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# FORMALITY AND HARD LEFSCHETZ PROPERTY OF ASPHERICAL MANIFOLDS

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**Abstract**

For a Lie group $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ with the semi-simple action $\phi\colon \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$, we show that if $\Gamma$ is a finite extension of a lattice of $G$ then $K(\Gamma, 1)$ is formal. Moreover we show that a compact symplectic aspherical manifold with the fundamental group $\Gamma$ satisfies the hard Lefschetz property. By those results we give many examples of formal solvmanifolds satisfying the hard Lefschetz property but not admitting Kähler structures.

## 1. Introduction

Formal spaces (see Definition 5.3) in the sense of Sullivan are important in de Rham homotopy theory. Well-known examples of formal spaces are compact Kähler manifolds (see [9]). Suppose $\Gamma$ is a torsion-free finitely generated nilpotent group. Then $K(\Gamma, 1)$ is formal if and only if $\Gamma$ is abelian by Hasegawa's theorem in [11]. But in case $\Gamma$ is a virtually polycyclic (see Definition 2.1) group, the formality of $K(\Gamma, 1)$ is more complicated. One of the purposes of this paper is to apply the way of the algebraic hull of $\Gamma$ to study the formality of $K(\Gamma, 1)$. For a torsion-free virtually polycyclic group $\Gamma$, we have a unique algebraic group $\mathbf{H}_\Gamma$ with an injective homomorphism $\psi\colon \Gamma \to \mathbf{H}_\Gamma$ so that:
(1) $\psi(\Gamma)$ is Zariski-dense in $\mathbf{H}_\Gamma$.
(2) The centralizer $Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma))$ of $\mathbf{U}(\mathbf{H}_\Gamma)$ is contained in $\mathbf{U}(\mathbf{H}_\Gamma)$.
(3) $\dim \mathbf{U}(\mathbf{H}_\Gamma) = \operatorname{rank} \Gamma$.
Such $\mathbf{H}_\Gamma$ is called the algebraic hull of $\Gamma$. We call the unipotent radical of $\mathbf{H}_\Gamma$ the unipotent hull of $\Gamma$ and denote it by $\mathbf{U}_\Gamma$. In [3], Baues constructed a compact aspherical manifold $M_\Gamma$ with the fundamental group $\Gamma$ which is called the standard $\Gamma$-manifold by the algebraic hull of $\Gamma$. And he gave the way of computation of the de Rham cohomology of $M_\Gamma$. By using these results, we prove:

**Proposition 1.1.** *If the unipotent hull $\mathbf{U}_\Gamma$ of $\Gamma$ is abelian, $K(\Gamma, 1)$ is formal.*

So we would like to know criteria for $\mathbf{U}_\Gamma$ to be abelian. We prove the following theorem.

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**Theorem 1.2.** *Let $\Gamma$ be a torsion-free virtually polycyclic group. Then the following two conditions are equivalent*:
(1) $\mathbf{U}_\Gamma$ *is abelian.*
(2) $\Gamma$ *is a finite extension group of a lattice of a Lie group* $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ *such that the action* $\phi\colon \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ *is semi-simple.*

Therefore we have:

**Corollary 1.3.** *If* $\Gamma$ *satisfies the condition* (2) *in* Theorem 1.2, *then* $K(\Gamma, 1)$ *is formal.*

REMARK 1. A lattice $\Gamma$ of $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ is the form $\Gamma' \ltimes_\phi \Gamma''$ such that $\Gamma'$ and $\Gamma''$ are lattices of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and the action $\phi$ of $\Gamma'$ preserves $\Gamma''$.

As well as formality the hard Lefschetz property (see Definition 5.5) is an important property of a compact Kähler manifold. We have the following proposition.

**Proposition 1.4.** *Let $M$ be a compact symplectic aspherical manifold with the torsion-free virtually polycyclic fundamental group* $\Gamma$. *If the unipotent hull* $\mathbf{U}_\Gamma$ *is abelian, then $M$ satisfies the hard Lefschetz property.*

Hence we have:

**Corollary 1.5.** *If* $\Gamma$ *satisfies the condition* (2) *in* Theorem 1.2, *then a compact symplectic aspherical manifold with the fundamental group* $\Gamma$ *satisfies the hard Lefschetz property*.

In [5], Benson and Gordon showed that a compact symplectic aspherical manifold with the torsion-free nilpotent fundamental group $\Gamma$ satisfies the hard Lefschetz property if and only if $\Gamma$ is abelian.

As we see in [11] and [5], formality and the hard Lefschetz property are strong criteria for aspherical manifolds to admit Kähler structures. But by the results of this paper, we can obtain many non-Kähler formal aspherical manifolds satisfying the hard Lefschetz property.

Let $M$ be a compact aspherical manifold with the virtually polycyclic fundamental group. In [4], Baues and Cortés showed that if $M$ admits a Kähler structure then the fundamental group of $M$ is virtually abelian (this result is an extension of the result in [1] and [12]). Let $G$ be a simply connected solvable Lie group. We say that $G$ is of type (I) if for any $g \in G$ all eigenvalues of the adjoint operator $\mathrm{Ad}_g$ have absolute value 1. In [2] it was proved that a lattice of a simply connected solvable Lie group $G$ is virtually nilpotent if and only if $G$ is type (I). Hence we have:

**Corollary 1.6.** *Let $\Gamma$ be a finite extension group of a lattice of a Lie group $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ such that the action $\phi$: $\mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ is semi-simple and $G$ is not of type* (I). *Then a compact aspherical manifold $M$ with the fundamental group $\Gamma$ is formal but admits no Kähler structure. If $M$ admits a symplectic structure, then $M$ satisfies the hard Lefschetz property.*

REMARK 2. In [12], Hasegawa showed that a simply connected solvable Lie group $G$ with a virtually abelian lattice such that $G/\Gamma$ admits Kähler structure can be written as $G = \mathbb{R}^{2k} \ltimes_\phi \mathbb{C}^l$ such that

$$\phi(t_j)((z_1, \dots, z_l)) = \left(e^{\sqrt{-1}\theta_1^j t_j} z_1, \dots, e^{\sqrt{-1}\theta_l^j t_j} z_l\right),$$

where each $e^{\sqrt{-1}\theta_i^j}$ is a root of unity.

Solvmanifolds are homogeneous spaces of connected solvable Lie groups. These are examples of aspherical manifolds with the polycyclic fundamental groups. In particular for a simply connected solvable Lie group $G$ with a lattice $\Gamma$, the solvmanifold $G/\Gamma$ is a compact aspherical manifold with the fundamental group $\Gamma$. As generalizations of solvmanifolds we define infra-solvmanifolds. Let $G$ be a simply connected solvable Lie group. Consider the group $\mathrm{Aut}(G) \ltimes G$ of affine transformations of $G$ and the projection $p$: $\mathrm{Aut}(G) \ltimes G \to \mathrm{Aut}(G)$. An infra-solvmanifold is a manifold of the form $G/\Delta$ for a torsion-free subgroup $\Delta$ of $\mathrm{Aut}(G) \ltimes G$ such that $p(\Delta)$ is contained in a compact subgroup of $\mathrm{Aut}(G)$. In [3] Baues showed that every compact infra-solvmanifold is diffeomorphic to a standard $\Gamma$-manifold and for any torsion-free virtually polycyclic group $\Gamma$ the standard $\Gamma$-manifold is diffeomorphic to an infra-solvmanifold $G/\Gamma$ such that $\Gamma \subset \mathrm{Aut}(G) \ltimes G$ is a discrete subgroup and $p(\Gamma)$ is finite. Thus for any $\Gamma$ satisfying the condition (2) in Theorem 1.2 we have a compact infra-solvmanifold $G/\Gamma$ for some $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ such that the action $\phi$: $\mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ is semi-simple.

