Watanabe, H.
Osaka J. Math.
51 (2014), 425–438

ON THE GENERAL TRANSFORMATION OF THE WIRTINGER INTEGRAL

HUMIHIKO WATANABE

(Received March 16, 2011, revised September 14, 2012)

Abstract

The Wirtinger integral is the uniformization to the upper half plane H of the hypergeometric function defined on the complex projective line $\mathbb{P}^1$. In [5] we established the transformation formulas of the Wirtinger integral for the linear fractional transformations $\tau \to \tau + 2$ and $\tau \to -1/\tau$ with the aide of the theory of theta functions. As a corollary we obtain the transformation formulas of the Wirtinger integral for the linear fractional transformations $\tau \to \tau + 2$ and $\tau \to \tau/(-2\tau + 1)$ which are identified with generators of the principal congruence subgroup $\Gamma(2)$ modulo center. These formulas correspond to the monodromy matrices of the hypergeometric function for generators of the fundamental group of $\mathbb{P}^1$ minus three points. The purpose of this paper is to generalize this result, that is, we establish the transformation formula of the Wirtinger integral for a general element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\Gamma(2)$, which corresponds to a general monodromy matrix of the hypergeometric function.

1. Introduction

Following the notation of Chandrasekharan [1], we introduce the four theta functions $\theta(v, \tau)$, $\theta_i(v, \tau)$ $(i = 1, 2, 3)$ by

$$\begin{aligned}
\theta(v, \tau) &= \frac{1}{i} \sum_{n=-\infty}^{+\infty} (-1)^n e^{(n+1/2)^2\pi i\tau} e^{(2n+1)\pi i v},\\
\theta_1(v, \tau) &= \sum_{n=-\infty}^{+\infty} e^{(n+1/2)^2\pi i\tau} e^{(2n+1)\pi i v},\\
\theta_2(v, \tau) &= \sum_{n=-\infty}^{+\infty} (-1)^n e^{n^2\pi i\tau} e^{2n\pi i v},\\
\theta_3(v, \tau) &= \sum_{n=-\infty}^{+\infty} e^{n^2\pi i\tau} e^{2n\pi i v},
\end{aligned}$$

which are defined for all $(v, \tau) \in \mathbb{C} \times H$, where $\mathbb{C}$ denotes the complex plane. Mumford [2] adopts the symbols θ_{00}, θ_{01}, θ_{10}, θ_{11} to denote the theta functions above. The relations

2010 Mathematics Subject Classification. Primary 34M35; Secondary 33C05.

between the two notations are as follows: $\theta(v,\tau) = -\theta_{11}(v,\tau)$, $\theta_1(v,\tau) = \theta_{10}(v,\tau)$, $\theta_2(v,\tau) = \theta_{01}(v,\tau)$, $\theta_3(v,\tau) = \theta_{00}(v,\tau)$. Note that $\theta(v,\tau)$ has a simple zero at $v=0$, $\theta_1(v,\tau)$ at $v=1/2$, $\theta_2(v,\tau)$ at $v=\tau/2$, and $\theta_3(v,\tau)$ at $v=(1+\tau)/2$. In this paper we also use the following abbreviations: $\theta_i(v) = \theta_i(v,\tau)$, $\theta_i = \theta_i(0,\tau)$, etc. Moreover we set $T(v)^{p,q,r,s} = T(v,\tau)^{p,q,r,s} = \theta(v,\tau)^p\theta_1(v,\tau)^q\theta_2(v,\tau)^r\theta_3(v,\tau)^s$.

We define two functions $z_1(\tau)$, $z_2(\tau)$, which we called *Wirtinger integrals* in our papers [5], [6] (see also [7], [8], [9]), by

$$\begin{aligned} z_1(\tau) = {} & \frac{\theta_3^2}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})} \\ & \times \int^{((1/2)+,0+,(1/2)-,0-)} T(v,\tau)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\,dv, \\ z_2(\tau) = {} & \frac{\theta_3^2}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\beta-\gamma)})} \\ & \times \int^{((1/2)+,0+,(1/2)-,0-)} T(v,\tau)^{2\beta-2\gamma+3,-2\beta-1,2\alpha-1,2\gamma-2\alpha-1}\,dv, \end{aligned}$$

where we assume that the parameters α, β, γ satisfy the conditions $\alpha, \gamma-\alpha, \gamma-\beta, \beta \notin (1/2)\mathbb{Z}$, and $((1/2)+, 0+, (1/2)-, 0-)$ denotes a Pochhammer cycle with base point $v = v_0$, where $\arg v_0 = \arg((1/2)-v_0) = 0$, turning first around $v=1/2$ once anticlockwisely, second around $v=0$ once anticlockwisely, third around $v=1/2$ once clockwisely, and lastly around $v=0$ once clockwisely. These functions are the lifts of Gauss' hypergeometric functions of *SL* type to the upper half plane, and form a fundamental system of solutions for the lift of Gauss' hypergeometric differential equation of *SL* type to the upper half plane. Note that $z_1(\tau)$ and $z_2(\tau)$ are transformed to each other by the involution ι defined by $\iota(\alpha,\beta,\gamma) = (\beta-\gamma+2, \alpha-\gamma, -\gamma+2)$. Let $\Gamma(2)$ be the principal congruence subgroup of level 2, and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\Gamma(2)$. Without loss of generality we may assume $c > 0$. The problem which we will study in this paper is as follows:

PROBLEM 1.1. Determine the constants A and B with respect to τ such that $z_1((a\tau+b)/(c\tau+d)) = Az_1((\tau)+Bz_2(\tau))$.

