SEMIGROUPS AND THE STRUCTURE OF WEIGHTED CONVOLUTION ALGEBRAS

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1. INTRODUCTION

Weighted convolution algebras on the half line $\mathbb{R}^+ = [0,\infty)$ arise naturally as the domains of the operational calculus maps determined by semigroups of bounded operators on a Banach space. For a continuous semigroup in a Banach algebra, which exists in every Banach algebra with bounded approximate identity [15, Th. 3.1, pp.35–36], this operational calculus map becomes an algebra homomorphism [15, pp.38–40], so weighted convolution algebras also arise as domains of Banach algebra homomorphisms. In fact, one of the motivations for the study of weighted convolution algebras is to learn more about the operational calculus map (see for example [15, Prob. 3.8, p.40], [11], [12, Th. 5.1]) and to use this map and structural features of weighted convolution algebras to study semigroups of operators and Banach algebras with a bounded approximate identity (see for instance [5], [15]). In this paper we report on research, much of it done jointly with F. Ghahramani and J.P. McClure, which, in a sense, goes in the other direction. We study homomorphisms and other properties of weighted convolution algebras by using semigroups of convolution operators defined on them.

Suppose that w is a positive Borel function on $[0,\infty)$ and that both w and $1/w$ are locally essentially bounded. Then $L^1(w)$ is the Banach space of (equivalence classes of) locally integrable functions f on $[0,\infty)$ for which fw is in $L^1(\mathbb{R}^+)$, with the inherited norm $\|f\| = \int_0^\infty |f(t)|\, w(t)\, dt$. Similarly, $M(w)$ is the space of locally finite measures for which the norm $\|\mu\| = \int_0^\infty w(t)\, d|\mu|(t) < \infty$. As usual, we consider $L^1(w)$ as a subspace of $M(w)$ by identifying f in $L^1(w)$ with $f(t)\, dt$ in $M(w)$. We are interested in the case that $L^1(w)$ is an algebra under the convolution product

$f*g(x) = \int_0^x f(x-t)\, g(t)\, dt$. In this case we can normalize w, without changing the elements or the topology of $L^1(w)$, so that: (i) w is right continuous; (ii) $w(x+y) \leq w(x)\, w(y)$; (iii) $w(0) = 1$ [12, Th. 2.1]. Such a normalized w will be called a *weight* (these are the strongly algebraic weights of [12]). With this normalization, $M(w)$ is also an algebra under convolution and it can be identified with the multiplier algebra of $L^1(w)$ and with the dual space of $C_0(1/w)$ (see [12, Th. 2.2] which just adds a few finishing touches to [7, Section 1] and [16, pp.303–306]). Here $C_0(1/w)$ is the Banach space of continuous functions on $\mathbb{R}^+$ for which $\lim_{t\to\infty} (f(t)/w(t)) = 0$ and with norm $\|f\| = \sup |f(t)/w(t)|$. With these identifications $M(w)$ has not only its usual topology, but also a weak* and strong operator topology, and if $\{\mu_t\}_{t\geq 0}$ is a semigroup of elements of $M(w)$, $\{\mu_t\}$ becomes identified with the semigroup of bounded operators $f \longrightarrow \mu_t*f$ on $L^1(w)$.

We are interested in the following questions and the relation between them:

(1)   When is a convolution semigroup $\{\mu_t\}_{t\geq 0}$ in $M(w)$ strongly continuous as a semigroup of operators on $L^1(w)$?

(2)   For which f in $L^1(w)$ is the principal ideal $L^1(w)*f$ dense in $L^1(w)$?

(3)   When is a continuous homomorphism $\phi : L^1(w_1) \longrightarrow L^1(w_2)$ particularly nice in the sense that it satisfies the equivalent conditions of Theorem 1.2 below?

The relations between these questions are described in Theorem (1.2) below, but first we need to make some definitions and elaborate on the questions.

