

CONICAL OPEN MAPPING THEOREMS AND REGULARITY

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Abstract. Suppose T is a continuous linear operator between two Hilbert spaces X and Y and let K be a closed convex nonempty cone in X . We investigate the possible existence of $\delta > 0$ such that $\delta B_Y \cap T(K) \subseteq T(B_X \cap K)$, where B_X, B_Y denote the closed unit balls in X and Y respectively. This property, which we call openness relative to K , is a generalization of the classical openness of linear operators. We relate relative openness to Jameson's property (G), to the strong conical hull intersection property, to bounded linear regularity, and to metric regularity. Our results allow a simple construction of two closed convex cones that have the strong conical hull intersection property but fail to be boundedly linearly regular.

AMS subject classifications. Primary 90C25; Secondary 47B99, 49N15, 52A05, 52A07.

Key words. Bounded linear regularity, linear regularity, conical hull intersection property, metric regularity, normal cone, open mapping theorem, property (G), strong CHIP, strong conical hull intersection property.

1. Introduction. Throughout this paper, we assume

$$X \text{ and } Y \text{ are two real Hilbert spaces and } T : X \rightarrow Y$$

is a continuous linear operator. The celebrated *open mapping theorem* (see, for instance [8, Theorem III.12.1]) proclaims the existence of a positive $\delta > 0$ such that

$$\delta B_Y \subseteq T(B_X)$$

provided that T is onto. (Here and elsewhere B_X and B_Y stand for the closed unit balls in X and Y , respectively.)

Our aim in this paper is to discuss a more general openness property relative to a cone. We also show how this property closely relates to Jameson's property (G), to strong CHIP, to bounded linear regularity, and to metric regularity.

Specifically, we assume throughout that

$$K \text{ is a closed convex cone in } X$$

and we investigate under which conditions one can be assured of the existence of $\delta > 0$ such that

$$\delta B_Y \cap T(K) \subseteq T(B_X \cap K).$$

If this is the case, as is when $K = X$ and T is onto by the open mapping theorem, then we say that T is *open relative to* K .

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Our main results can be summarized as follows.

- R1** If T is open relative to K , then $T(K)$ is closed (Theorem 2.1).
- R2** $T(K)$ can be closed without T being open relative to K (Example 2.1).
- R3** Suppose L_1, L_2 are two closed convex cones such that $\{L_1, L_2\}$ has strong CHIP but is not boundedly linearly regular. Then one can construct T and K as in **R2** (Theorem 3.1).
- R4** If T and K are as in **R2**, then one can construct an example of L_1, L_2 as in **R3** (Theorem 3.2).
- R5** Metric regularity in form of injectivity of $T|_K$ yields openness of T relative to K (Corollary 4.1).

By combining **R2** and **R4**, we obtain two cones that have strong CHIP but fail to be boundedly linearly regular — the construction in [6] is much more involved.

We conclude by fixing notation, which essentially follows Rockafellar's classical [15]. Suppose S is a set in X . Then $\text{cl } S$ (resp. $\text{int } S$, $\text{ri } S$, $\text{conv } S$, $\text{cone } S$, S^\ominus , S^\perp) stands for the *closure* (resp. *interior*, *relative interior*, *convex hull*, *conical hull*, *(negative) polar cone*, *orthogonal complement*) of S . We write $d(\cdot, S)$ for the *distance function* corresponding to S : $x \mapsto \inf\{\|x - s\| : s \in S\}$. For us a *cone* is a *nonempty* set closed under multiplication by nonnegative reals; in particular, every cone contains 0. If $x \in S$, then $N_S(x) = (S - x)^\ominus$ is the *normal cone* of S at x . The *indicator function* of S is denoted ι_S (0 in S ; $+\infty$ outside S). The *conjugate* (or *transpose*) of a linear operator T is denoted T^* . Finally, f^* stands for the (*Fenchel*) *conjugate* of a function f .

2. Basic properties.

$T(K)$ is open relative to $K \Rightarrow T(K)$ is closed.

DEFINITION 2.1. We say that T is *open relative to K* , if there exists some $\delta > 0$ such that $\delta B_Y \cap T(K) \subseteq T(B_X \cap K)$.