Once Problem 1 is established, we have an analogous formula for $z_2((a\tau+b)/(c\tau+d))$ by applying the involution ι to the formula for $z_1((a\tau+b)/(c\tau+d))$. Since the group $\Gamma(2)$ modulo center is isomorphic to the fundamental group of the Riemann sphere minus three points (which is the defining region of Gauss' hypergeometric differential equation), the constants A, B are identified with entries of the monodromy matrix for the element of the fundamental group corresponding to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let $Z(\tau)$ be a vector-valued function defined by

$$Z(\tau)=\left(\frac{2\pi\,\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}z_1(\tau),\ \frac{2\pi\alpha\Gamma(2-\gamma)}{\Gamma(-\beta)\Gamma(1+\beta-\gamma)}z_2(\tau)\right),$$

where $\Gamma(z)$ denotes the Gamma function. According to the results of [5], for any even integer m, the matrices $M_0(m)$ and $M_1(m)$ defined by $Z(\tau+m)=Z(\tau)M_0(m)$ and $Z(\tau/(-m\tau+1))=Z(\tau)M_1(m)$ are given by

$$M_0(m)=\begin{bmatrix} e^{m\pi i(\gamma-1)/2} & 0 \\ 0 & e^{m\pi i(1-\gamma)/2}\end{bmatrix}$$

and

$$M_1(m)=\begin{bmatrix} \dfrac{\sin\pi(\gamma-\alpha)\sin\pi(\gamma-\beta)e^{m\pi i(\alpha+\beta-\gamma+1)/2}-\sin\pi\alpha\sin\pi\beta e^{m\pi i(\gamma-\alpha-\beta-1)/2}}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)} & \dfrac{2\pi i\,\Gamma(1-\gamma)\Gamma(2-\gamma)\sin(m\pi(\alpha+\beta-\gamma+1)/2)}{\Gamma(-\alpha)\Gamma(-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\gamma)\sin\pi(\gamma-\alpha-\beta)} \\ \dfrac{2\pi i\,\Gamma(\gamma-1)\Gamma(\gamma)\sin(m\pi(\alpha+\beta-\gamma+1)/2)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\sin\pi(\gamma-\alpha-\beta)} & \dfrac{\sin\pi(\gamma-\alpha)\sin\pi(\gamma-\beta)e^{m\pi i(\gamma-\alpha-\beta-1)/2}-\sin\pi\alpha\sin\pi\beta e^{m\pi i(\alpha+\beta-\gamma+1)/2}}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)}\end{bmatrix}.$$

Let $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ be an element of the principal congruence subgroup $\varGamma(2)$. We assume that $d>1$. It is well-known (e.g. [4]) that there exist $2n+1$ even integers $m_0, m_1, m_2, \ldots, m_{2n}$ $(n\geq 1)$ such that the linear fractional transformation $(a\tau+b)/(c\tau+d)$ is written in the form:

$$\frac{a\tau+b}{c\tau+d}=m_0-\cfrac{1}{m_1-\cfrac{1}{m_2-\cfrac{}{\ddots-\cfrac{1}{m_{2n}+\tau}}}}.$$

This is also written in the form

$$\frac{a\tau+b}{c\tau+d}=(\varphi_{m_0}\circ\psi_{m_1}\circ\varphi_{m_2}\circ\psi_{m_3}\circ\cdots\circ\psi_{m_{2n-1}}\circ\varphi_{m_{2n}})(\tau),$$

where $\varphi_m(\tau)=\tau+m$ and $\psi_m(\tau)=\tau/(-m\tau+1)$. Setting $Z((a\tau+b)/(c\tau+d))=Z(\tau)M$ with a two-by-two matrix M, we have

(1.1) $$M=M_0(m_0)M_1(m_1)M_0(m_2)M_1(m_3)\cdots M_1(m_{2n-1})M_0(m_{2n}).$$

The coefficients A and B in Problem 1.1 are given as entries of this matrix M. It seems, however, to be difficult to derive the explicit formulas of the entries of M directly from calculating the product formula (1.1). In the next section we formulate our idea for giving the explicit formulas of the coefficients A and B.

2. How to determine the coefficients A and B

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\Gamma(2)$ such that $c > 0$. The following formulas are well-known (e.g. [3], [4]):

$$\theta\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \left(\frac{c}{d}\right) e^{\pi i(3d-2+bd-cd)/4} \sqrt{\frac{c\tau+d}{i}} e^{c\pi i v^2/(c\tau+d)} \theta(v,\tau),$$

$$\theta_1\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \left(\frac{c}{d}\right) e^{\pi i(d-4+2c+bd-2cd)/4} \sqrt{\frac{c\tau+d}{i}} e^{c\pi i v^2/(c\tau+d)} \theta_1(v,\tau),$$

$$\theta_2\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \left(\frac{c}{d}\right) e^{\pi i(3d-4+2a-ab+bd-cd)/4} \sqrt{\frac{c\tau+d}{i}} e^{c\pi i v^2/(c\tau+d)} \theta_2(v,\tau),$$