Recall that a semigroup of bounded operators $\{U(t)\}_{t\geq 0}$ on a Banach space X is *strongly continuous* (or is a C_0 semigroup) if $U(t)x$ is continuous for $t \geq 0$ for all x in X [5, def. VIII.1.1, p.614] (actually it is enough to assume continuity at $t = 0$, [14, Cor. 2.3, p.4]). The convolution semigroups we consider will not obviously be strongly continuous at $t = 0$ (this is what question (1) asks). Hence we need to consider more general semigroups and therefore we say that $\{U(t)\}$ is *almost continuous* if $U(t)x$ is

continuous for $t > 0$ for all x and if $\|U(t)\|$ is bounded as $t \to 0+$. For an almost continuous semigroup $\{U(t)\}$, we let $w(t) = \|U(t)\|$ and define the *operational calculus map*

$$(1.1) \qquad \phi(f) = \int_0^\infty f(t)\, U(t)\, dt\,.$$

The integral is a strong Bochner integral, [13, p.85], but because of our continuity assumptions $\phi(f)x = \int_0^\infty f(t)\, U(t)x\, dt$ is actually just a (vector valued) improper Riemann integral. The operational calculus map is clearly a bounded linear map from $L^1(w)$ to the bounded operators on X, and it has an obvious extension to $M(w)$. It is not hard to show that ϕ, and its extension, are algebra homomorphisms, [13, pp.435–437]. One can renorm X with an equivalent norm so that some $\{e^{-rt}\, U(t)\}$ is a contraction semigroup, [14, pp.19–20]. In this case the weight w will be appropriately normalized, [11, Lemma (2.1), p.131].

Suppose now that $L^1(w_1)$ and $L^1(w_2)$ are weighted convolution algebras on $\mathbb{R}^+$ and let $\phi : L^1(w_1) \to L^1(w_2)$ be a non–zero continuous homomorphism. Then ϕ has a unique extension with the same norm to a homomorphism, which we continue to call ϕ, between the corresponding measure algebras [12, Th. 3.4]. Thus if $\{\delta_t\}$ is the semigroup of point masses in $M(w_1)$, which, as a semigroup of operators on $L^1(w_1)$, is just the right–translation semigroup, then $\mu_t = \phi(\delta_t)$ is a semigroup in $M(w_2)$ and is in fact almost continuous (see [12, Th. 3.6], or the discussion after Theorem (2.1) below). The operational calculus map determined by $\{\mu_t\}$ according to formula (1.1) is just the homomorphism ϕ, [12, Th. 3.6(c)] (cf. [15, pp.38–39]). Conversely, starting with an almost continuous semigroup $\{\mu_t\}$ in $L^1(w_2)$ and letting $w(t) = \|\mu_t\|$, the operational calculus map $\phi(f) = \int_0^\infty f(t)\, \mu_t\, dt$ defines a homomorphism from $L^1(w_1)$ to $M(w_2)$, [12, Th. 3.17], but it is not always easy to determine if ϕ maps $L^1(w_1)$ into $L^1(w_2)$.

If f is a locally integrable function on $\mathbb{R}^+$, we follow the usual terminology and let $\alpha(f)$ be the infimum of the support of f, and similarly for locally finite measures.

The function f in $L^1(w)$ with $\alpha(f) = d$ is said to be *standard* in $L^1(w)$, if the closure of the principal ideal $L^1(w)*f$ is the *standard ideal* $L^1(w)_d = \{g \in L^1(w) : \alpha(g) \geq d\}$. For $\alpha(f) = 0$ this just says $L^1(w)*f$ is dense, as in question (2) above. We can now state the result relating questions (1), (2), and (3).