Theorem 2.1. *If T is open relative to K , then $T(K)$ is closed.*

Proof. Obtain $\delta > 0$ such that $\delta B_Y \cap T(K) \subseteq T(B_X \cap K)$ and pick an arbitrary $y \in \text{cl } T(K)$. After scaling if necessary, we assume without loss of generality that $\|y\| < \delta$. Now $B_X \cap K$ is weakly compact, hence so is $T(B_X \cap K)$. In particular, $T(B_X \cap K)$ is closed. Hence $y \in \text{cl}(\delta B_Y \cap T(K)) \subseteq \text{cl}(T(B_X \cap K)) = T(B_X \cap K) \subseteq T(K)$ and the result follows. \square

If K happens to be a *subspace*, then the converse implication in Theorem 2.1 holds true (by the open mapping theorem). In the next subsection, we will show how this can go wrong for a cone K .

T is open relative to $K \not\Rightarrow T(K)$ is closed.

EXAMPLE 2.1. Let $X := \mathbf{R}^4$, $Y := \mathbf{R}^3$, and $T : X \rightarrow Y : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$. Consider the curve

$$y(t) := \frac{t}{\sqrt{2}}(\cos(2\pi t), \sin(2\pi t), 1), \quad t \in [0, 1]$$

and define in X the closed convex cone

$$K := \text{cl cone}\{(y(t), \sqrt{1 - t^2}) : t \in [0, 1]\}.$$

Then $T(K)$ coincides with the *icecream cone* (see Figure 2.1)

$$L := \{y \in Y : y_1^2 + y_2^2 \leq y_3^2, y_3 \geq 0\},$$

and for every $t \in (0, 1]$, the ray generated by the vector $y(t)$ is an extreme ray of L . Despite the closedness of $T(K)$, the mapping T is not open relative to K .

Proof. Let $z(t) := (y(t), \sqrt{1-t^2})$, for every $t \in [0, 1]$. Then $\|y(t)\| = t$ and $\|z(t)\| = 1$. Clearly, $\{z(t) : t \in [0, 1]\}$ is compact, hence (by [15, Theorem 17.2]) so is

$$C := \text{conv} \{z(t) : t \in [0, 1]\}.$$

Claim 1: $\min_{t \in [0, 1]} \max\{t/\sqrt{2}, \sqrt{1-t^2}\} = 1/\sqrt{3}$. Calculus.

Claim 2: $\min \|C\| > 0$.

Every element of C is of the form

$$x = \sum_{i=1}^n \lambda_i (t_i \cos(2\pi t_i)/\sqrt{2}, t_i \sin(2\pi t_i)/\sqrt{2}, t_i/\sqrt{2}, \sqrt{1-t_i^2}),$$

for some $t_i \in [0, 1]$ and $\lambda_i > 0$ with $\sum_{i=1}^n \lambda_i = 1$. We now distribute the indices $i \in \{1, \dots, n\}$ as follows: $i \in I_3 : \Leftrightarrow t_i/\sqrt{2} \geq \sqrt{1-t_i^2}$ and $I_4 := \{1, \dots, n\} \setminus I_3$. In view of Claim 1, we deduce that the third and fourth components of x satisfy

$$x_3 \geq \sum_{i \in I_3} \lambda_i \frac{t_i}{\sqrt{2}} \geq \sum_{i \in I_3} \lambda_i \frac{1}{\sqrt{3}} \quad \text{and} \quad x_4 \geq \sum_{i \in I_4} \lambda_i \sqrt{1-t_i^2} \geq \sum_{i \in I_4} \lambda_i \frac{1}{\sqrt{3}}.$$

Subclaim: $\max\{x_3, x_4\} \geq 1/(2\sqrt{3})$.

If the Subclaim were false, then (using the last displayed inequalities) $1/\sqrt{3} = \sum_{i \in I_3} \lambda_i/\sqrt{3} + \sum_{i \in I_4} \lambda_i/\sqrt{3} \leq x_3 + x_4 < 1/(2\sqrt{3}) + 1/(2\sqrt{3}) = 1/\sqrt{3}$, which is absurd. The Subclaim thus holds.