*Notations and terminology*: Let $k$ be a subfield of $\mathbb{C}$. A group $\mathbf{G}$ is called $k$-algebraic group if $\mathbf{G}$ is a Zariski-closed subgroup of $GL_n(\mathbb{C})$ which is defined by polynomials with coefficients in $k$. Let $\mathbf{G}(k)$ denote the set of $k$-points of $\mathbf{G}$ and $\mathbf{U}(\mathbf{G})$ the maximal Zariski-closed unipotent normal $k$-subgroup of $\mathbf{G}$ called the unipotent radical of $\mathbf{G}$. A general reference is [7]. In this paper, algebraic groups are always written in the bold face.

## 2. Algebraic hulls

In this section we explain the algebraic hulls of polycyclic groups or simply connected solvable Lie groups.

DEFINITION 2.1. A group $\Gamma$ is *polycyclic* if it admits a sequence

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$$

of subgroups such that each $\Gamma_i$ is normal in $\Gamma_{i-1}$ and $\Gamma_{i-1}/\Gamma_i$ is cyclic. We set $\operatorname{rank}\Gamma = \sum_{i=1}^{i=k} \operatorname{rank}\Gamma_{i-1}/\Gamma_i$ which is independent of the choice of a sequence $\Gamma_i$.

There are close relations between polycyclic groups and solvable Lie groups.

**Theorem 2.2** ([21, Proposition 3.7, Theorem 4.28]). *Let $G$ be a simply connected solvable Lie group and $\Gamma$ a lattice in $G$. Then $\Gamma$ is torsion-free polycyclic and* $\dim G = \operatorname{rank}\Gamma$. *Conversely every polycyclic group admits a finite index normal subgroup which is isomorphic to a lattice in a simply connected solvable Lie group.*

Let $\Gamma$ be a virtually polycyclic group and $\Gamma'$ be a finite index polycyclic subgroup. We set $\operatorname{rank}\Gamma = \operatorname{rank}\Gamma'$.

DEFINITION 2.3. Let $k$ be a subfield $\mathbb{C}$. Let $\Gamma$ be a torsion-free virtually polycyclic group (resp. simply connected solvable Lie group). Then a $k$-algebraic group $\mathbf{H}_\Gamma$ is a *$k$-algebraic hull* of $\Gamma$ if there exists an injective homomorphism $\psi\colon \Gamma \to \mathbf{H}_\Gamma(k)$ and $\mathbf{H}_\Gamma$ satisfies the following conditions:
(1) $\psi(\Gamma)$ is Zariski-dense in $\mathbf{H}_\Gamma$.
(2) $Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma)) \subset \mathbf{U}(\mathbf{H}_\Gamma)$.
(3) $\dim \mathbf{U}(\mathbf{H}_\Gamma) = \operatorname{rank}\Gamma$ (resp. $\dim\Gamma$).

**Theorem 2.4** ([3, Theorem A.1, Corollary A.3], [21, Proposition 4.40, Lemma 4.41]). *Let $\Gamma$ be a torsion-free virtually polycyclic group* (*resp. simply connected solvable Lie group*). *Then there exists a $\mathbb{Q}$-algebraic* (*resp. $\mathbb{R}$-algebraic*) *hull of $\Gamma$ and for any subfield $k \subset \mathbb{C}$ which contains $\mathbb{Q}$* (*resp. $\mathbb{R}$*) *a $k$-algebraic hull of $\Gamma$ is unique up to $k$-algebraic group isomorphism.*

We call the unipotent radical of $\mathbf{H}_\Gamma$ the unipotent hull of $\Gamma$ and denote it by $\mathbf{U}_\Gamma$.

**Lemma 2.5.** *Let $\Gamma$ be a torsion-free virtually polycyclic group and $\Delta$ a finite index subgroup of $\Gamma$. Let $\psi\colon \Gamma \to \mathbf{H}_\Gamma$ be the $k$-algebraic hull of $\Gamma$ and $\mathbf{G}$ the Zariski-closure of $\psi(\Delta)$ in $\mathbf{H}_\Gamma$. Then the algebraic group $\mathbf{G}$ is the $k$-algebraic hull of $\Delta$ and we have $\mathbf{U}_\Delta = \mathbf{U}_\Gamma$.*

Proof. Let $\mathbf{H}^0_\Gamma$ be the identity component of $\mathbf{H}_\Gamma$. Since $\mathbf{G}$ is a closed finite index subgroup of $\mathbf{H}_\Gamma$, we have $\mathbf{H}^0_\Gamma \subset \mathbf{G}$. Since $\Gamma$ is virtually polycyclic, $\mathbf{H}^0_\Gamma$ is solvable. Hence we have $\mathbf{U}(\mathbf{H}_\Gamma) = (\mathbf{H}^0_\Gamma)_{\mathrm{unip}} = \mathbf{U}(\mathbf{G})$. Since $\operatorname{rank}\Gamma = \operatorname{rank}\Delta$, we have

$$\dim \mathbf{U}(\mathbf{G}) = \operatorname{rank}\Delta,$$

and we have

$$Z_{\mathbf{G}'}(\mathbf{U}(\mathbf{G})) \subset Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma)) \subset \mathbf{U}(\mathbf{H}_\Gamma) = \mathbf{U}(\mathbf{G}).$$

Hence the lemma follows. $\square$

**Lemma 2.6** ([21, Proof of Theorem 4.34])**.** *Let $G$ be a simply connected solvable Lie group with a lattice $\Gamma$. Let $\psi\colon G \to \mathbf{H}_G$ be the $\mathbb{R}$-algebraic hull of $G$ and $\mathbf{H}'$ the Zariski-closure of $\psi(\Gamma)$ in $\mathbf{H}_G$. Then $\mathbf{H}'$ is the $\mathbb{R}$-algebraic hull of $\Gamma$ and we have $\mathbf{U}_G = \mathbf{U}_\Gamma$.*

## 3. Cohomology computations of aspherical manifolds with virtually torsion-free polycyclic fundamental groups

Let $\Gamma$ be a torsion-free virtually polycyclic group and $\mathbf{H}_\Gamma$ the $\mathbb{Q}$-algebraic hull of $\Gamma$. Denote $H_\Gamma = \mathbf{H}_\Gamma(\mathbb{R})$. Let $U_\Gamma$ be the unipotent radical of $H_\Gamma$ and let $T$ be a maximal reductive subgroup. Then $H_\Gamma$ decomposes as a semi-direct product $H_\Gamma = T \ltimes U_\Gamma$. Let $\mathfrak{u}$ be the Lie algebra of $U_\Gamma$. Since the exponential map $\exp\colon \mathfrak{u} \to U_\Gamma$ is a diffeomorphism, $U_\Gamma$ is diffeomorphic to $\mathbb{R}^n$ such that $n = \operatorname{rank}\Gamma$. The splitting $H_\Gamma = T \ltimes U_\Gamma$ gives rise to the affine action $\alpha\colon H_\Gamma \to \mathrm{Aut}(U_\Gamma) \ltimes U_\Gamma$ such that $\alpha$ is an injective homomorphism.

In [3] Baues constructed a compact aspherical manifold $M_\Gamma = \alpha(\Gamma)\backslash U_\Gamma$ with $\pi_1(M_\Gamma) = \Gamma$. We call $M_\Gamma$ a standard $\Gamma$-manifold.

**Theorem 3.1** ([3, Theorem 1.2])**.** *Standard $\Gamma$-manifold is unique up to diffeomorphism.*

Let $A^*(M_\Gamma)$ be the de Rham complex of $M_\Gamma$. Then $A^*(M_\Gamma)$ is the set of the $\Gamma$-invariant differential forms $A^*(U_\Gamma)^\Gamma$ on $U_\Gamma$. Let $\left(\bigwedge \mathfrak{u}^*\right)^T$ be the left-invariant forms on $U_\Gamma$ which are fixed by $T$. Since $\Gamma \subset H_\Gamma = T \ltimes U_\Gamma$, we have the inclusion

$$\left(\bigwedge \mathfrak{u}^*\right)^T = A^*(U_\Gamma)^{H_\Gamma} \subset A^*(U_\Gamma)^\Gamma = A^*(M_\Gamma).$$

**Theorem 3.2** ([3, Theorem 1.8])**.** *This inclusion induces a cohomology isomorphism.*

## 4. Proof of Theorem 1.2

**4.1. The embeddings of solvable Lie algebras in splittable Lie algebras.** The idea of this subsection is based on [22]. Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{n} = \{X \in \mathfrak{g} \mid \mathrm{ad}_X \text{ is nilpotent}\}$. Then $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{g}$ and called the nilradical of $\mathfrak{g}$.