$$\begin{aligned}\theta_3\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= \left(\frac{c}{d}\right) e^{\pi i(3d-3+2a+2c-ab-ad-bc+bd-2cd)/4} \sqrt{\frac{c\tau+d}{i}} \\ &\quad \times e^{c\pi i v^2/(c\tau+d)} \theta_3(v,\tau),\end{aligned}$$

where $\left(\frac{c}{d}\right)$ denotes Legendre–Jacobi's symbol. Substitution $\tau \to (a\tau+b)/(c\tau+d)$ and $v \to v/(c\tau+d)$ makes the integral representation for $z_1(\tau)$ given in Section 1 into

$$\begin{aligned} & z_1\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= \frac{1}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})} \theta_3\left(0, \frac{a\tau+b}{c\tau+d}\right)^2 \\ &\quad \times \int^{(((c\tau+d)/2)+,0+,((c\tau+d)/2)-,0-)} T\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} \frac{dv}{c\tau+d}. \end{aligned}$$

Applying the transformation formulas above for theta functions to the integral representation for $z_1((a\tau+b)/(c\tau+d))$, we have

(2.1)
$$\begin{aligned} z_1\left(\frac{a\tau+b}{c\tau+d}\right) &= \frac{e^{\pi i d(b+c)/2} e^{\pi i\alpha} e^{\pi i d(\alpha-\gamma)} e^{\pi i c(2-d)(\gamma-\alpha-\beta)/2} e^{\pi i a(b-2)\gamma/2} \theta_3^2}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})} \\ &\quad \times \int^{(((c\tau+d)/2)+,0+,((c\tau+d)/2)-,0-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\, dv \end{aligned}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ with $c > 0$. *In the rest of this paper we concentrate our attention on the case* $0 < c < d$. Applying the reasoning for this case to the other one $d < c$

which we do not refer to in this paper, we would obtain a result similar to the main theorem stated later. Noticing that the integers c, d are relatively prime to each other, we define integers k_ν by

$$k_0 = 0, \quad k_\nu = \left[\frac{d\nu}{c}\right] + 1 \quad (1 \le \nu \le c-1), \quad k_c = d,$$

where for a real number x the symbol $[x]$ denotes the maximal integer not exceeding to x. The branch cuts L_ν ($\nu = 0, \dots, c$) of the integrand $T(v)^{2\alpha-1, 2\gamma-2\alpha-1, 2\beta-2\gamma+3, -2\beta-1}$ of the integral in (2.1) are given as follows. Namely, L_0 is the union of two rays on $\mathbb{C}$ defined by the equation $v = s$ for the real parameter s such that $s \le 0$, $1/2 \le s$; if $\nu \ge 1$, L_ν is the union of two rays on $\mathbb{C}$ defined by $v = s + (\nu/2)\tau$ for the real parameter s such that $s \le (k_\nu - 1)/2$, $k_\nu/2 \le s$. We set $l_{\nu+1} = k_{\nu+1} - k_\nu$. Let $I_{\mu\nu}$, J_ν $(0 \le \mu \le l_{\nu+1} - 1,\ 0 \le \nu \le c-1)$ be given by

(2.2)

$$I_{\mu\nu} = \frac{1}{(1-\varepsilon((k_\nu + \mu + 1 + \nu\tau)/2))(1-\varepsilon((k_\nu + \mu + \nu\tau)/2))} \times \int^{((1/2)+,0+,(1/2)-,0-)} T\left(v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv,$$

and

$$J_\nu = \frac{1}{(1-\varepsilon((k_{\nu+1} + \nu\tau)/2))(1-\varepsilon((k_{\nu+1} + (\nu+1)\tau)/2))} \times \int^{((\tau/2)+,0+,(\tau/2)-,0-)} T\left(v + \frac{k_{\nu+1} + \nu\tau}{2}\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv, \tag{2.3}$$

where $\varepsilon((m + n\tau)/2)$, m and n being integers, denotes the local monodromy of the function $T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}$ along a small circle turning around the center $v = (m + n\tau)/2$ in the anticlockwise direction, and therefore it coincides with one of the four quantities $e^{4\pi i\alpha}$, $e^{4\pi i(\gamma-\alpha)}$, $e^{4\pi i(\beta-\gamma)}$, $e^{-4\pi i\beta}$. In the integral representation (2.2) we take the base point $v = v_0$ of the cycle $((1/2)+, 0+, (1/2)-, 0-)$ in such a manner that $\arg v_0 = 0$ and $\arg(v_0 - 1/2) = \pi$; in the integral representation (2.3) we take the base point $v = v_0$ of the cycle $((\tau/2)+, 0+, (\tau/2)-, 0-)$ in such a manner that $\arg v_0 = \arg\tau$ and $\arg(v_0 - \tau/2) = \arg\tau + \pi$. Then the integral in (2.1) has the following decomposition:

(2.4)

$$\frac{1}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})} \int^{(((c\tau+d)/2)+,0+,(c\tau+d)/2)-,0-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\, dv = \sum_{\nu=0}^{c-1} (I_\nu + J_\nu),$$

where

$$I_\nu = \sum_{\mu=0}^{l_{\nu+1}-1} I_{\mu\nu}. \tag{2.5}$$

Thus, Problem 1.1 follows immediately from the following:

PROBLEM 2.1. Determine the constants P_ν, Q_ν, R_ν, S_ν with respect to τ such that $I_\nu = P_\nu z_1(\tau)/\theta_3^2 + Q_\nu z_2(\tau)/\theta_3^2$ and $J_\nu = R_\nu z_1(\tau)/\theta_3^2 + S_\nu z_2(\tau)/\theta_3^2$.