Claim 2 now follows immediately from the Subclaim. It also implies (directly or by [15, Corollary 9.6.1]) that

$$K = \text{cone} \{(y(t), \sqrt{1-t^2}) : t \in [0, 1]\} = [0, +\infty) \cdot C.$$

Hence $T(K) = \text{cone} \{y(t) : t \in [0, 1]\} = L$, since every ray generated by $y(t)$ with $t \in (0, 1]$ is clearly an extreme ray of the icecream cone L . (See Figure 2.1.) It only remains to show that T is not open relative to K . Now fix an arbitrary $\delta > 0$ and let $t^* := \min\{1, \delta/2\}$. Since $\|y(t^*)\| = t^*$, we have $2y(t^*) \in \delta B_Y \cap L$. Because the ray generated by $y(t^*)$ is an extreme ray of L , there is only one element in K that is also a pre-image of $2y(t^*)$ under T : $2z(t^*)$. However, since $\|z(t^*)\| = 1$, the vector $2z(t^*)$ does not belong to $B_X \cap K$. We thus have shown $2y(t^*) \in (\delta B_Y \cap T(K)) \setminus T(B_X \cap K)$. Therefore, since $\delta > 0$ was chosen arbitrarily, the operator T is not open relative to K . \square

Jameson's property (G). We now show that Jameson's property (G) (see [13]), a geometric property of a collection of closed convex cones, can be expressed in the relative openness framework.

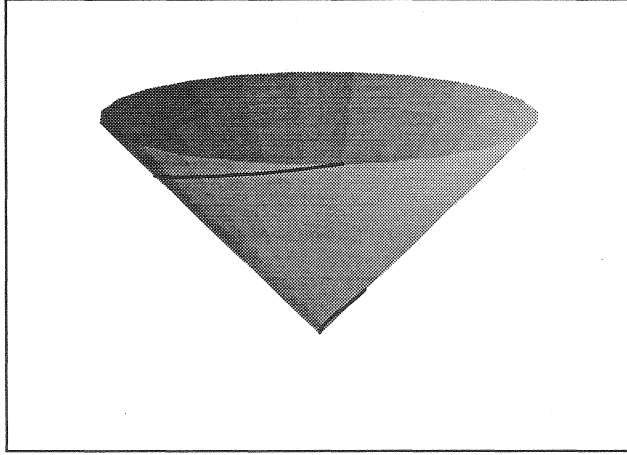


FIG. 2.1. The icecream cone L , which is also a three-dimensional cross section of K , and the curve $y(t)$.

DEFINITION 2.2. Suppose K_1 and K_2 are two closed convex cones in X . Then $\{K_1, K_2\}$ has *property (G)*, if there exists $\delta > 0$ such that $\delta B_X \cap (K_1 + K_2) \subseteq (B_X \cap K_1) + (B_X \cap K_2)$.

Our first relates property (G) of two cones to the openness property. We omit the simple proof.

Proposition 2.1. Suppose K_1, K_2 are two closed convex cones in X . Then $\{K_1, K_2\}$ has *property (G)* if and only if the sum operator

$$T_\Sigma : X \times X \rightarrow X : (x_1, x_2) \mapsto x_1 + x_2$$

is open relative to $K_1 \times K_2$.

Corollary 2.1. Suppose K_1, K_2 are two closed convex cones in X . If $\{K_1, K_2\}$ has *property (G)*, then $K_1 + K_2$ is closed.

Proof. Combine Theorem 2.1 with Proposition 2.1. □

REMARK 2.1. Suppose X is finite dimensional and K_1, K_2 are two closed convex cones in X . Let $K := K_1 + K_2$. We now show that $\{K_1, K_2\}$ has property (G) whenever K is polyhedral. According to [5, Corollary 2.10], $\{K_1, K_2\}$ has property (G) if and only if there exists $\alpha > 0$ such that the map $q : K \rightarrow [0, +\infty)$, defined by

$$q(x) := \min\{\max\{\|x_1\|, \|x_2\|\} : x_1 \in K_1, x_2 \in K_2, x_1 + x_2 = x\},$$

satisfies $q(x) \leq \alpha\|x\|$, $\forall x \in K$. It is elementary to verify that q is sublinear, hence convex. Now assume in addition that K is polyhedral. Then $K \cap B_\infty$ is polyhedral as well, where B_∞ denotes the closed unit ball with respect to the max-norm in X . In fact, $K \cap B_\infty$ is a polytope. It follows from [15, Theorem 32.2] that $\sup q(K \cap B_\infty) < \infty$. Hence $\sup q(K \cap B_X) =: \alpha^* < \infty$. Since q is positively homogeneous, we conclude that $q \leq \alpha^*\|\cdot\|$ on K . Therefore, $\{K_1, K_2\}$ has property (G) as claimed.