EQUIVALENCE THEOREM (1.2). *Suppose that* $\phi : L^1(w_1) \longrightarrow L^1(w_2)$ *is a continuous non-zero homomorphism between weighted convolution algebras on* $\mathbb{R}^+$. *Let* $\mu_t = \phi(\delta_t)$ *and let* $\{e_n\}$ *be a bounded approximate identity in* $L^1(w_2)$. *Then* $\{\mu_t\}$ *is an almost continuous semigroup on* $L^1(w_2)$. *Moreover, the following are equivalent:*

(a) $\{\mu_t\}$ *is a strongly continuous semigroup in* $L^1(w_2)$.

(c)   *The closure of the range of* ϕ *contains a non-zero standard element of* $L^1(w_2)$.

(d)   *Whenever* $L^1(w)*f$ *is dense in* $L^1(w_1)$, *then* $L^1(w_2)*\phi(f)$ *is dense in* $L^1(w_2)$.

(e) $\{\phi(e_n)\}$ *is a bounded approximate identity in* $L^1(w_2)$.

(h)   *The extension* $\phi : M(w_1) \longrightarrow M(w_2)$ *is continuous in the strong operator topologies.*

The above *equivalence theorem* is just [9, Th. (2.2)], with three of the equivalent conditions, (b), (f), and (g), omitted. A homomorphism which satisfies the above equivalent conditions will be called a *standard homomorphism*. Since the equivalence theorem is proved in [9], we will not give a complete proof here. Rather we will study arbitrary almost continuous semigroups in weighted convolution algebras, mentioning whenever parts of the equivalence theorem follow from continuity of semigroups. In particular we will not prove the two deepest implications: (a) $\Rightarrow$ (d), which involves determining the generator of the semigroup $\{\mu_t\}$; and (e) $\Rightarrow$ (h), which uses the Cohen factorization theorem for modules.

In section 2 we investigate almost continuity and continuity of convolution semigroups on $L^1(w)$, mostly by adapting results which are given in [12] and [9] for

the semigroups $\mu_t = \phi(\delta_t)$ for a homomorphism ϕ. Theorem (2.1) gives characterizations and properties of almost continuous convolution semigroups. Theorem (2.3) gives a criterion for an almost continuous convolution semigroup to be strongly continuous, and Theorem (2.8) gives a class of $L^1(w)$ for which all almost continuous convolution semigroups are strongly continuous. The Equivalence Theorem above shows that if all almost continuous convolution semigroups on $L^1(w)$ are strongly continuous, then all homomorphisms into $L^1(w)$ are standard. Theorem (2.9) gives a converse of this result and Corollary (2.11) is an application of this converse.

In section 3 we examine the function $t \mapsto \alpha(\mu_t)$ for convolution semigroups and as an application show that every non–zero continuous endomorphism of a radical $L^1(w)$ is one–to–one. The results in section 3 are adapted from [12], [8].

We now collect for easy reference some useful computational facts about $L^1(w)$ and $M(w)$. For proofs, or references to standard proofs, see Lemmas 3.2, 4.1, and 4.7 of [12]. Recall that we always assume that the weight w is normalized in such a way that $M(w)$ is identified both with the dual space of $C_0(1/w)$ and with the multiplier algebra of $L^1(w)$.

LEMMA (1.3). *For the weighted convolution algebras* $L^1(w) \subseteq M(w)$ *we have the following*:

(a) $\alpha(\mu*\nu) = \alpha(\mu) + \alpha(\nu)$, *so that, in particular* $M(w)$ *is an integral domain.*

(b) *If the net* $\{\mu_n*\nu\}$ *converges to* λ *in the strong operator topology of* $M(w)$, *then* $\alpha(\nu) \le \alpha(\lambda)$ *and* $\limsup(\alpha(\mu_n)) \le \alpha(\lambda) - \alpha(\mu)$.

(c) *If* $\{\mu_n\}$ *is a bounded net in* $M(w)$ *and* $\nu \ne 0$, *then* $\{\mu_n\}$ *converges to* μ *in the weak* topology of* $M(w) = C_0(1/w)^*$ *if and only if* $\{\mu_n*\nu\}$ *converges weak* to* $\mu*\nu$.