In fact, the constants A and B in Problem 1.1 are written by

$$A = e^{\pi id(b+c)/2}e^{\pi i\alpha}e^{\pi id(\alpha-\gamma)}e^{\pi ic(2-d)(\gamma-\alpha-\beta)/2}e^{\pi ia(b-2)\gamma/2}\sum_{\nu=0}^{c-1}(P_\nu + R_\nu),$$

$$B = e^{\pi id(b+c)/2}e^{\pi i\alpha}e^{\pi id(\alpha-\gamma)}e^{\pi ic(2-d)(\gamma-\alpha-\beta)/2}e^{\pi ia(b-2)\gamma/2}\sum_{\nu=0}^{c-1}(Q_\nu + S_\nu).$$

We will give the explicit formulas for I_ν and J_ν in Theorems 4.1 and 5.1.

3. Auxiliary formulas

In this section we introduce some auxiliary formulas which are applied to the study of the integrals I_ν and J_ν.

Lemma 3.1. *Let p, q, r, s be complex constants but not integers. Then we have*

$$\int^{((1/2)+,0+,(1/2)-,0-)} T(v)^{p,q,r,s}\,dv = \int^{((1/2)+,0+,(1/2)-,0-)} T(v)^{q,p,s,r}\,dv. \tag{3.1}$$

We omit the proof.

Lemma 3.2. *We have*:

$$\begin{aligned}&\frac{1}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\beta-\gamma)})}\int^{((\tau/2)+,0+,(\tau/2)-,0-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\,dv\\ &= \frac{1-e^{2\pi i(\alpha-\gamma)}}{1-e^{-2\pi i\gamma}}z_1(\tau)/\theta_3^2 + \frac{e^{\pi i(\alpha-\gamma)}(e^{\pi i\beta}-e^{-\pi i\beta})}{1-e^{-2\pi i\gamma}}z_2(\tau)/\theta_3^2,\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\frac{1}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\gamma-\alpha)})}\int^{((\tau/2)+,0+,(\tau/2)-,0-)} T(v)^{2\gamma-2\alpha-1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3}\,dv\\ &= \frac{1-e^{-2\pi i\alpha}}{1-e^{-2\pi i\gamma}}z_1(\tau)/\theta_3^2 + \frac{e^{\pi i(-\alpha-\beta+\gamma)}-e^{\pi i(-\alpha+\beta-\gamma)}}{1-e^{-2\pi i\gamma}}z_2(\tau)/\theta_3^2.\end{aligned} \tag{3.3}$$

Proof. Applying Cauchy's theorem on the contour integral to the integration of $T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}$ along the parallelogram with verices 0, $1/2$, $(1+\tau)/2$, $\tau/2$, we have

(3.4)
$$\begin{aligned}
&\frac{1}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\beta-\gamma)})}\int^{(0+,(\tau/2)+,0-,(\tau/2)-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\,dv\\
&+\frac{1}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})}\int^{((1/2)+,0+,(1/2)-,0-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\,dv\\
&-\frac{e^{2\pi i(\alpha-\gamma)}}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\gamma-\alpha)})}\int^{((\tau/2)+,0+,(\tau/2)-,0-)} T(v)^{2\gamma-2\alpha-1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3}\,dv\\
&-\frac{e^{\pi i(\alpha+\beta-\gamma)}}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\beta-\gamma)})}\int^{(0+,(1/2)+,0-,(1/2)-)} T(v)^{2\beta-2\gamma+3,-2\beta-1,2\alpha-1,2\gamma-2\alpha-1}\,dv\\
&=0.
\end{aligned}$$

Substitution $\alpha\to\gamma-\alpha$ and $\beta\to\gamma-\beta-2$ makes (3.4) into

(3.5)
$$\begin{aligned}
&\frac{1}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\gamma-\alpha)})}\int^{(0+,(\tau/2)+,0-,(\tau/2)-)} T(v)^{2\gamma-2\alpha-1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3}\,dv\\
&+\frac{1}{(1-e^{4\pi i(\gamma-\alpha)})(1-e^{4\pi i\alpha})}\int^{((1/2)+,0+,(1/2)-,0-)} T(v)^{2\gamma-2\alpha-1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3}\,dv\\
&-\frac{e^{-2\pi i\alpha}}{(1-e^{-4\pi i(\gamma-\beta)})(1-e^{4\pi i\alpha})}\int^{((\tau/2)+,0+,(\tau/2)-,0-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\,dv\\
&-\frac{e^{\pi i(\gamma-\alpha-\beta)}}{(1-e^{4\pi i(\beta-\gamma)})(1-e^{-4\pi i\beta})}\int^{(0+,(1/2)+,0-,(1/2)-)} T(v)^{-2\beta-1,2\beta-2\gamma+3,2\gamma-2\alpha-1,2\alpha-1}\,dv\\
&=0.
\end{aligned}$$

Combining (3.1), (3.4), (3.5), we have (3.2) and (3.3). □

One can prove the following lemma similarly.