If one of the cones is actually a subspace, then another attractive characterization is available:

FACT 2.1. *Suppose Z is a closed subspace of X and denote the (orthogonal) projection onto Z^\perp by T_\perp . Then:*

- (i) T_\perp is open relative to K if and only if $\{K, Z\}$ has property (G).
- (ii) $T_\perp(K)$ is closed if and only if $K + Z$ is.

Proof. (i): is [6, Proposition 2.6]. (The setting in [6] is finite dimensional, but the proof of this result works in Hilbert space equally well.)

(ii): The kernel of T_\perp equals Z . By [12, Lemma 17.H], $T_\perp(K)$ is closed if and only if $K + Z$ is. \square

3. Bounded linear regularity and strong CHIP.

DEFINITION 3.1. Suppose K_1, K_2 are two closed convex cones in X . Then $\{K_1, K_2\}$ is *linearly regular*, if there exists $\kappa > 0$ such that

$$d(x, K_1 \cap K_2) \leq \kappa \max\{d(x, K_1), d(x, K_2)\}, \quad \forall x \in X$$

If only for every bounded subset S of X there exists $\kappa_S > 0$ such that the last inequality holds for every $x \in S$, then $\{K_1, K_2\}$ is *boundedly linearly regular*.

REMARK 3.1. The definition makes sense and is useful for (finitely many) closed convex intersecting sets; see [3], [4, Section 5], and [5].

In our conical setting, (bounded) linear regularity was characterized in [5, Theorem 6.7] as follows:

FACT 3.1. *The following are equivalent:*

- (i) $\{K_1, K_2\}$ is linearly regular.
- (ii) $\{K_1, K_2\}$ is boundedly linearly regular.
- (iii) $\{K_1^\ominus, K_2^\ominus\}$ has property (G).

DEFINITION 3.2. Suppose K_1, K_2 are two closed convex cones in X . Then $\{K_1, K_2\}$ has *strong CHIP*, if $N_{K_1 \cap K_2}(x) = N_{K_1}(x) + N_{K_2}(x)$, $\forall x \in K_1 \cap K_2$.

REMARK 3.2. The notion strong CHIP, where CHIP stands for “conical hull intersection property”, was coined by Deutsch and co-workers in their studies of constraint approximation problems. See [5], [6], [7], [9], [10], and [11] for further information. Again, the definition of strong CHIP makes sense for a finite collection of closed convex intersecting sets.

Strong CHIP allows the following simple characterization, taken from [5, Proposition 6.4]:

FACT 3.2. *Suppose K_1, K_2 are two closed convex cones in X . Then $\{K_1, K_2\}$ has strong CHIP if and only if $K_1^\ominus + K_2^\ominus$ is closed.*

It follows from [5, Theorem 3.6] that bounded linear regularity is stronger than strong CHIP:

FACT 3.3. *Suppose K_1, K_2 are two closed convex cones in X . If $\{K_1, K_2\}$ is boundedly linearly regular, then it has strong CHIP.*

The following two results connect bounded linear regularity and strong CHIP to openness relative to a cone. Firstly, we show how properties of two cones give rise to openness properties of the sum operator.

Theorem 3.1. *Suppose L_1, L_2 are two closed convex cones in X . Let $K_1 := L_1^\ominus$, $K_2 := L_2^\ominus$, and define the sum operator $T_\Sigma : X \times X \rightarrow X : (x_1, x_2) \mapsto x_1 + x_2$. Then:*
(i) $\{L_1, L_2\}$ is boundedly linearly regular if and only if T_Σ is open relative to $K_1 \times K_2$.
(ii) $\{L_1, L_2\}$ has strong CHIP if and only if $T_\Sigma(K_1 \times K_2)$ is closed.