**Lemma 4.1** ([18, p. 58])**.** *We have $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$.*

Let $D(\mathfrak{g})$ denote the space of the derivations of $\mathfrak{g}$. By the Jordan decomposition, we have the decomposition $\mathrm{ad}_X = d_X + n_X$ such that $d_X$ is a semi-simple operator and $n_X$ is a nilpotent operator.

**Lemma 4.2** ([22, Proposition 3]). *We have $d_X$, $n_X \in D(\mathfrak{g})$.*

Then we have the homomorphism $f\colon \mathfrak{g} \to D(\mathfrak{g})$ such that $f(X) = d_X$ for $X \in \mathfrak{g}$. Since $\ker f = \mathfrak{n}$, we have $\operatorname{Im} f \cong \mathfrak{g}/\mathfrak{n}$.

Let $\bar{\mathfrak{g}} = \operatorname{Im} f \ltimes \mathfrak{g}$ and $\bar{\mathfrak{n}} = \{X - d_X \in \bar{\mathfrak{g}} \mid X \in \mathfrak{g}\}$. Since $\operatorname{ad}_{X-d_X} = \operatorname{ad}_X - d_X$ on $\mathfrak{g}$, $\operatorname{ad}_{X-d_X}$ is a nilpotent operator. So $\bar{\mathfrak{n}}$ consists of nilpotent elements.

**Proposition 4.3.** *We have $d_X(\bar{\mathfrak{n}}) \subset \mathfrak{n}$ for any $X \in \mathfrak{g}$, $\bar{\mathfrak{n}}$ is a nilpotent ideal of $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{g}} = \operatorname{Im} f \ltimes \bar{\mathfrak{n}}$.*

Proof. By Lie's theorem, we have a basis $X_1, \dots, X_l$ of $\mathfrak{n} \otimes \mathbb{C}$ such that $\operatorname{ad}_{\mathfrak{g}}$ on $\mathfrak{n}$ are represented by upper triangular matrices. Then for any $X \in \mathfrak{g}$, we have

$$\begin{aligned}
&\operatorname{ad}_X(X_1) = a_{X,1}X_1,\\
&\operatorname{ad}_X(X_2) = a_{X,2}X_2 + b_{X,12}X_1,\\
&\dots\\
&\operatorname{ad}_X(X_l) = a_{X,l}X_l + b_{X,l-1l}X_{l-1} + \cdots + b_{X,1l}X_1.
\end{aligned}$$

We take $X_{l+1}, \dots, X_{l+m}$ such that $X_1, \dots, X_l, X_{l+1}, \dots, X_{l+m}$ is a basis of $\mathfrak{g} \otimes \mathbb{C}$. By Lemma 4.1, we have $\operatorname{ad}_X(X_i) \in \mathfrak{n}$. Hence we have

$$\begin{aligned}
&\operatorname{ad}_X(X_{l+1}) = b_{X,ll+1}X_l + \cdots + b_{X,1l+1}X_1,\\
&\dots\\
&\operatorname{ad}_X(X_{l+m}) = b_{X,ll+m}X_l + \cdots + b_{X,1l+m}X_1.
\end{aligned}$$

Then we have

$$\begin{aligned}
&d_X(X_i) = a_{X,i}X_i, \quad 1 \le i \le l,\\
&d_X(X_i) = 0, \quad l+1 \le i \le l+m.
\end{aligned}$$

Hence we have $d_X(\mathfrak{g}) \subset \mathfrak{n}$ and $d_X(\bar{\mathfrak{n}}) \subset \mathfrak{n}$. This implies $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] \subset \mathfrak{n}$. In particular, $\bar{\mathfrak{n}}$ is an ideal of $\bar{\mathfrak{g}}$. Since $\bar{\mathfrak{n}}$ consists of nilpotent elements, $\bar{\mathfrak{n}}$ is a nilpotent ideal. By $\bar{\mathfrak{g}} = \{d_X + Y - d_Y \mid X, Y \in \mathfrak{g}\}$, we have $\bar{\mathfrak{g}} = \operatorname{Im} f \ltimes \bar{\mathfrak{n}}$. □

By this proposition, we have the inclusion $i\colon \mathfrak{g} \to D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$ given by $i(X) = d_X + X - d_X$ for $X \in \mathfrak{g}$.

**4.2. Constructions of algebraic hulls of simply connected solvable Lie groups.** Let $G$ be a simply connected solvable Lie group and $\mathfrak{g}$ the Lie algebra of $G$. Let $N$ be the maximal normal nilpotent subgroup of $G$ which corresponds to the nilradical $\mathfrak{n}$ of $\mathfrak{g}$. Consider the injection $i\colon \mathfrak{g} \to \operatorname{Im} f \ltimes \bar{\mathfrak{n}} \subset D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$ constructed in the last subsection. Let $\bar{N}$ be the simply connected Lie group which corresponds to $\bar{\mathfrak{n}}$. Since the Lie algebra of $\mathrm{Aut}(\bar{N}) \ltimes \bar{N}$ is $D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$, we have the Lie group homomorphism $I: G \to \mathrm{Aut}(\bar{N}) \ltimes \bar{N}$ induced by the injective homomorphism $i: \mathfrak{g} \to D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$.

**Lemma 4.4.** *The homomorphism $I: G \to \mathrm{Aut}(\bar{N}) \ltimes \bar{N}$ is injective.*

Proof. Since the restriction of $i: \mathfrak{g} \to D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$ on $\mathfrak{n}$ is injective, the restriction $I: G \to \mathrm{Aut}(\bar{N}) \ltimes \bar{N}$ on $N$ is also injective. Let $T_f$ be the subgroup of $\mathrm{Aut}(\bar{N})$ which corresponds to $\mathrm{Im}\, f$. We have $I: G \to T_f \ltimes \bar{N}$. By Proposition 4.3, $\bar{\mathfrak{g}}/\mathfrak{n} = \mathrm{Im}\, f \oplus \bar{\mathfrak{n}}/\mathfrak{n}$. So we have the induced map $I: G/N \to T_f \times \bar{N}/N$ and it is sufficient to show that this map is injective. Let $j: \mathrm{Im}\, f \oplus \bar{\mathfrak{n}}/\mathfrak{n} \to \bar{\mathfrak{n}}/\mathfrak{n}$ be the projection and $J: T_f \times \bar{N}/N \to \bar{N}/N$ be the homomorphism which corresponds to $j$. Since the composition

$$j \circ i(X \bmod \mathfrak{n}) = X - d_X \mod \mathfrak{n}$$

is surjective, $j \circ i: \mathfrak{g}/\mathfrak{n} \to \bar{\mathfrak{n}}/\mathfrak{n}$ is an isomorphism. Since $G/N$ and $\bar{N}/N$ are simply connected abelian groups, $J \circ I: G/N \to \bar{N}/N$ is also an isomorphism. Hence $I: G/N \to T_f \times \bar{N}/N$ is injective. □

A simply connected nilpotent Lie group is considered as the real points of a unipotent $\mathbb{R}$-algebraic group (see [19, p. 43]) by the exponential map. We have the unipotent $\mathbb{R}$-algebraic group $\bar{\mathbf{N}}$ with $\bar{\mathbf{N}}(\mathbb{R}) = \bar{N}$. We identify the group $\mathrm{Aut}_a(\bar{\mathbf{N}})$ of automorphisms of algebraic groups with $\mathrm{Aut}(\mathfrak{n}_{\mathbb{C}})$ and $\mathrm{Aut}_a(\bar{\mathbf{N}})$ has the $\mathbb{R}$-algebraic group structure with $\mathrm{Aut}_a(\bar{\mathbf{N}})(\mathbb{R}) = \mathrm{Aut}(N)$. So we have the $\mathbb{R}$-algebraic group $\mathrm{Aut}_a(\bar{\mathbf{N}}) \ltimes \bar{N}$. By the above lemma, we have the injection $I: G \to \mathrm{Aut}(N) \ltimes N = \mathrm{Aut}_a(\bar{\mathbf{N}}) \ltimes \bar{N}(\mathbb{R})$. Let $\mathbf{G}$ be the Zariski-closure of $I(G)$ in $\mathrm{Aut}_a(\bar{\mathbf{N}}) \ltimes \bar{N}$.