Lemma 3.3. *We have*:

$$\begin{aligned}
&\frac{1}{(1-e^{4\pi i(\beta-\gamma)})(1-e^{4\pi i\alpha})}\int^{((\tau/2)+,0+,(\tau/2)-,0-)} T(v)^{2\beta-2\gamma+3,-2\beta-1,2\alpha-1,2\gamma-2\alpha-1}\,dv\\
&=\frac{e^{\pi i(\alpha+\beta-\gamma)}-e^{\pi i(-\alpha+\beta+\gamma)}}{1-e^{2\pi i\gamma}}z_1(\tau)/\theta_3^2+\frac{1-e^{2\pi i\beta}}{1-e^{2\pi i\gamma}}z_2(\tau)/\theta_3^2,
\end{aligned}\tag{3.6}$$

$$\begin{aligned}
&\frac{1}{(1-e^{4\pi i(\gamma-\alpha)})(1-e^{-4\pi i\beta})}\int^{((\tau/2)+,0+(\tau/2)-,0-)} T(v)^{-2\beta-1,2\beta-2\gamma+3,2\gamma-2\alpha-1,2\alpha-1}\,dv\\
&=\frac{e^{\pi i(-\alpha-\beta+\gamma)}-e^{\pi i(\alpha-\beta+\gamma)}}{1-e^{2\pi i\gamma}}z_1(\tau)/\theta_3^2+\frac{1-e^{2\pi i(\gamma-\beta)}}{1-e^{2\pi i\gamma}}z_2(\tau)/\theta_3^2.
\end{aligned}\tag{3.7}$$

4. The integrals I_ν

The explicit formulas for the integrals I_ν are as follows:

Theorem 4.1. (i) *We have*

$$
\begin{aligned}
I_0 = e^{\pi i(2\alpha-\gamma)/2}\Bigg\{&\cos\frac{\pi}{2}(\gamma-2\alpha)\cdot\frac{1-e^{\pi i(1-\gamma)k_1}}{1-e^{\pi i(1-\gamma)}} \\
&+ i\sin\frac{\pi}{2}(\gamma-2\alpha)\cdot\frac{1-(-1)^{k_1}e^{\pi i(1-\gamma)k_1}}{1+e^{\pi i(1-\gamma)}}\Bigg\}z_1(\tau)/\theta_3^2.
\end{aligned}
\tag{4.1}
$$

(ii) *We have*

$$
\begin{aligned}
I_1 = &-e^{\pi i(2\alpha-5\gamma)/2}e^{2\pi i(\alpha+\beta-\gamma)}e^{\pi i(\gamma-1)k_1} \\
&\times\left\{\cos\Theta_1\cdot\frac{1-e^{\pi i(1-\gamma)l_2}}{1-e^{\pi i(1-\gamma)}} + i\sin\Theta_1\cdot\frac{1-(-1)^{l_2}e^{\pi i(1-\gamma)l_2}}{1+e^{\pi i(1-\gamma)}}\right\}z_2(\tau)/\theta_3^2,
\end{aligned}
\tag{4.2}
$$

where

$$\Theta_1 = \pi\left\{2\alpha-\gamma-\frac{(-1)^{k_1}}{2}(2\beta-\gamma)\right\}.$$

(iii) *If ν is a positive even integer, then we have*

$$
\begin{aligned}
I_\nu = &\, e^{\pi i(2\alpha-\gamma)/2}e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_\nu} \\
&\times\left\{\cos\Theta_\nu\cdot\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}} + i\sin\Theta_\nu\cdot\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}z_1(\tau)/\theta_3^2,
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
\Theta_\nu = \pi\Bigg\{&(-1)^{l_\nu}(2\beta-\gamma)+(-1)^{l_\nu+l_{\nu-1}}(2\alpha-\gamma)+\cdots+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}(2\beta-\gamma) \\
&+\frac{2-(-1)^{k_\nu}}{2}(2\alpha-\gamma)\Bigg\}.
\end{aligned}
$$

(iv) *If ν is an odd integer and $\nu>1$, then we have*

$$
\begin{aligned}
I_\nu = &-e^{\pi i(2\alpha-5\gamma)/2}e^{-2\pi i\gamma(l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(\gamma-1)k_\nu} \\
&\times\left\{\cos\Theta_\nu\cdot\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}} + i\sin\Theta_\nu\cdot\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}z_2(\tau)/\theta_3^2,
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
\Theta_\nu = \pi\Bigg\{&2\alpha-\gamma+(-1)^{l_\nu}(2\beta-\gamma)+(-1)^{l_\nu+l_{\nu-1}}(2\alpha-\gamma)+\cdots \\
&+(-1)^{l_\nu+\cdots+l_3}(2\beta-\gamma)+(-1)^{l_\nu+\cdots+l_2}(2\alpha-\gamma)-\frac{(-1)^{k_\nu}}{2}(2\beta-\gamma)\Bigg\}.
\end{aligned}
$$