Proof. (i): $\{L_1, L_2\}$ is boundedly linearly regular $\Leftrightarrow \{K_1, K_2\}$ has property (G) (by Fact 3.1) $\Leftrightarrow T_\Sigma$ is open relative to $K_1 \times K_2$ (by Proposition 2.1). (ii): $\{L_1, L_2\}$ has strong CHIP $\Leftrightarrow K_1 + K_2$ is closed (by Fact 3.2) $\Leftrightarrow T_\Sigma(K_1 \times K_2)$ is closed. \square

REMARK 3.3. In [6, Section 3], the reader can find two closed convex cones L_1, L_2 in \mathbb{R}^4 such that $\{L_1, L_2\}$ has strong CHIP but is not boundedly linearly regular. In view of Theorem 3.1, we may now construct $T : \mathbb{R}^8 \rightarrow \mathbb{R}^4$ such that T is not open relative to some closed convex cone $K \subseteq \mathbb{R}^8$, but with $T(K)$ closed. However, Example 2.1 is a much simpler construction between two spaces of lower dimension.

REMARK 3.4. In \mathbb{R}^n with $n \leq 3$, consider two arbitrary closed convex cones L_1, L_2 . According to Bakan [1], $\{L_1, L_2\}$ has strong CHIP if and only if $\{L_1, L_2\}$ is boundedly linearly regular. Hence the example from [6, Section 3] mentioned in Remark 3.3 is optimal in the sense that it could not reside in any space of smaller dimension.

Theorem 3.2. *Suppose $T : X \rightarrow Y$ is a continuous linear operator and K is a closed convex cone in X . In $X \times Y$, let $K_1 := X \times 0$, $K_2 := \text{gra}(T|_K)$, and denote the corresponding polar cones by L_1, L_2 , respectively. Then K_1, K_2 are two closed convex cones in $X \times Y$. Moreover*

- (i) T is open relative to K if and only if $\{L_1, L_2\}$ is boundedly linearly regular.
- (ii) $T(K)$ is closed if and only if $\{L_1, L_2\}$ has strong CHIP.

Proof. It is easy to check that K_1, K_2 are closed convex cones.

- (i) In view of Fact 3.1, it suffices to show that

$T(K)$ is open relative to $K \Leftrightarrow \{K_1, K_2\}$ has property (G).

“ \Rightarrow ”: Obtain $\delta > 0$ such that $\delta B_Y \cap T(K) \subseteq T(B_X \cap K)$.

Claim: $\delta B_{X \times Y} \cap (K_1 + K_2) \subseteq (1 + \delta)[(B_{X \times Y} \cap K_1) + (B_{X \times Y} \cap K_2)]$.

Indeed, let $(x, 0) + (k, Tk) \in K_1 + K_2$ with $x \in X$, $k \in K$, and $\|x + k\|^2 + \|Tk\|^2 \leq \delta^2$. Then $\|x + k\| \leq \delta$ and $Tk \in \delta B_Y \cap T(K)$. By assumption, there exists $k^* \in B_X \cap K$ such that $Tk^* = Tk$. Let $x^* := x + k - k^*$. Then $\|(x^*, 0)\| = \|x^*\| \leq \|x + k\| + \|k^*\| \leq \delta + 1$ and $\|(k^*, Tk^*)\|^2 \leq 1 + \delta^2 \leq (1 + \delta)^2$. Hence $(x, 0) + (k, Tk) = (x^* + k^*, Tk^*) =$

$(x^*, 0) + (k^*, Tk^*) \in (1 + \delta)[(B_{X \times Y} \cap K_1) + (B_{X \times Y} \cap K_2)]$. The Claim thus holds. After dividing the inclusion of the Claim by $(1 + \delta)$, we see that $\{K_1, K_2\}$ has property (G). “ \Leftarrow ”: Pick $\delta > 0$ such that $\delta B_{X \times Y} \cap (K_1 + K_2) \subseteq (B_{X \times Y} \cap K_1) + (B_{X \times Y} \cap K_2)$. Fix an arbitrary $k \in K$ with $Tk \in \delta B_Y$. Then $(-k, 0) \in K_1$ and $(k, Tk) \in K_2$, hence $(0, Tk) \in \delta B_{X \times Y} \cap (K_1 + K_2)$. By assumption, there exist $x^* \in B_X$ and $k^* \in K$ such that $(0, Tk) = (x^*, 0) + (k^*, Tk^*)$. It follows that $k^* = -x^* \in B_X$ and $Tk^* = Tk$, which yields $Tk \in T(B_X \cap K)$. Since k has been chosen arbitrarily, we conclude $\delta B_Y \cap T(K) \subseteq T(B_X \cap K)$ and so T is open relative to K . Altogether, statement (i) is verified.