**Lemma 4.5.** *We have $\mathbf{U}(\mathbf{G}) = \bar{N}$.*

Proof. Let $\mathbf{T}$ be the Zariski-closure of $T_f$ in $\mathrm{Aut}_a(\bar{\mathbf{N}})$. Then $\mathbf{G} \subset \mathbf{T} \ltimes \bar{\mathbf{N}}$. Since $\mathbf{G}$ is connected solvable and $\mathbf{T}$ consists of semi-simple automorphisms, we have $\mathbf{U}(\mathbf{G}) = \mathbf{G} \cap \bar{\mathbf{N}}$. By this, it is sufficient to show $\dim \mathbf{U}(\mathbf{G}) = \dim \bar{\mathbf{N}}$. Let $\mathbf{N}$ be the Zariski-closure of $I(N)$. By $I(N) \subset \bar{N}$, we have $\mathbf{U}(\mathbf{G})/\mathbf{N} = \mathbf{U}(\mathbf{G}/\mathbf{N})$. Thus it is sufficient to show $\mathbf{U}(\mathbf{G}/\mathbf{N}) = G/N$. Consider the induced map $I: G/N \to T_f \times \bar{N}/N$ as the proof of Lemma 4.4. The Zariski-closure of $I(G/N)$ in $\mathbf{T} \times \bar{\mathbf{N}}/\mathbf{N}$ is $\mathbf{G}/\mathbf{N}$. Since $\mathbf{T} \times \bar{\mathbf{N}}/\mathbf{N}$ is commutative, the projection $\mathbf{T} \times \bar{\mathbf{N}}/\mathbf{N} \to \bar{\mathbf{N}}/\mathbf{N}$ is an $\mathbb{R}$-algebraic group homomorphism. Since we showed that $J \circ I: G/N \to \bar{N}/N$ is isomorphism In the proof of Lemma 4.4, the image $J \circ I(G/N)$ is Zariski-dense in $\bar{\mathbf{N}}/\mathbf{N}$. This implies $\bar{\mathbf{N}}/\mathbf{N} = \mathbf{U}(\mathbf{G}/\mathbf{N})$. Hence the lemma follows. □

By this lemma we have the following proposition.

**Proposition 4.6.** *$\mathbf{G}$ is the algebraic hull of $G$ and the Lie algebra of the unipotent hull $\mathbf{U}_G$ is $\bar{\mathfrak{n}}_{\mathbb{C}}$.*

Proof. We show that $\mathbf{G}$ satisfies the properties of the algebraic hull of $G$. We have $\dim \mathbf{U}(\mathbf{G}) = \dim \bar{\mathbf{N}} = \dim G$. Let $(t, x) \in Z_{\mathbf{G}}(\mathbf{U}(\mathbf{G})) \subset \mathrm{Aut}_a \bar{\mathbf{N}} \ltimes \bar{N}$. Since $\mathbf{U}(\mathbf{G}) = \mathbf{N}$ and t is a semi-simple automorphism, we have $t(y) = y$ for any $y \in \bar{\mathbf{N}}$. So we have $t = \mathrm{id}_{\bar{\mathbf{N}}}$. We have $Z_{\mathbf{G}}(\mathbf{U}(\mathbf{G})) \subset \mathbf{U}(\mathbf{G})$. Hence the proposition follows. □

**4.3. Proof of Theorem 1.2.** We first prove:

**Theorem 4.7.** *Let $G$ be a simply connected solvable Lie group. Then $\mathbf{U}_G$ is abelian if and only if $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ such that the action $\phi\colon \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ is semi-simple.*

Proof. Consider the inclusion $i\colon \mathfrak{g} \to \mathrm{Im}\, f \ltimes \bar{\mathfrak{n}}$. By the above argument, the Lie algebra of $\mathbf{U}_G$ is $\bar{\mathfrak{n}}_{\mathbb{C}}$. Suppose $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ such that the action $\phi\colon \mathbb{R}^n \to \mathrm{Aut}\, \mathbb{R}^m$ is semi-simple. It is sufficient to show $\bar{\mathfrak{n}} = \{X - d_X \mid X \in \mathfrak{g}\} \subset \mathrm{Im}\, f \ltimes \mathfrak{g}$ is an abelian Lie algebra. Let $X, Y \in \mathfrak{g}$ and $X = X_1 + X_2$, $Y = Y_1 + Y_2$ be the decompositions induced by the semi-direct product $\mathfrak{g} = \mathbb{R}^n \ltimes_{\phi_*} \mathbb{R}^m$. Then we have $d_{X_2} = 0$, $d_{Y_2} = 0$, $[X_1, Y_1] = 0$ and $[X_2, Y_2] = 0$ by the assumption. Hence we have

$$[X - d_X, Y - d_Y] = [X_1, Y_2] + [X_2, Y_1] - d_{X_1}(Y_2) + d_{Y_1}(X_2).$$

Since the action $\phi_*$ is semi-simple, we have $d_{X_1}(Y_2) = [X_1, Y_2]$ and $d_{Y_1}(X_2) = [Y_1, X_2]$. Therefore we have $[X - d_X, Y - d_Y] = 0$. This implies $\bar{\mathfrak{n}}$ is abelian.

Conversely we assume $\mathbf{U}_G$ is abelian. By Proposition 4.6, $\bar{\mathfrak{n}}$ is abelian. By $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$, $\mathfrak{g}$ is two-step solvable. By [8, Lemma 4.1], we have the decomposition $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{g}^\infty$ for some nilpotent subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ where $\mathfrak{g}^\infty = \bigcap \mathfrak{g}^i$ for the lower central series $\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots$ of $\mathfrak{g}$. Since $\bar{\mathfrak{n}}$ is abelian, the subspace $\{X - d_X \mid X \in \mathfrak{a}\}$ is a abelian subalgebra of $\bar{\mathfrak{n}}$. Since $\mathfrak{a}$ is nilpotent, the Lie algebra $\{X - d_X \mid X \in \mathfrak{a}\}$ is identified with $\mathfrak{a}$. Hence $\mathfrak{a}$ is abelian. Finally we show that the action of $\mathfrak{a}$ on $\mathfrak{g}^\infty$ is semi-simple. We suppose that $\mathrm{ad}_X$ on $\mathfrak{g}^\infty$ is not semi-simple for some $X \in \mathfrak{a}$. Then the action of $\mathrm{ad}_X - d_X$ on $\mathfrak{g}^\infty$ is non-trivial. Since we have $\bar{\mathfrak{n}} = \{X - d_X \mid X \in \mathfrak{g}\} \subset \mathrm{Im}\, f \ltimes \bar{\mathfrak{n}}$, we have $[\bar{\mathfrak{n}}, \mathfrak{a}] \neq \{0\}$. This contradicts $\bar{\mathfrak{n}}$ is abelian. Hence the action of $\mathfrak{a}$ on $\mathfrak{g}^\infty$ is semi-simple and we have the theorem. □

Proof of Theorem 1.2. By Theorem 2.2, we have a finite index subgroup of $\Gamma$ which is isomorphic to a lattice of some simply connected solvable Lie group $G$. By Lemma 2.5 and 2.6, we have $\mathbf{U}_\Gamma = \mathbf{U}_G$. Hence by Theorem 4.7 we have the theorem. □

REMARK 3. A virtually polycyclic group $\Gamma$ has the maximal nilpotent normal subgroup called the nilradical of $\Gamma$. Since the nilradical of $\Gamma$ is contained in $\mathbf{U}_\Gamma$ (see

[3, Proposition A.7]), if $\mathbf{U}_\Gamma$ is abelian then the nilradical of $\Gamma$ is also abelian. But the converse is not true. Consider $G = \mathbb{R} \ltimes_\phi \mathbb{R}^4$ with

$$\phi(t) = \begin{pmatrix} e^{rt} & 0 & 0 & 0 \\ 0 & e^{-rt} & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then for some $r \neq 0$ $G$ has a lattice $\mathbb{Z} \ltimes_\phi \Gamma''$ for a lattice $\Gamma''$ of $\mathbb{R}^4$. We have $\mathbf{U}_\Gamma = \mathbf{U}_G = \mathbb{C}^2 \times \mathbf{U}_3(\mathbb{C})$ and it is not abelian. On the other hand the nilradical of $\Gamma$ (resp. $G$) is isomorphic to $\mathbb{Z}^4$ (resp. $\mathbb{R}^4$).