We prove (4.3) only because the other formulas are proved similarly. Let ν be a positive even integer. Let $T_{\pm}(v+(k_\nu+\mu)/2+(\nu/2)\tau)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}$ be the branches of $T(v+(k_\nu+\mu)/2+(\nu/2)\tau)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}$ analytically continued on the real ray $v \geq -\mu/2$ minus half periods from the upper side for the plus sign and from the lower side for the minus sign, respectively, such that they coincide with each other on the real interval $-(\mu+1)/2 < v < -\mu/2$. Then we have

$$\begin{aligned} &T_+\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} \\ &=\frac{1+(-1)^\mu}{2}e^{\pi i(1-\gamma)\mu}e^{\pi i(2\alpha-2\gamma+1)}T\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3} \\ &\quad+\frac{1+(-1)^{\mu+1}}{2}e^{\pi i(1-\gamma)(\mu+1)}T\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} \end{aligned}$$

and

$$\begin{aligned} &T_-\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} \\ &=\frac{1+(-1)^\mu}{2}e^{\pi i(\gamma-1)\mu}e^{\pi i(2\gamma-2\alpha-1)}T\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3} \\ &\quad+\frac{1+(-1)^{\mu+1}}{2}e^{\pi i(\gamma-1)(\mu+1)}T\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}. \end{aligned}$$

Combining these relations, we have

$$\begin{aligned} &T_+\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} \\ &=e^{\pi i\{-2\gamma\mu+2\alpha-3\gamma+(-1)^\mu(2\alpha-\gamma)\}}T_-\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}. \end{aligned}$$

Now we have

$$\begin{aligned} &T_-\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} \\ &=e^{\pi i(\alpha+\beta-\gamma+1)}T_+\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3,-2\beta-1,2\alpha-1,2\gamma-2\alpha-1}. \end{aligned}$$

Combining the preceding two relations, we have

$$T_+\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} \tag{4.5}$$
$$= e^{\pi i\{-2\gamma\mu+2\alpha-3\gamma+(-1)^\mu(2\alpha-\gamma)\}}e^{\pi i(\alpha+\beta-\gamma+1)}$$
$$\times T_+\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3,-2\beta-1,2\alpha-1,2\gamma-2\alpha-1}.$$

By the same argument, we have

$$T_+\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3,-2\beta-1,2\alpha-1,2\gamma-2\alpha-1} \tag{4.6}$$
$$= e^{\pi i\{2\gamma(\mu+l_\nu)+2\beta+\gamma+(-1)^{\mu+l_\nu}(2\beta-\gamma)\}}e^{\pi i(\alpha+\beta-\gamma+1)}$$
$$\times T_+\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}.$$

From (4.5) and (4.6) we have

$$T_+\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}$$
$$= e^{\pi i\{2\gamma l_\nu+(-1)^\mu(2\alpha-\gamma)+(-1)^{\mu+l_\nu}(2\beta-\gamma)\}}e^{4\pi i(\alpha+\beta-\gamma)}$$
$$\times T_+\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}.$$

Repeating the same procedure, we arrive at the following

Lemma 4.2. *For a positive even integer ν we have*

(4.7)
$$T_+\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}$$
$$= e^{\pi i\{(-1)^\mu(2\alpha-\gamma)+(-1)^{\mu+l_\nu}(2\beta-\gamma)+(-1)^{\mu+l_\nu+l_{\nu-1}}(2\alpha-\gamma)+(-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}(2\beta-\gamma)+\cdots}$$
$$^{+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}(2\alpha-\gamma)+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3+l_2}(2\beta-\gamma)\}}$$
$$\times e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}T_+\left(v+\frac{k_\nu+\mu}{2}\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}.$$

Let us compute the integral I_ν for a positive even integer ν. Substituting (4.7) into (2.2) we have

$$\begin{aligned}(4.8)\qquad I_{\mu\nu} = {} & e^{\pi i\{(-1)^{\mu}(2\alpha-\gamma)+(-1)^{\mu+l_\nu}(2\beta-\gamma)+(-1)^{\mu+l_\nu+l_{\nu-1}}(2\alpha-\gamma)+(-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}(2\beta-\gamma)+\cdots}\\ & {}^{+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}(2\alpha-\gamma)+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3+l_2}(2\beta-\gamma)\}}\\ & \times \frac{e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}}{(1-\varepsilon((k_\nu+\mu)/2))(1-\varepsilon((k_\nu+\mu+1)/2))}\\ & \times \int^{((1/2)+,0+,(1/2)-,0-)} T\left(v+\frac{k_\nu+\mu}{2}\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv.\end{aligned}$$

Note that

$$\begin{aligned}(4.9)\qquad & \frac{1}{(1-\varepsilon((k_\nu+\mu)/2))(1-\varepsilon((k_\nu+\mu+1)/2))}\\ & \times \int^{((1/2)+,0+,(1/2)-,0-)} T\left(v+\frac{k_\nu+\mu}{2}\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv\\ & = \frac{1+(-1)^{\mu}}{2}\frac{e^{\pi i(1-\gamma)\mu}}{(1-\varepsilon(k_\nu/2))(1-\varepsilon((k_\nu+1)/2))}\\ & \quad \times \int^{((1/2)+,0+,(1/2)-,0-)} T\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv\\ & \quad + \frac{1-(-1)^{\mu}}{2}\frac{e^{\pi i(1-\gamma)\mu}e^{\pi i(2\alpha-\gamma)}}{(1-\varepsilon(k_\nu/2))(1-\varepsilon((k_\nu+1)/2))}\\ & \quad \times \int^{((1/2)+,0+,(1/2)-,0-)} T\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3} dv.\end{aligned}$$