(ii): Clearly, $K_1 + K_2 = X \times T(K)$. Hence $T(K)$ is closed $\Leftrightarrow K_1 + K_2$ is closed $\Leftrightarrow L_1^\ominus + L_2^\ominus$ is closed $\Leftrightarrow \{L_1, L_2\}$ has strong CHIP (by Fact 3.2). \square

REMARK 3.5. In Example 2.1, we constructed a (continuous) linear operator $T : \mathbf{R}^4 \rightarrow \mathbf{R}^3$ and a closed convex cone K in \mathbf{R}^4 such that $T(K)$ is closed, but T is not open relative to K . By Theorem 3.2, this yields two closed convex cones L_1, L_2 in \mathbf{R}^7 such that $\{L_1, L_2\}$ has strong CHIP but is not boundedly linearly regular. We note that these two cones were obtained much more easily than the example in [6, Section 3]. However, the latter example resides in \mathbf{R}^4 (rather than \mathbf{R}^7) and is, in a sense made precise in Remark 3.4, optimal.

4. A sufficient condition: metric regularity. In this final section, we assume for simplicity that

$$\boxed{X \text{ and } Y \text{ are Euclidean spaces.}}$$

Theorem 4.1. *T is open relative to K if and only if $T(K)$ is closed and there exist real $\kappa > 0$ and $\delta_Y > 0$ such that*

$$d(y, (T^*)^{-1}(K^\ominus)) \leq \kappa d(T^*y, K^\ominus), \quad \forall y \in \delta_Y B_Y.$$

Proof. In view of Theorem 2.1, we assume throughout this proof that $T(K)$ is closed.

Fix an arbitrary $\delta > 0$. Then

$$\delta B_Y \cap T(K) \subseteq T(B_X \cap K) \tag{4.1}$$

holds if and only if $\iota_{\delta B_Y} + \iota_{T(K)} \geq \iota_{T(B_X \cap K)}$. Since all functions appearing in the last inequality are closed, this last inequality is equivalent to

$$(\iota_{\delta B_Y} + \iota_{T(K)})^* \leq \iota_{T(B_X \cap K)}^*. \tag{4.2}$$

We now evaluate both sides of the last equation using convex calculus. Firstly, $\iota_{T(K)} = \iota_{K^\ominus} \circ T^*$ by [15, Corollary 16.3.1 and Theorem 14.1]. Secondly, $\iota_{\delta B_Y} = \delta \|\cdot\|$ by [15, Corollary 16.1.1 and Example following Corollary 13.2.2]. Note that $0 \in T(K) \cap \text{int}(\delta B_Y)$. Hence $\text{ri}(\text{dom } \iota_{T(K)}) \cap \text{int}(\text{dom } \iota_{\delta B_Y}) \neq \emptyset$ by [15, Corollary 6.3.2]. Using [15, Theorem 16.4], the left side of (4.2) thus evaluates to $\delta d(\cdot, (T^*)^{-1}(K^\ominus))$. Similarly, the right side of (4.2) becomes $d(T^*(\cdot), K^\ominus)$. Altogether, (4.2) and hence (4.1) are both equivalent to

$$d(y^*, (T^*)^{-1}(K^\ominus)) \leq \frac{1}{\delta} d(T^*y^*, K^\ominus), \quad \forall y^* \in Y.$$

The result follows, since both sides of the last inequality are positively homogeneous. \square

We now state a particularization of a powerful result on metric regularity, see [16, Example 9.44]:

FACT 4.1. (*metric regularity in constraint systems*) Suppose $F : Y \rightarrow X$ is a continuous linear operator and C is a closed convex cone in X . Define a set-valued map $\Omega : Y \rightarrow 2^X$ by $y \mapsto Fy - C$. Then metric regularity of Ω for $\bar{x} = 0$ at $\bar{y} = 0$ means the existence of real $\kappa > 0$, $\delta_X > 0$, and $\delta_Y > 0$ such that

$$d(y, \Omega^{-1}(x)) \leq \kappa d(Fy - x, C), \quad \forall y \in \delta_Y B_Y, \forall x \in \delta_X B_X;$$

in fact, this holds if and only if $F^*|_{C^\ominus}$ is one-to-one. In particular, if this regularity holds, then (by choosing $x = \bar{x} = 0$ in the above inequality)

$$d(y, F^{-1}(C)) \leq \kappa d(Fy, C), \quad \forall y \in \delta_Y B_Y.$$

Corollary 4.1. *If $T|_K$ is one-to-one, then T is open relative to K .*

Proof. It follows immediately from Fact 4.1 (with $F = T^*$ and $C = K^\ominus$) that there exist real $\kappa > 0$ and $\delta_Y > 0$ such that

$$d(y, (T^*)^{-1}(K^\ominus)) \leq \kappa d(T^*y, K^\ominus), \quad \forall y \in \delta_Y B_Y.$$

We now show that $T(K)$ is closed. Pick an arbitrary $\bar{y} \in \text{cl } T(K)$. Then there exists a sequence (k_n) in K with $Tk_n \rightarrow \bar{y}$. After passing to a subsequence if necessary, we assume without loss of generality that $L := \lim_n \|k_n\|$ exists in $[0, +\infty]$. If $L < +\infty$, then we can arrange (by compactness and after passing to a further subsequence if necessary) that $k_n \rightarrow \bar{k}$, for some $\bar{k} \in K$. But then $\bar{y} = T\bar{k} \in T(K)$. The remaining possibility is $L = +\infty$. Without loss of generality, we assume that $k_n/\|k_n\| \rightarrow k^* \in K \setminus \{0\}$. Then $0 \leftarrow Tk_n/\|k_n\| \rightarrow Tk^*$, which contradicts injectivity of $T|_K$. Hence this case never occurs and $T(K)$ is indeed closed. The result now follows from Theorem 4.1. \square

With some care, most results in this paper will be found to have analogues in more general Banach space settings. We conclude with the following illustrative variant of Corollary 4.1.

Theorem 4.2. *Suppose X, Y are Banach spaces, where X is reflexive, and $T : X \rightarrow Y$ is continuous and linear. Suppose further K is a closed convex cone in X with $T(K)$ closed. If $T|_K$ is one-to-one, then T is open relative to K .*

Proof. Let $L := T(K)$.

Claim: There exists $\delta > 0$ and $l^* \in L$ with $(l^* + \delta B_Y) \cap L \subseteq T(B_X \cap K)$.

Since $B_X \cap K$ is weakly compact, the set $T(B_X \cap K)$ is closed. Clearly, $L =$

$\bigcup_{n \in \{1, 2, \dots\}} nT(B_X \cap K)$. Now L in itself is a complete metric space and each $nT(B_X \cap K)$ is closed in Y and hence in L . Thus, by Baire's theorem (see, for instance [14, Theorem 8.14]), some $nT(B_X \cap K)$ has nonempty interior in L and hence there exists some l^* in the interior of $T(B_X \cap K)$ relative to L . The Claim thus holds.

Assume to the contrary that T is not open relative to K . Then there exists a sequence (x_n) in K such that $Tx_n \rightarrow 0$, but all x_n lie outside B_X . Define $k_n := x_n / \sqrt{\|Tx_n\|}$, for every $n \in \{1, 2, \dots\}$. Then (k_n) is a sequence in K with $\|k_n\| \rightarrow +\infty$ and $Tk_n \rightarrow 0$. Set $l_n := Tk_n$ and obtain (by the Claim) $k^* \in B_X \cap K$ with $Tk^* = l^*$. Then $l^* \leftarrow l^* + l_n \in L$. Hence eventually $T(k^* + k_n) = l^* + l_n \in (l^* + \delta B_Y) \cap L$. By injectivity of $T|_K$ and the Claim, we have $k^* + k_n \in B_X \cap K$ eventually. But this is absurd, since $\|k_n\| \rightarrow +\infty$. Therefore, the theorem is proven. \square

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