2. CONTINUITY OF SEMIGROUPS

Our first continuity result characterizes almost continuity of convolution semigroups. We see that for these semigroups relatively little is needed to guarantee almost continuity and that almost continuous convolution semigroups have additional continuity properties.

THEOREM (2.1). *Suppose that* $\{\mu_t\}$ *is a convolution semigroup in* $M(w)$. *If* $\|\mu_t\|$ *is bounded as* $t \to 0+$, *the following are equivalent*:

(a) $\{\mu_t\}$ *is almost continuous.*

(b) *For all* v *in* $M(w)$, $\mu_t * v$ *is weak*–continuous for* $t \geq 0$.

(c) $\lim_{t \to 0+} \mu_t = \delta_0$ *in the weak*–topology on* $M(w)$.

(d) *For some* $v \neq 0$ *in* $M(w)$ *and some* $a \geq 0$, *is* $\{\mu_t * v\}$ *weak*–continuous from the right at* $t = a$.

Proof.   We first note that one can take $a = 0$ in (d). For the formula $\mu_{a+h} * v = \mu_h * (\mu_a * v)$ shows that right weak*–continuity at a for some v is equivalent to weak*–continuity at 0 for some other measure. It now follows from (1.3) (c) that (c) and (d) are equivalent to each other and to (b) for right continuity. To obtain left continuity we use the formula

$$\|\mu_{t-s} * v - \mu_t * v\| \leq \|\mu_{t-s}\| \; \|v - \mu_s * v\|$$

in the usual way [14, p.4]. Since (a) clearly implies (d), we just need to show that (b) implies (a). We adapt the proof from our [12, Th. 3.6]. For all g in $L^1(w) \subseteq M(w)$, we have that $\mu_t * g$ is weak*–continuous and therefore weak*–measurable. Thus for all h in $L^1(w)$, the real–valued function $\|\mu_t * g - h\|$ is measurable, [13, Th. 3.5.2, p.72]. Since $\mu_t * g$ takes its values in the separable space $L^1(w)$, the usual proof of the Pettis measurability theorem, [13, Th. 3.5.3, pp.72–73], shows that $\mu_t * g$ is strongly measurable. Thus $\{\mu_t\}$ is a strongly measurable semigroup of bounded operators on the space

$L^1(w)$, and is therefore strongly continuous for $t > 0$, [12, Th. 10.2.3, p.305]. Thus $\{\mu_t\}$ is almost continuous, and the proof is complete.

The above theorem shows that the semigroup $\mu_t = \phi(\delta_t)$ in the Equivalence Theorem is almost continuous. The boundedness follows from $\|\mu_t\| \leq \|\phi\| \|\delta_t\| = \|\phi\| w_1(t)$. For continuity, choose some f in $L^1(w_1)$ for which $\phi(f)$ is not zero. Then in the norm topology, we have

$$(2.2) \qquad \lim_{t \to 0+} \mu_t * \phi(f) = \lim_{t \to 0+} \phi(\delta_t * f) = \phi(f)$$

by the continuity of ϕ. Thus $\{\mu_t\}$ satisfies condition (d) of Theorem (2.1).

We now give a criterion for an almost continuous convolution semigroup to be strongly continuous. The proof is adapted from [12, Cor. 3.13].

THEOREM (2.3). *Suppose that* $\{\mu_t\}$ *is an almost continuous semigroup in* $M(w)$. *If there is a standard* $g \neq 0$ *in* $L^1(w)$ *for which* $\lim_{t \to 0+} \mu_t * g = g$ *in norm, then* $\{\mu_t\}$ *is a strongly continuous semigroup of operators on* $L^1(w)$.