## 5. Formality and hard Lefschetz properties of aspherical manifolds

**5.1. Formality.** We review the definition of formality and prove Proposition 1.1.

DEFINITION 5.1. A *differential graded algebra* (called DGA) is a graded $\mathbb{R}$-algebra $A^*$ with the following properties:
(1) $A^*$ is graded commutative, i.e.

$$y \wedge x = (-1)^{p \cdot q} x \wedge y, \quad x \in A^p,\ y \in A^q.$$

(2) There is a differential operator $d\colon A \to A$ of degree one such that $d \circ d = 0$ and

$$d(x \wedge y) = dx \wedge y + (-1)^p x \wedge dy, \quad x \in A^p.$$

Let $A$ and $B$ be DGAs. If a morphism of graded algebra $\varphi\colon A \to B$ satisfies $d \circ \varphi = \varphi \circ d$, we call $\varphi$ a morphism of DGAs. If a morphism of DGAs induces a cohomology isomorphism, we call it a quasi-isomorphism.

DEFINITION 5.2. A and B are *weakly equivalent* if there is a finite diagram of DGAs

$$A \leftarrow C_1 \to C_2 \leftarrow \cdots \leftarrow C_n \to B$$

such that all the morphisms are quasi-isomorphisms.

Let $M$ be a smooth manifold. The de Rham complex $A^*(M)$ of $M$ is a DGA. The cohomology algebra $H^*(M, \mathbb{R})$ is a DGA with $d = 0$.

DEFINITION 5.3. A smooth manifold $M$ is *formal* if $A^*(M)$ and $H^*(M, \mathbb{R})$ are weakly equivalent.

**Proposition 5.4.** *Let $\Gamma$ be a torsion-free virtually polycyclic group. If the unipotent hull $U_\Gamma$ is abelian, the standard $\Gamma$-manifold $M_\Gamma$ is formal.*

Proof. We use same notations as in Section 3. If the $k$-unipotent hull of $\Gamma$ is abelian, $\left(\bigwedge \mathfrak{u}^*, d\right) = \left(\bigwedge \mathfrak{u}^*, 0\right)$. By Theorem 3.2, we have the diagram of DGAs

$$A^*(M_\Gamma) \leftarrow \left(\left(\bigwedge \mathfrak{u}^*\right)^T\right) = H^*(M_\Gamma)$$

such that the map $A^*(M_\Gamma) \leftarrow \left(\left(\bigwedge \mathfrak{u}^*\right)^T\right)$ is a quasi-isomorphism. Hence the proposition follows. □

Hence we have Proposition 1.1.

**5.2. The hard Lefschetz property.** We review the definition of the hard Lefschetz property and prove Proposition 1.4.

DEFINITION 5.5. Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. We say that $(M, \omega)$ satisfies the *hard Lefschetz property* if the linear map

$$[\omega^{n-i}]\wedge\colon H^i(M, \mathbb{R}) \to H^{2n-i}(M, \mathbb{R})$$

is an isomorphism for any $0 \leq i \leq n$.

Proof of Proposition 1.4. As in the proof of Proposition 1.1, we have an isomorphism $\left(\bigwedge \mathfrak{u}^*\right)^T \cong H^*(M,\mathbb{R})$. Consider the cohomology class of a symplectic form $\omega$ on $M$. We have $\omega_0 \in \left(\bigwedge^2 \mathfrak{u}^*\right)^T$ which represents the cohomology class $[\omega] \in H^2(M,\mathbb{R})$. Since $\omega_0^n \neq 0$ for $2n = \dim \mathfrak{u} = \dim M$, $\omega_0$ is a symplectic form on the vector space $\mathfrak{u}$. Since the linear map

$$\omega_0^{n-i}\wedge\colon \bigwedge \mathfrak{u}^i \to \bigwedge \mathfrak{u}^{2n-i}$$

is injective for any $0 \leq i \leq n$ by the hard Lefschetz property of a torus, the restriction

$$\omega_0^{n-i}\wedge\colon \left(\bigwedge \mathfrak{u}^i\right)^T \to \left(\bigwedge \mathfrak{u}^{2n-i}\right)^T$$

is also injective and so

$$[\omega^{n-i}]\wedge\colon H^i(M, \mathbb{R}) \to H^{2n-i}(M, \mathbb{R})$$

is injective and thus it is an isomorphism by the Poincaré duality. Hence we have the proposition. □

## 6. Examples

EXAMPLE 1. Let $G = \mathbb{R} \ltimes_{\phi} \mathbb{R}^2$ with $\phi(t) = \begin{pmatrix} e^{rt} & 0 \\ 0 & e^{-rt} \end{pmatrix}$. Then for some $r \neq 0$, $\phi(1)$ is conjugate to an element of $SL_2(\mathbb{Z})$. Hence we have a lattice $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^2$. $G \times \mathbb{R}$ has a left-invariant symplectic form. In [10] (see also [20]) by direct computations Fernandez and Gray showed that $G/\Gamma \times S^1$ is formal and satisfies the hard Lefschetz property and admits no Complex structure. This is also a simple example for the result of this paper.

EXAMPLE 2. Let $G = \mathbb{C} \ltimes_{\phi} \mathbb{C}^2$ with $\phi(x) = \begin{pmatrix} e^{x} & 0 \\ 0 & e^{-x} \end{pmatrix}$. Then the cochain complex $\left(\bigwedge \mathfrak{g}^*, d\right)$ of the Lie algebra of $G$ is given by:

$$
\begin{aligned}
&\mathfrak{g}^* = \langle x_1, x_2, y_1, y_2, z_1, z_2 \rangle, \\
&dx_1 = dx_2 = 0, \\
&dy_1 = -x_1 \wedge y_1 + x_2 \wedge y_2, \quad dy_2 = -x_2 \wedge y_1 - x_1 \wedge y_2, \\
&dz_1 = x_1 \wedge z_1 - x_2 \wedge z_2, \quad dz_2 = x_1 \wedge z_2 + x_2 \wedge z_1.
\end{aligned}
$$

We have an invariant symplectic form $\omega = x_1 \wedge x_2 + z_1 \wedge y_1 + y_2 \wedge z_2$. For some $p, q \in \mathbb{R}$ $\phi(p\mathbb{Z} + \sqrt{-1}q\mathbb{Z})$ is conjugate to a subgroup of $SL_4(\mathbb{Z})$ and hence we have a lattice $\Gamma = (p\mathbb{Z} + \sqrt{-1}q\mathbb{Z}) \ltimes \Gamma''$ for a lattice $\Gamma''$ of $\mathbb{C}^2$ (see [15] and [13]). For any lattice $\Gamma$, $G/\Gamma$ is complex, symplectic with the hard Lefschetz property and formal but not Kähler.

REMARK 4. For a Lie group $G$ in Example 2, the de Rham cohomology of $G/\Gamma$ depends on a choice of a lattice $\Gamma$. Under some conditions, the de Rham cohomology of a solvmanifold $G/\Gamma$ is isomorphic to the cohomology of Lie algebra $\mathfrak{g}$ of $G$ (see [14], [21, Section 7]). But for a general solvmanifold $G/\Gamma$ it is difficult to compute the de Rham cohomology of $G/\Gamma$. By the results of this paper, for a Lie group $G = \mathbb{R}^n \ltimes_{\phi} \mathbb{R}^m$ with the semi-simple action $\phi$, we can say that $G/\Gamma$ is formal and hard Lefschetz for any lattice $\Gamma$ even if an isomorphism $H^*(G/\Gamma, \mathbb{R}) \cong H^*(\mathfrak{g})$ fails to hold.