Combining (4.8) with (4.9) and substituting the resulting equality into (2.5), we have after some calculation

(4.10)

$$\begin{aligned}I_\nu = {} & e^{\pi i(2\alpha-\gamma)}e^{\pi i\{(-1)^{l_\nu}(2\beta-\gamma)+(-1)^{l_\nu+l_{\nu-1}}(2\alpha-\gamma)+\cdots+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}(2\beta-\gamma)\}}\\ & \times e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}\\ & \times \frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}+\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}\frac{1}{(1-\varepsilon(k_\nu/2))(1-\varepsilon((k_\nu+1)/2))}\\ & \times \int^{((1/2)+,0+,(1/2)-,0-)} T\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv\\ & + e^{-\pi i\{(-1)^{l_\nu}(2\beta-\gamma)+(-1)^{l_\nu+l_{\nu-1}}(2\alpha-\gamma)+\cdots+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}(2\beta-\gamma)\}}\\ & \times e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}\\ & \times \frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}-\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}\frac{1}{(1-\varepsilon(k_\nu/2))(1-\varepsilon((k_\nu+1)/2))}\\ & \times \int^{((1/2)+,0+,(1/2)-,0-)} T\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3} dv.\end{aligned}$$

Now we have

(4.11)

$$\frac{1}{(1-\varepsilon(k_\nu/2))(1-\varepsilon((k_\nu+1)/2))} \times \int^{((1/2)+,0+,(1/2)-,0-)} T\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv$$
$$= \frac{e^{\pi i(1-\gamma)k_\nu} e^{\{1-(-1)^{k_\nu}\}\pi i(2\alpha-\gamma)/2}}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})} \int^{((1/2)+,0+,(1/2)-,0-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\, dv$$

and

(4.12)

$$\frac{1}{(1-\varepsilon(k_\nu/2))(1-\varepsilon((k_\nu+1)/2))} \times \int^{((1/2)+,0+(1/2)-,0-)} T\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3} dv$$
$$= \frac{e^{\pi i(1-\gamma)k_\nu} e^{\{1-(-1)^{k_\nu}\}\pi i(\gamma-2\alpha)/2}}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})} \int^{((1/2)+,0+,(1/2)-,0-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\, dv.$$

Substituting (4.11) and (4.12) into (4.10), we have

$$\begin{aligned}
I_\nu = {} & e^{\pi i(2\alpha-\gamma)} e^{\pi i\{(-1)^{l_\nu}(2\beta-\gamma)+(-1)^{l_\nu+l_{\nu-1}}(2\alpha-\gamma)+\cdots+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}(2\beta-\gamma)\}} \\
& \times e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_\nu} e^{\{1-(-1)^{k_\nu}\}\pi i(2\alpha-\gamma)/2} \\
& \times \frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}} + \frac{1-(-1)^{l_{\nu+1}} e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\} z_1(\tau)/\theta_3^2 \\
& + e^{-\pi i\{(-1)^{l_\nu}(2\beta-\gamma)+(-1)^{l_\nu+l_{\nu-1}}(2\alpha-\gamma)+\cdots+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}(2\beta-\gamma)\}} \\
& \times e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_\nu} e^{\{1-(-1)^{k_\nu}\}\pi i(\gamma-2\alpha)/2} \\
& \times \frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}} - \frac{1-(-1)^{l_{\nu+1}} e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\} z_1(\tau)/\theta_3^2,
\end{aligned}$$

from which (4.3) follows immediately.

5. The integrals J_ν

The explicit formulas for integrals J_ν are as follows:

Theorem 5.1. (i) *We have*

$$J_0 = e^{\pi i(1-\gamma)k_1} e^{\pi i\alpha} \frac{\sin[\pi\{\gamma - (-1)^{k_1}(2\alpha-\gamma)\}/2]}{\sin\pi\gamma} z_1(\tau)/\theta_3^2 + e^{\pi i(1-\gamma)k_1} e^{\pi i\alpha} \frac{\sin[\pi\{\gamma + (-1)^{k_1}(2\beta-\gamma)\}/2]}{\sin\pi\gamma} z_2(\tau)/\theta_3^2. \tag{5.1}$$

(ii) *If ν is a positive even integer, then we have*

(5.2)
$$\begin{aligned} J_\nu &= e^{\pi i\{(-1)^{l_{\nu+1}}(2\alpha-\gamma)+(-1)^{l_{\nu+1}+l_\nu}(2\beta-\gamma)+\cdots+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3}(2\alpha-\gamma)+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3+l_2}(2\beta-\gamma)\}} \\ &\quad\times e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_{\nu+1}} e^{\pi i\alpha} \\ &\quad\times \frac{\sin[\pi\{\gamma+(-1)^{k_{\nu+1}}(\gamma-2\alpha)\}/2]}{\sin\pi\gamma} z_1(\tau)/\theta_3^2 \\ &\quad+ e^{\pi i\{(-1)^{l_{\nu+1}}(2\alpha-\gamma)+(-1)^{l_{\nu+1}+l_\nu}(2\beta-\gamma)+\cdots+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3}(2\alpha-\gamma)+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3+l_2}(2\beta-\gamma)\}} \\ &\quad\times e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_{\nu+1}} e^{\pi i\alpha} \\ &\quad\times \frac{\sin[\pi\{\gamma+(-1)^{k_{\nu+1}}(2\beta-\gamma)\}/2]}{\sin\pi\gamma} z_2(\tau)/\theta_3^2. \end{aligned}$$