Proof. In order to use the arguments in this proof later, we postpone invoking the hypothesis that $\mu_t * g \longrightarrow g$ for some standard g until the end of the proof. For the almost continuous semigroup $\{\mu_t\}$ in $M(w)$, we define the *convergence ideal*

$$(2.4) \qquad I = I(\mu_t) = \{f \in L^1(w) : \lim_{t \to 0+} \mu_t * f = f\}.$$

It is clear that I is an ideal and, since μ_t as bounded as $t \longrightarrow 0+$, that I is closed in norm. Also since $\mu_{t+s} * f = \mu_s * (\mu_t * f)$, it follows from the definition of almost continuity that I contains all $\mu_t * f$, and in particular that $I \neq \{0\}$. It thus follows from Lemma (1.3) (b) that $\lim_{t \to 0+} \alpha(\mu_t) = 0$. Hence

$$\alpha(I) = \inf\{\alpha(f) : f \in I\} = 0.$$

In the current theorem, we are assuming that I contains a standard element. Hence I itself must be a standard ideal, [10, Lemma 8.2, p.548]. Since $\alpha(I) = 0$, I must then be all of $L^1(w)$, and the proof is complete.

In the next result we collect information about the convergence ideal, when we do not assume that the semigroup is strongly continuous. For much more information in the case that $\mu_t = \phi(\delta_t)$ see [9, Th. (2.4)].

COROLLARY (2.5). *If I is the convergence ideal of an almost continuous semigroup $\{\mu_t\}$ in $M(w)$, then I is a norm–closed weak*–dense ideal in $L^1(w)$ and*

$$I = cl[\cup_{t>0} \mu_t * L^1(w)].$$

Proof. The proof of the previous theorem shows that I is a norm–closed ideal containing all $\mu_t * L^1(w)$ and hence containing the closure of $\cup_{t>0} \mu_t * L^1(w)$. Each f in I satisfies $f = \lim_{t \to 0+} \mu_t * f$ and is thus a limit of elements in $\cup_{t>0} \mu_t * L^1(w)$, giving the reverse inclusion. Finally, the weak*–continuity of the semigroup, given in Theorem (2.1) (b) above, shows that I is dense in the relative weak*–topology of $L^1(w) \subseteq M(w)$.

Notice that formula (2.2) shows that for $\mu_t = \phi(\delta_t)$ the convergence ideal is a closed ideal containing the range of ϕ. Thus Theorem (2.3) gives the implication (c) $\Rightarrow$ (a) in the Equivalence Theorem.

The next result shows that one can always obtain strong continuity by passing to a larger algebra (cf. [12, Cor. (3.16)]).

COROLLARY (2.6). *If $\{\mu_t\}$ is an almost continuous semigroup in $M(w_1)$, then there is an $M(w_2) \supseteq M(w_1)$ for which $\{\mu_t\}$ is a strongly continuous semigroup on $L^1(w_2)$.*

Proof. Let g be a non–zero element of $L^1(w_1)$ for which $\lim\limits_{t\to 0+} \mu_t * g = g$ in the norm of $L^1(w_1)$. By [10, Th. (6.5), p.549], there is an $L^1(w_2) \supseteq L^1(w_1)$ in which g is a standard element. Since $L^1(w_1)$ is continuously embedded in $L^1(w_2)$, we have $\mu_t * g \longrightarrow g$ in the norm of $L^1(w_2)$. Hence it follows from Theorem (2.3) that $\{\mu_t\}$ is a strongly continuous semigroup on $L^1(w_2)$. This completes the proof.

The most interesting question is: for which $L^1(w)$ is every almost continuous homomorphism continuous? One sufficient condition is that all closed ideals are standard, and Domar [4] has shown that this holds if w is logarithmically convex and satisfies a suitable growth condition. On the other hand Dales and McClure [3] have constructed radical algebras $L^1(w)$ with nonstandard closed ideals. By a simple adaptation of the arguments in our joint paper [9], we can show that for so–called regulated weights [1], [2] all almost continuous semigroups are continuous, even though the Dales–McClure counterexamples can have regulated weights.