EXAMPLE 3. Let $G = \mathbb{R} \ltimes_{\phi} \mathbb{R}^4$ with

$$
\phi(t) = \begin{pmatrix}
e^{pt}\cos(qt) & -e^{pt}\sin(qt) & 0 & 0 \\
e^{pt}\sin(qt) & e^{pt}\cos(qt) & 0 & 0 \\
0 & 0 & e^{-pt}\cos(-qt) & -e^{-pt}\sin(-qt) \\
0 & 0 & e^{-pt}\sin(-qt) & e^{-pt}\cos(-qt)
\end{pmatrix}.
$$

Then for $p, q$ as Example 2, $\phi(1)$ is conjugate to an element of $SL_4(\mathbb{Z})$ and hence $G$ has a lattice $\Gamma = \mathbb{Z} \ltimes \Gamma''$ for a lattice $\Gamma''$ of $\mathbb{R}^4$. The cochain complex $\left(\bigwedge (\mathfrak{g} \oplus \mathbb{R})^*, d\right)$

of the Lie algebra of $G \times \mathbb{R}$ is given by:

$$(\mathfrak{g} \oplus \mathbb{R})^* = \langle w, x_1, x_2, x_3, x_4, y \rangle,$$
$$dx_1 = -pw \wedge x_1 + qw \wedge x_2, \quad dx_2 = -qw \wedge x_1 - pw \wedge x_2,$$
$$dx_3 = pw \wedge x_3 - qw \wedge x_4, \quad dx_4 = qw \wedge x_3 + pw \wedge x_4.$$

We have a left-invariant symplectic form $\omega = w \wedge y + x_1 \wedge x_3 + x_4 \wedge x_2$. We regard $w + \sqrt{-1}y$, $x_1 + \sqrt{-1}x_2$, $x_3 + \sqrt{-1}x_4$ as (1,0)-forms, we obtain a left-invariant complex structure. By the result of this paper, for any lattice $\Gamma$, $G/\Gamma \times S^1$ is formal and any symplectic form on $G/\Gamma \times S^1$ satisfies the hard Lefschetz property.

REMARK 5. In [6], Bock studies formality and the hard Lefschetz property of solvmanifolds of dimension $\leq 6$ by direct computations. The cohomology of $G/\Gamma$ may vary for a choice of $\Gamma$ and Bock does not decide whether $G/\Gamma \times S^1$ is formal and satisfies the hard Lefschetz property.

By combining the above examples we obtain:

EXAMPLE 4. Let $G = \mathbb{R}^2 \ltimes_\phi \mathbb{R}^{2k+4(l+m+n)}$ such that

$$\begin{aligned}
\phi(t_1, t_2) = & \bigoplus_{i=1}^{k} \begin{pmatrix} \cos a_i t_1 & -\sin a_i t_1 \\ \sin a_i t_1 & \cos a_i t_1 \end{pmatrix} \\
& \oplus \bigoplus_{i=1}^{l} \begin{pmatrix} e^{b_i t_1} & 0 & 0 & 0 \\ 0 & e^{-b_i t_1} & 0 & 0 \\ 0 & 0 & e^{b_i t_1} & 0 \\ 0 & 0 & 0 & e^{-b_i t_1} \end{pmatrix} \\
& \oplus \bigoplus_{i=1}^{m} \begin{pmatrix} e^{c_i t_1}\cos(d_i t_2) & -e^{c_i t_1}\sin(d_i t_2) & 0 & 0 \\ e^{c_i t_1}\sin(d_i t_2) & e^{c_i t_1}\cos(d_i t_2) & 0 & 0 \\ 0 & 0 & e^{-c_i t_1}\cos(-d_i t_2) & -e^{-c_i t_1}\sin(-d_i t_2) \\ 0 & 0 & e^{-c_i t_1}\sin(-d_i t_2) & e^{-c_i t_1}\cos(-d_i t_2) \end{pmatrix} \\
& \oplus \bigoplus_{i=1}^{n} \begin{pmatrix} e^{e_i t_1}\cos(f_i t_1) & -e^{e_i t_1}\sin(f_i t_1) & 0 & 0 \\ e^{e_i t_1}\sin(f_i t_1) & e^{e_i t_1}\cos(f_i t_1) & 0 & 0 \\ 0 & 0 & e^{-e_i t_1}\cos(-f_i t_1) & -e^{-e_i t_1}\sin(-f_i t_1) \\ 0 & 0 & e^{-e_i t_1}\sin(-f_i t_1) & e^{-e_i t_1}\cos(-f_i t_1) \end{pmatrix}.
\end{aligned}$$

We write $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ for matrices $A$, $B$.

We suppose $a_i = 2\pi/K_i$ for $K_i = 2, 3, 4,$ or $6$, $b_i = rL_i$ for $r$ as Example 1 and $L_i \in \mathbb{Z}$, $c_i = pM_i$, $d_i = qM_i'$, $e_i = pN_i$ and $f_i = qN_i'$ for $p, q$ as Example 2 and $M_i, M_i', N_i, N_i' \in \mathbb{Z}$. Then each component of $\phi(\mathbb{Z}^2)$ for the direct product is conjugate to a subgroup of $SL_2(\mathbb{Z})$ or $SL_4(\mathbb{Z})$ and hence we have a lattice $\Gamma = \mathbb{Z}^2 \ltimes \Gamma''$ for a lattice $\Gamma''$ of $\mathbb{R}^{2k+4(l+m+n)}$. The cochain complex $\left(\bigwedge \mathfrak{g}^*, d\right)$ of the Lie algebra of $G$ is given by:

$$
\begin{aligned}
&\mathfrak{g}^* = \langle u_1, u_2, w_1, \ldots, w_{2k}, x_1, \ldots, x_{4l}, y_1, \ldots, y_{4m}, z_1, \ldots, z_{4n}\rangle,\\
&du_1 = du_2 = 0,\\
&dw_{2i-1} = a_i u_1 \wedge w_{2i}, \quad dw_{2i} = -a_i u_1 \wedge w_{2i-1} \quad (1 \le i \le k),\\
&dx_{2i-1} = -b_i u_1 \wedge x_{2i-1}, \quad dx_{2i} = b_i u_1 \wedge x_{2i} \quad (1 \le i \le 2l),\\
&dy_{4i-3} = -c_i u_1 \wedge y_{4i-3} + d_i u_2 \wedge y_{4i-2}, \quad dy_{4i-2} = -d_i u_2 \wedge y_{4i-3} - c_i u_1 \wedge y_{4i-2},\\
&dy_{4i-1} = c_i u_1 \wedge y_{4i-1} - d_i u_2 \wedge y_{4i}, \quad dy_{4i} = d_i u_2 \wedge y_{4i-1} + c_i u_1 \wedge y_{4i} \quad (1 \le i \le m),\\
&dz_{4i-3} = -e_i u_1 \wedge z_{4i-3} + f_i u_1 \wedge z_{4i-2}, \quad dz_{4i-2} = -f_i u_1 \wedge z_{4i-3} - e_i u_1 \wedge z_{4i-2},\\
&dz_{4i-1} = e_i u_1 \wedge z_{4i-1} - f_i u_1 \wedge z_{4i}, \quad dz_{4i} = f_i u_1 \wedge z_{4i-1} + e_i u_1 \wedge z_{4i} \quad (1 \le i \le n).
\end{aligned}
$$

$G$ has a left-invariant symplectic form

$$
\begin{aligned}
\omega = u_1 \wedge u_2 &+ \sum_{i=1}^{k} w_{2i-1} \wedge w_{2i} + \sum_{i=1}^{2l} x_{2i-1} \wedge x_{2i}\\
&+ \sum_{i=1}^{m}(y_{4i-3} \wedge y_{4i-1} + y_{4i} \wedge y_{4i-2}) + \sum_{i=1}^{n}(z_{4i-3} \wedge z_{4i-1} + z_{4i} \wedge z_{4i-2}).
\end{aligned}
$$

Regarding

$$
\begin{aligned}
&u_1 + \sqrt{-1}u_2,\\
&w_{2i-1} + \sqrt{-1}w_{2i} \quad (1 \le i \le k),\\
&x_{4i-3} + \sqrt{-1}x_{4i-1}, \quad x_{4i-2} + \sqrt{-1}x_{4i} \quad (1 \le i \le l),\\
&y_{2i-1} + \sqrt{-1}y_{2i} \quad (1 \le i \le 2m),\\
&z_{2i-1} + \sqrt{-1}z_{2i} \quad (1 \le i \le 2n)
\end{aligned}
$$

as $(1, 0)$-forms, we have a left-invariant complex structure on $G$. By the results of this paper, for any lattice $\Gamma$ of $G$, $G/\Gamma$ is formal and satisfies the hard Lefschetz property but admits no Kähler structure.