(iii) *If ν is a positive odd integer, then we have*

$$\begin{aligned} J_\nu &= -e^{\pi i\{(-1)^{l_{\nu+1}}(2\alpha-\gamma)+(-1)^{l_{\nu+1}+l_\nu}(2\beta-\gamma)+\cdots+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_2}(2\alpha-\gamma)\}} \\ &\quad\times e^{-2\pi i\gamma(l_{\nu+1}+l_{\nu-1}+\cdots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(\gamma-1)k_{\nu+1}} e^{\pi i(\alpha-3\gamma)} \\ &\quad\times \frac{\sin[\pi\{\gamma+(-1)^{k_{\nu+1}}(\gamma-2\alpha)\}/2]}{\sin\pi\gamma} z_1(\tau)/\theta_3^2 \\ &\quad- e^{\pi i\{(-1)^{l_{\nu+1}}(2\alpha-\gamma)+(-1)^{l_{\nu+1}+l_\nu}(2\beta-\gamma)+\cdots+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_2}(2\alpha-\gamma)\}} \\ &\quad\times e^{-2\pi i\gamma(l_{\nu+1}+l_{\nu-1}+\cdots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(\gamma-1)k_{\nu+1}} e^{\pi i(\alpha-3\gamma)} \\ &\quad\times \frac{\sin[\pi\{\gamma+(-1)^{k_{\nu+1}}(2\beta-\gamma)\}/2]}{\sin\pi\gamma} z_2(\tau)/\theta_3^2. \end{aligned} \tag{5.3}$$

In fact, let ν be a positive even integer. By the same argument as in Section 4, we have

Lemma 5.2. *For a positive even integer ν we have*

$$\begin{aligned} &T_+\left(v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} \\ &= e^{\pi i\{(-1)^{l_{\nu+1}}(2\alpha-\gamma)+(-1)^{l_{\nu+1}+l_\nu}(2\beta-\gamma)+\cdots+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3}(2\alpha-\gamma)+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3+l_2}(2\beta-\gamma)\}} \\ &\quad\times e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} T_+\left(v+\frac{k_{\nu+1}}{2}\right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}. \end{aligned} \tag{5.4}$$

Substitution of (5.4) into (2.3) and some calculation makes J_ν into

$$
\begin{aligned}
J_\nu &= e^{\pi i\{(-1)^{l_{\nu+1}}(2\alpha-\gamma)+(-1)^{l_{\nu+1}+l_\nu}(2\beta-\gamma)+\cdots+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3}(2\alpha-\gamma)+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3+l_2}(2\beta-\gamma)\}} \\
&\quad \times e^{2\pi i\gamma(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} \\
&\quad \times \left\{ \frac{1+(-1)^{k_{\nu+1}}}{2} \frac{e^{\pi i(1-\gamma)k_{\nu+1}}}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\beta-\gamma)})} \right. \\
&\qquad \times \int^{((\tau/2)+,0+,(\tau/2)-,0-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\,dv \\
&\qquad + \frac{1-(-1)^{k_{\nu+1}}}{2} \frac{e^{\pi i(1-\gamma)k_{\nu+1}} e^{\pi i(2\alpha-\gamma)}}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\gamma-\alpha)})} \\
&\qquad \left. \times \int^{((\tau/2)+,0+,(\tau/2)-,0-)} T(v)^{2\gamma-2\alpha-1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3}\,dv \right\}.
\end{aligned}
\tag{5.5}
$$

Applying the formulas of Lemma 3.2 to (5.5), we have (5.2) immediately. One can derive the other formulas (5.1) and (5.3) similarly.

ACKNOWLEDGEMENT. The author would like to thank the referee for giving him valuable comments on the manuscript of this paper. This work was supported by JSPS. KAKENHI (19540158).

References

[1] K. Chandrasekharan: Elliptic Functions, Springer, Berlin, 1985.
[2] D. Mumford: Tata Lectures on Theta, I, Birkhäuser Boston, Boston, MA, 1983.
[3] H. Rademacher: Topics in Analytic Number Theory, Springer, New York, 1973.
[4] J. Tannery and J. Molk: Eléments de la Théorie des Fonctions Elliptiques, II, Chelsea, New York, 1972.
[5] H. Watanabe: *Transformation relations of matrix functions associated to the hypergeometric function of Gauss under modular transformations*, J. Math. Soc. Japan **59** (2007), 113–126.
[6] H. Watanabe: *Twisted homology and cohomology groups associated to the Wirtinger integral*, J. Math. Soc. Japan **59** (2007), 1067–1080.
[7] W. Wirtinger: *Zur Darstellung der hypergeometrischen Function durch bestimmte Integrale*, Akad. Wiss. Wien. S.-B. IIa **111** (1902), 894–900.
[8] Erdélyi (ed.): Higher Transcendental Functions, I, McGraw-Hill, New York, 1953.
[9] M.J. Kampé de Fériet: La Fonction Hypergéométrique, Mémorial des sciences mathématiques **85**, Gauthier-Villars, Paris, 1937.

Kitami Institute of Technology
165, Koencho, Kitami 0908507, Hokkaido
Japan
e-mail: hwatanab@cs.kitami-it.ac.jp