Recall, [1, def. 1.3, p.81], that the weight w is *regulated at* $a \geq 0$ if $\lim\limits_{x\to\infty} w(x+y)/w(x) = 0$ for all $y > a$. For such weights, $L^1(w)$ must be radical [1, p.82]. We now collect, with references to the literature, the topological facts we need about $L^1(w)$ with regulated weights.

LEMMA (2.7). *If the weight w is regulated at $a \geq 0$, then for all g in $L^1(w)$ with $\alpha(g) \geq a$ we have*:

(a)   *Convolution by g is a compact operator on $L^1(w)$.*

(b)   *Convolution by g is a compact operator on $M(w)$.*

(c)   *If $\{\lambda_n\}$ is a bounded net in $M(w)$ for which $\{\lambda_n * g\}$ converges in the weak*–topology to $\lambda * g$, then $\lambda_n * g$ converges in norm to $\lambda * g$.*

Part (a) is the fundamental result of Bade and Dales [1, Lemma 1.4 and Th. 2.2].

Part (b) is an easy consequence of this [9, Lemma (3.1)], and (c) follows easily from (b), [9, Th. (3.2)]. We can now prove the promised continuity theorem (cf. [9, Th. (3.4)]).

THEOREM (2.8). *If w is a regulated weight, then every almost continuous convolution semigroup $\{\mu_t\}$ on $L^1(w)$ is strongly continuous.*

Proof. Suppose w is regulated at a and choose $g \neq 0$ standard with $\alpha(g) \geq a$ (for instance $g = \delta_a * u$ where $u(x) \equiv 1$). Then $\{\mu_t * g\}$ converges in the weak*–topology to g as $t \longrightarrow 0+$, by Theorem (2.1) (d). Hence, by Lemma (2.6) (c), $\lim_{t \to 0+} \mu_t * g = g$ in norm. But g is standard, so $\{\mu_t\}$ is strongly continuous by Theorem (2.3). This completes the proof.

It follows from the Equivalence Theorem (1.2) that, if all almost continuous convolution semigroups on $L^1(w)$ are strongly continuous (which would happen for regulated weights), then every homomorphism to $L^1(w)$ is standard. The following shows that the converse is true.

THEOREM (2.9). *For the convolution algebra $L^1(w)$, the following are equivalent:*

(a) *Every almost continuous semigroup in $M(w)$ is a strongly continuous semigroup on $L^1(w)$.*

(b) *Every continuous non–zero homomorphism from any convolution algebra $L^1(w_1)$ to $L^1(w)$ is standard.*

(c) *Every continuous non–zero endomorphism of $L^1(w)$ is standard.*

Proof. Because of the Equivalence Theorem it is enough to prove that (c) implies (a). So suppose that $\{\mu_t\}$ is an almost continuous semigroup. Let α_t be one of the standard summability kernels on $\mathbb{R}^+$ (like $t \mapsto x^{t-1}/\Gamma(t)$) [15, chapter 1]. Since $\lim_{t \to \infty} \|\mu^t\|^{1/t}$ exists

and is finite, we can find an $r > 0$ for which $\{e^{-rt}\alpha_t\}$ and $\{(e^{-rt}\alpha_t)*\mu_t\}$ are bounded semigroups in $L^1(w)$.

Now define the semigroup $\beta_t = (e^{-rt}\alpha_t)*\mu_t*\delta_t$. Since $\{e^{-rt}\alpha_t\}$ is in $L^1(w)$ and is norm–continuous for $t > 0$, $\{\beta_t\}$ is also in $L^1(w)$, and an easy argument shows that it is norm–continuous for $t > 0$. Also by our choice of r , we have $\|\beta_t\| = O(\|\delta_t\|) = O(w(t))$. Thus when we define the operational calculus map $\phi(f) = \int_0^\infty f(t)\,\beta_t\,dt$ as a homomorphism from $L^1(w)$ to $M(w)$, we have not just a strong Bochner integral but a uniform Bochner integral (in fact an improper Riemann integral). Hence $\phi(f) \in L^1(w)$; i.e. ϕ is an endomorphism with $\phi(\delta_t) = \beta_t$. It then follows from condition (c) that β_t is a strongly continuous semigroup. Since both $\{e^{-rt}\alpha_t\}$ and $\{\delta_t\}$ are strongly continuous, we have that $\{\mu_t\}$ is strongly continuous by the following simple lemma.

