of spectral set functions. Let $\mathcal{Q}_\ell$ be a quasialgebra of sets in the space Ω_ℓ and $P_\ell : \mathcal{Q}_\ell \to \text{BL}(E)$ a spectral set function, $\ell = 1, 2, $.

We say that the spectral set functions P_1 and P_2 commute if $P_1(X_1)P_2(X_2) = P_2(X_2)P_1(X_1)$, for any sets $X_1 \in \mathcal{Q}_1$ and $X_2 \in \mathcal{Q}_2$.

PROPOSITION 7. *Let* $\Omega = \Omega_1 \times \Omega_2$ *and* $\mathcal{Q} = \{X_1 \times X_2 : X_1 \in \mathcal{Q}_1, X_2 \in \mathcal{Q}_2\}$. *Assume that the spectral set functions* $P_\ell : \mathcal{Q}_\ell \to \text{BL}(E)$, $\ell = 1, 2$, *are closable and that they commute. Let* $P(X_1 \times X_2) = P_1(X_1)P_2(X_2)$, *for every* $X_1 \in \mathcal{Q}_1$ *and* $X_2 \in \mathcal{Q}_2$.

Then $P : \mathcal{Q} \to \text{BL}(E)$ *is a closable spectral set function.*

Proof. It is obvious that P is a spectral set function. Also, a set $Y \subset \Omega$ is P-null if and only if there exist a P_1-null set Y_1 and a P_2-null set Y_2 such that $Y \subset (Y_1 \times \Omega_2) \cup \Omega_1 \times Y_2)$.

Given any functions, g and h, on the spaces Ω_1 and Ω_2, respectively, let us denote by $g \otimes h$ the function, f, on Ω such that $f(\omega) = g(\omega_1) h(\omega_2)$, for every $\omega = (\omega_1, \omega_2)$, $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. Let, further, $[g]_{P_1} \otimes [h]_{P_2} = [g \otimes h]_P$, for any such functions, g and h.

If Δ_ℓ is the structure space of the Banach algebra $L^\infty(P_\ell)$, $\ell = 1,2$, then $\Delta = \Delta_1 \times \Delta_2$ is the structure space of the Banach algebra $L^\infty(P)$. For, a function, f, on Ω is Q-simple if and only if

$$f = \sum_{j=1}^{k} g_j \otimes h_j,$$

with some $k = 1,2,\ldots$ and functions $g_j \in \text{sim}(Q_1)$ and $h_j \in \text{sim}(Q_2)$, $j = 1,2,\ldots,k$. Furthermore, by definition, the family of elements of the form $[f]_P$, where $f \in \text{sim}(Q)$, is dense in $L^\infty(P)$.

Let $A_0(P) = \{P(f) : f \in \text{sim}(Q)\}$. Because the map $P : \text{sim}(Q) \to \text{BL}(E)$ is linear and $\|f\|_\infty \leq \|P(f)\|$, for every $f \in \text{sim}(Q)$, there is a linear norm-decreasing map $\Phi_0 : A_0(P) \to L^\infty(P)$ such that $\Phi_0(P(f)) = [f]_P$, for every $f \in \text{sim}(Q)$. Because $A(P)$ is the closure of $A_0(P)$ and the space $L^\infty(P)$ is complete, there is a linear norm-decreasing map $\Phi : A(P) \to L^\infty(P)$ such that $\Phi(P(f)) = [f]_P$, for every $f \in \text{sim}(Q)$. To prove that the spectral set function P is closable, by Proposition 3, it suffices to show that the map Φ is injective.

Let $\Phi_\ell : A(P_\ell) \to L^\infty(P_\ell)$ be the injective map, whose existence is guaranteed by Proposition 3, such that $\|\Phi_\ell(T_\ell)\|_\infty \leq \|T_\ell\|$, for every $T_\ell \in A(P_\ell)$, and $\Phi_\ell(P_\ell(f_\ell)) = [f_\ell]_{P_\ell}$, for every $f_\ell \in \text{sim}(Q)$, $\ell = 1,2$. A simple argument based on the density of $A_0(P)$ in $A(P)$ then shows that $\Phi(T_1 T_2) = \Phi_1(T_1) \otimes \Phi_2(T_2)$, for every $T_1 \in A(P_1)$ and $T_2 \in A(P_2)$. Furthermore, because Φ_ℓ is an injective homomorphism with range dense in $L^\infty(P)$, the structure space of the algebra $A(P_\ell)$

can be identified with Δ_ℓ, $\ell = 1,2$. Consequently, the structure space of $A(P)$ can be identified with Δ. Thus, if we identify elements of the algebras $A(P)$ and $L^\infty(P)$ with their respective Gelfand transforms, Φ becomes the identity map on a dense subalgebra of $C(\Delta)$.

It is well-known that the product of two commuting σ-additive spectral measures is not necessarily a σ-additive spectral measure. The first example demonstrating this phenomenon was constructed by S. Kakutani (in [7]). Proposition 7 shows that closable spectral set functions do not suffer by this defect. It thereby gives us a method of constructing a wealth of closable spectral set functions.

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Centre for Mathematical Analysis
Australian National University
GPO Box 4
Canberra ACT 2601
Australia