EXAMPLE 5 (Oeljeklaus–Toma manifolds). We apply the result of this paper to non-Kähler complex manifolds constructed by Oeljeklaus and Toma in [17]. Let $K$ be a finite extension field of $\mathbb{Q}$ with the degree $s + 2t$ for positive integers $s, t$. Suppose $K$ admits embeddings $\sigma_1, \ldots, \sigma_s, \sigma_{s+1}, \ldots, \sigma_{s+2t}$ into $\mathbb{C}$ such that $\sigma_1, \ldots, \sigma_s$ are real embeddings and $\sigma_{s+1}, \ldots, \sigma_{s+2t}$ are complex ones satisfying $\sigma_{s+i} = \bar{\sigma}_{s+i+t}$ for $1 \le i \le t$. We can choose $K$ admitting such embeddings (see [17]). Denote $\mathcal{O}_K$ the ring of algebraic integers of $K$, $\mathcal{O}_K^*$ the group of units in $\mathcal{O}_K$ and

$$\mathcal{O}_K^{*+} = \{a \in \mathcal{O}_K^* : \sigma_i > 0 \text{ for all } 1 \leq i \leq s\}.$$

Define $l\colon \mathcal{O}_K^{*+} \to \mathbb{R}^{s+t}$ by

$$l(a) = (\log|\sigma_1(a)|, \dots, \log|\sigma_s(a)|, 2\log|\sigma_{s+1}(a)|, \dots, 2\log|\sigma_{s+t}(a)|)$$

for $a \in \mathcal{O}_K^{*+}$. Then by Dirichlet's units theorem, $l(\mathcal{O}_K^{*+})$ is a lattice in the vector space $L = \left\{x \in \mathbb{R}^{s+t} \;\middle|\; \sum_{i=1}^{s+t} x_i = 0\right\}$. For the projection $p\colon L \to \mathbb{R}^s$ given by the first $s$ coordinate functions. Then we have a subgroup $U$ with the rank $s$ of $\mathcal{O}_K^{*+}$ such that $p(l(U))$ is a lattice in $\mathbb{R}^s$. We have the action of $U \ltimes \mathcal{O}_K$ on $H^s \times \mathbb{C}^t$ such that

$$\begin{aligned}
&(a, b) \cdot (x_1 + \sqrt{-1}y_1, \dots, x_s + \sqrt{-1}y_s, z_1, \dots, z_t) \\
&= (\sigma_1(a)x_1 + \sigma_1(b) + \sqrt{-1}\sigma_1(a)y_1, \dots, \sigma_s(a)x_s + \sigma_s(b) + \sqrt{-1}\sigma_s(a)y_s, \\
&\quad \sigma_{s+1}(a)z_1 + \sigma_{s+1}(b), \dots, \sigma_{s+t}(a)z_t + \sigma_{s+t}(b)).
\end{aligned}$$

In [17] it is proved that the quotient $X(K, U) = H^s \times \mathbb{C}^t / U \ltimes \mathcal{O}_K$ is compact. We call this complex manifold a Oeljeklaus–Toma (OT) manifold with $(s, t)$. By this construction we give solvmanifold-presentations $G/\Gamma$ of OT-manifolds with $(s, t)$. We consider $p(l(U)) \ltimes_\phi (\mathbb{R}^s \times \mathbb{C}^t)$ with

$$\begin{aligned}
&\phi(t_1, \dots, t_s) \\
&= \begin{pmatrix}
e^{t_1} & & & & & \\
& \ddots & & & & \\
& & e^{t_s} & & & \\
& & & \sigma_{s+1} \circ \sigma_1^{-1}(e^{t_1}) & & \\
& & & & \ddots & \\
& & & & & \sigma_{s+t} \circ \sigma_1^{-1}(e^{t_1})
\end{pmatrix}
\end{aligned}$$

for $(t_1, \dots, t_s) \in p(l(U))$. Then for some lattice $\Gamma''$ of $\mathbb{R}^s \times \mathbb{C}^t$, we have $p(l(U)) \ltimes_\phi \Gamma'' \cong U \ltimes \mathcal{O}_K$. Since $p(l(U))$ is a lattice of $\mathbb{R}^s$, we have an extension of $\phi$ on $\mathbb{R}^s$ and $U \ltimes \mathcal{O}_K$ can be seen as a lattice of $\mathbb{R}^s \ltimes_\phi (\mathbb{R}^s \times \mathbb{C}^t)$. Thus OT-manifolds are formal complex solvmanifolds not admitting Kähler structure.

REMARK 6. For $t = 1$, OT-manifolds $X(K, U)$ admit LCK (locally conformal Kähler) structures.

REMARK 7. We call $X(K, U)$ simple type if the action of $U$ on $\mathcal{O}$ admits no proper non-trivial submodule of lower rank. If $X(K, U)$ is simple type, then in [17] it is proved that the second Betti number is $b_2 = s(s-1)/2$. Then the second cohomology

$H^2(X(K,U),\mathbb{R})$ is spanned by $\{[dt_i \wedge dt_j]\}_{1\le i<j\le s}$ and hence simple type OT-manifolds admit no symplectic structure.

EXAMPLE 6. Infra-solvmanifolds appear in study of geometries of 3-manifolds. See [23] for the general theory of geometries of 3-manifolds. A compact aspherical 3-manifold $M$ with the virtually solvable fundamental group admits a one of the three geometries $E^3$, *Nil*, *Sol* i.e. $M$ is diffeomorphic to $G/\Gamma$ such that $G$ is $\mathbb{R}^3$, $\mathbf{U}_3(\mathbb{R})$ or $\mathbb{R}\ltimes_\phi \mathbb{R}^2$ as an Example 1 with a left-invariant metric and $\Gamma \subset C\ltimes G$ is a lattice for the group $C$ of isometric automorphisms of $G$. In the $E^3$ case, $\Gamma$ is virtually abelian by Bieberbach's first theorem. In the *Sol* case, $C$ is finite (see [23]). Hence a compact 3-manifold $M$ admitting the geometry $E^3$ or *Sol* is formal.

## 7. Remarks

In this section we give an example of a formal standard $\Gamma$-manifold with the hard Lefschetz property such that $U_\Gamma$ is not abelian. In addition this is also an example of formal manifold satisfying the hard Lefschetz property such that it is finitely covered by a non-formal manifold not satisfying the hard Lefschetz property. We notice that compact manifolds finitely covered by non-Kähler manifolds are not Kähler.