LEMMA (2.10). *Suppose that* $\{T(t)\}$ *is a strongly continuous semigroup of operator on a Banach space* X *and that* $\{\delta(t)\}$ *is a commuting semigroup with* $\|\delta(t)\|$ *bounded as* $t \longrightarrow 0+$. *Then* $\{\delta(t)\}$ *is a strongly continuous semigroup if and only if* $U(t) = \delta(t)\,T(t)$ *is a strongly continuous semigroup.*

Proof. The lemma, and hence Theorem (2.9) as well, are immediate consequences of the formula $U(t)x - x = \delta(t)(T(t)x - x) + (\delta(t)x - x)$.

As an application of Theorem (2.9), we prove the following result.

COROLLARY (2.11). *If* I *is the convergence ideal of an almost continuous semigroup* $\{\mu_t\}$ *on* $L^1(w)$, *then there is a* g *in* I *with* $\alpha(g) = 0$.

Proof. From the proof of the last theorem, it is clear that we can assume that $\mu_t = \phi(\delta_t)$ for some non–zero continuous automorphism of $L^1(w)$. (Just replace $\{\mu_t\}$ with $\{\beta_t\}$ if necessary.) Let f be a standard element of $L^1(w)$ with $\alpha(f) = 0$ (for instance, let

$f(t) = e^{-rt}$ for e^{-rt} in $L^1(w)$). Then $g = \phi(f)$ has $\alpha(g) = 0$, [12, Lemma 4.5], and g belongs to I by formula (2.2).

One can also give an alternate proof of the above corollary which does not use Theorem (2.9). For each $a > 0$, let $I_a = \{g \in I : \alpha(g) \geq a\}$. Then I_a is a closed subspace of I. In the proof of Theorem (2.9) we showed that $\alpha(I) = 0$, so that each I_a is a proper closed subspace. Thus $\underset{a>0}{\cup} I_a = \underset{n=1}{\overset{\infty}{\cup}} I_{1/n}$ is a first–category subset of I, so by the Baire category theorem there is a g in I but in no I_a. For such g, $\alpha(g)$ must be zero.

3. THE SUPPORT FUNCTION

In this section we survey some results related to $\{\alpha(\mu_t)\}$ for an almost continuous semigroup. The results are adapted from [12, section 4], where many of the proofs are adapted from [8, pp.344–348]. We also give an application from [12, Section 5] showing that certain homomorphisms are one–to–one. Parts (a) and (b) of the next lemma are given for more general semigroups in [12, Th. 4.3].

LEMMA (3.1). *Suppose that* $\{\mu_t\}$ *is an almost continuous convolution semigroup on* $L^1(w)$. *Then we have:*

(a)   *There is a number* $A \geq 0$ *for which* $\alpha(\mu_t) = At$ *for* $t \geq 0$.

(b)   *There is a complex number* b *for which* $\mu_t(At) = b^t$ *for* $t \geq 0$.

(c)   *If* $L^1(w)$ *is a radical algebra and if there is a number* c *for which* $\underset{t \to \infty}{\limsup} (\|\mu_t\|/w(ct))^{1/t} \leq \infty$, *then* $A \geq c$.

Proof. Let $\beta(t) = \alpha(\mu_t)$. It follows from the Titchmarsh convolution theorem, given in Lemma (1.3) (a), above, that $\beta(s+t) = \beta(s) + \beta(t)$. Since μ_t is almost continuous, there

is $g \neq 0$ with $\lim_{t \to 0+} \mu_t * g = g$, so it follows from Lemma (1.3) (b) that $\lim_{t \to 0+} \beta(t) = 0$.