Let $\Gamma = \mathbb{Z}\ltimes_\phi \mathbb{Z}^2$ such that for $t\in\mathbb{Z}$

$$\phi(t) = \begin{pmatrix} (-1)^t & (-1)^t t \\ 0 & (-1)^t \end{pmatrix}.$$

**Lemma 7.1.** *The algebraic hull of $\Gamma$ is given by $\mathbf{H}_\Gamma = \{\pm 1\}\ltimes \mathbf{U}_3(\mathbb{C})$ such that*

$$(-1)\cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & (-1)z \\ 0 & 1 & (-1)y \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. We have the inclusion

$$\Gamma \cong \left( (-1)^x, \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) \subset \{\pm 1\}\ltimes \mathbf{U}_3(\mathbb{C}).$$

Then $\Gamma$ is Zariski-dense in $\{\pm 1\}\ltimes \mathbf{U}_3(\mathbb{C})$ and $\operatorname{rank}\Gamma = 3 = \dim \mathbf{U}_3(\mathbb{C})$. Since the action of $\{\pm 1\}$ on $\mathbf{U}_3(\mathbb{C})$ is faithful, the centralizer of $\mathbf{U}_3(\mathbb{C})$ is contained in $\mathbf{U}_3(\mathbb{C})$. Hence the lemma follows. □

We have $\mathbf{H}_\Gamma(\mathbb{R}) = \{\pm 1\}\ltimes U_\Gamma$ such that $U_\Gamma = \mathbf{U}_3(\mathbb{R})$. Let $\mathfrak{u}$ be the Lie algebra of $U_\Gamma$. We have $\mathfrak{u} = \langle X_1, X_2, X_3\rangle$ such that the bracket is given by

$$[X_1, X_2] = -[X_2, X_1] = X_3.$$

The $\{\pm 1\}$-action on $\mathfrak{u}$ is given by

$$(-1)\cdot X_1 = X_1, \quad (-1)\cdot X_i = -X_i, \quad i = 2, 3.$$

Let $x_1, x_2, x_3$ be the basis of $\mathfrak{u}^*$ which is dual to $X_1, X_2, X_3$. Then the DGA $\left(\bigwedge \mathfrak{u}^*\right)^{\{\pm 1\}}$ is the subalgebra of $\bigwedge \mathfrak{u}^*$ generated by $\{x_1, x_2 \wedge x_3\}$ and the derivation on $\left(\bigwedge \mathfrak{u}^*\right)^{\{\pm 1\}}$ is trivial. Let $M_\Gamma$ be the standard $\Gamma$-manifold. Then by Theorem 3.2, we have the quasi-isomorphism $\left(\bigwedge \mathfrak{u}^*\right)^{\{\pm 1\}} \to A^*(M_\Gamma)$. Since the derivation on $\left(\bigwedge \mathfrak{u}^*\right)^{\{\pm 1\}}$ is trivial, we have the isomorphism $\left(\bigwedge \mathfrak{u}^*\right)^{\{\pm 1\}} \cong H^*(M)$. Hence we have:

**Proposition 7.2.** *$M_\Gamma$ is formal.*

REMARK 8. Since $U_\Gamma$ is not abelian, the converse of Proposition 5.4 is not true.

REMARK 9. We have the finite index subgroup $2\mathbb{Z} \ltimes \mathbb{Z}^2$ which is nilpotent. So $\Gamma$ is virtually nilpotent but not virtually abelian. By the result of [11], $K(2\mathbb{Z} \ltimes \mathbb{Z}^2, 1)$ is not formal. But for the finite extension group $\Gamma$, $K(\Gamma, 1)$ is formal.

REMARK 10. Since $\{\pm 1\}$ acts isometrically on $U_\Gamma$ with the invariant metric, $M_\Gamma$ admits the *Nil* geometry. So we have a formal 3-dimensional compact manifold admitting the *Nil* geometry.

Let $\Delta = \Gamma \times \mathbb{Z}$. Then we have $H_\Delta = H_\Gamma \times \mathbb{R}$ and $U_\Delta = U_\Gamma \times \mathbb{R}$. As above we have the quasi-isomorphism inclusion $\left(\bigwedge \mathfrak{u}^*\right)^{\{\pm 1\}} \otimes \bigwedge(y) \subset A^*(M_\Delta)$. Let $\omega = x_1 \wedge y + x_2 \wedge x_3$. Then $\omega$ is a symplectic form on $M_\Delta$. Since $H^1(M_\Delta, \mathbb{R}) \cong \langle x_1, y\rangle$ and $H^3(M_\Delta, \mathbb{R}) \cong \langle x_1 \wedge x_2 \wedge x_3, x_2 \wedge x_3 \wedge y\rangle$, the linear map $[\omega]\wedge\colon H^1(M_\Delta, \mathbb{R}) \to H^3(M_\Delta, \mathbb{R})$ is an isomorphism and hence we have the following proposition.

**Proposition 7.3.** *$M_\Gamma \times S^1$ satisfies the hard Lefschetz property.*

REMARK 11. $\Delta$ is a finite extension group of the non-abelian nilpotent group $2\mathbb{Z} \ltimes \mathbb{Z}^2 \times \mathbb{Z}$ as Remark 9. By the result of [5], a compact $K(2\mathbb{Z} \ltimes \mathbb{Z}^2 \times \mathbb{Z}, 1)$-manifold is not a Lefschetz 4-manifold. Thus $M_\Delta$ is a example of a Lefschetz 4-manifold with non-Lefschetz finite covering space. In [16, Example 3.4], Lin showed the existence of Lefschetz 4-manifolds with non-Lefschetz finite covering space. $M_\Delta$ is a simpler and more constructive example.

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## References


[1] D. Arapura: *Kähler solvmanifolds*, Int. Math. Res. Not. (2004), 131–137.
[2] L. Auslander: *An exposition of the structure of solvmanifolds*, I, *Algebraic theory*, Bull. Amer. Math. Soc. **79** (1973), 227–261.
[3] O. Baues: *Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups*, Topology **43** (2004), 903–924.
[4] O. Baues and V. Cortés: *Aspherical Kähler manifolds with solvable fundamental group*, Geom. Dedicata **122** (2006), 215–229.
[5] C. Benson and C.S. Gordon: *Kähler and symplectic structures on nilmanifolds*, Topology **27** (1988), 513–518.
[6] C. Bock: *On Low-dimensional solvmanifolds*, preprint arXiv:0903.2926 (2009).
[7] A. Borel: Linear Algebraic Groups, second edition, Springer, New York, 1991.
[8] D. Burde, K. Dekimpe and K. Vercammen: *Complete LR-structures on solvable Lie algebras*, J. Group Theory **13** (2010), 703–719.
[9] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245–274.
[10] M. Fernández and A. Gray: *Compact symplectic solvmanifolds not admitting complex structures*, Geom. Dedicata **34** (1990), 295–299.
[11] K. Hasegawa: *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc. **106** (1989), 65–71.
[12] K. Hasegawa: *A note on compact solvmanifolds with Kähler structures*, Osaka J. Math. **43** (2006), 131–135.
[13] K. Hasegawa: *Small deformations and non-left-invariant complex structures on six-dimensional compact solvmanifolds*, Differential Geom. Appl. **28** (2010), 220–227.
[14] A. Hattori: *Spectral sequence in the de Rham cohomology of fibre bundles*, J. Fac. Sci. Univ. Tokyo Sect. I **8** (1960), 289–331.
[15] I. Nakamura: *Complex parallelisable manifolds and their small deformations*, J. Differential Geometry **10** (1975), 85–112.
[16] Y. Lin: *Examples of non-Kähler Hamiltonian circle manifolds with the strong Lefschetz property*, Adv. Math. **208** (2007), 699–709.
[17] K. Oeljeklaus and M. Toma: *Non-Kähler compact complex manifolds associated to number fields*, Ann. Inst. Fourier (Grenoble) **55** (2005), 161–171.
[18] A.L. Onishchik and E.B. Vinberg: Lie Groups and Lie Algebras, III, Springer, Berlin, 1994.
[19] A.L. Onishchik and E.B. Vinberg: Lie Groups and Lie Algebras, II, Springer, Berlin, 2000.
[20] A. Tralle and J. Oprea: Symplectic Manifolds with no Kähler Structure, Lecture Notes in Mathematics **1661**, Springer, Berlin, 1997.
[21] M.S. Raghunathan: Discrete Subgroups of Lie Groups, Springer, New York, 1972.
[22] B.E. Reed: *Representations of solvable Lie algebras*, Michigan Math. J. **16** (1969), 227–233.
[23] P. Scott: *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.



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