It is now easy to see, [17], that $\beta(t) = \alpha(\mu_t) = At$ for some $A \geq 0$.

Now let $k(t) = \mu_t(At)$ and write $\mu_t = k(t) \delta_{At} + \lambda_t$. Since $\alpha(\lambda_t) \geq At$ and $\lambda_t(At) = 0$, we have, for all measures ν, that $\lambda_t * \nu(At + \alpha(\nu)) = 0$. Thus multiplying out the expressions for μ_s and μ_t gives $k(s+t) = k(s) k(t)$ (cf. [8, p.348]). Also, since δ_{At} and λ_t are mutually singular measures, $\|k(t) \delta_{At}\| = |k(t)| \, w(At) \leq \|\mu_t\|$, which is bounded near 0. It is now easy to see that $k(t) = b^t$ for some b, [17]. This completes the proof of part (b).

Bade and Dales [1, Th. 3.6, p.99] show that if f in $L^1(w)$ satisfies $\limsup(\|f^{*n}\|/w(cx))^{1/n} < \infty$, then $\alpha(f) \geq c$. Choose some g with $\alpha(g) = 0$, then applying the Bade–Dales result to $f = g*\mu_1$ gives $A = \alpha(\mu) \geq c$ (cf. the proof of [12, Th. 4.7)]. This completes the proof.

The number A in the above lemma is called the *character of the semigroup* $\{\mu_t\}$. When ϕ is a homomorphism the *character of* ϕ is the character of the semigroup $\{\phi(\delta_t)\}$. The distinction between positive and zero character is important, and the best results require positive character. We omit the proof of the next result which is part of [12, Th. 4.9].

THEOREM (3.2). *Suppose that* $\phi : L^1(w_1) \to L^1(w_2)$ *is a continuous non–zero homomorphism with character* A. *If* A *is positive, then*

(a) $\alpha(\phi(\mu)) = A\alpha(\mu)$ *for all* μ *in* $M(w_1)$.

(b)   *If for any* μ *in* $M(w_1)$ *with* $\alpha(\mu) > 0$ *the measure* $\phi(\mu)$ *has non–zero mass at* $\alpha(\phi(\mu))$, *then whenever* λ *in* $M(w_1)$ *has point mass at* $\alpha(\lambda) > 0$ *we also have* $\phi(\lambda)[\alpha(\phi(\lambda))] \neq 0$.

As an application we have the following result taken from Theorem 4.7 and Corollary 5.3 of [12].

THEOREM (3.3). *Suppose that ϕ is a continuous non–zero endomorphism of the radical convolution algebra $L^1(w)$. Then ϕ has character $A \geq 1$ and ϕ is one–to–one.*

Proof. Let $\mu_t = \phi(\delta_t)$ and let A be the character of $\{\mu_t\}$ and ϕ. Since $\|\mu_t\| \leq \|\phi\| \|\delta_t\| = \|\phi\| \, w(t)$, it follows from Lemma (3.1) (c) that $A \geq 1$. Now suppose that g is a non–zero element of $L^1(w)$ with $\phi(g) = 0$, and let $a = \alpha(g)$. Choose some $b > a$ and let $g_1 = g\chi_{[0,b)}$ be the "restriction" of g to $[0,b)$; also let $g_2 = g - g_1 = g\chi_{[b,\infty)}$. Since $\alpha(g_1) \neq \alpha(-g_2)$, it follows from Theorem (3.2) (a) that $\alpha(\phi(g_1)) \neq \alpha(\phi(-g_2))$. But this contradicts our assumption that $0 = \phi(g_1) - \phi(-g_2)$. This completes the proof.

In [12], the fact that ϕ is one–to–one is obtained as a special case of a result [12, Th. 5.1] which gives a sufficient condition for the operational calculus map of formula (1.1) to be injective for an almost continuous quasinilpotent semigroup of operators on a Banach space.

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