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THE HYPERBOLIC REGION FOR HYPERBOLIC BOUNDARY VALUE PROBLEMS

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Abstract

The well-posedness of hyperbolic initial boundary value problems is linked to the occurrence of zeros of the so-called Lopatinskiĭ determinant. For an important class of problems, the Lopatinskiĭ determinant vanishes in the hyperbolic region of the frequency domain and nowhere else. In this paper, we give a criterion that ensures that the hyperbolic region coincides with the projection of the forward cone. We give some examples of strictly hyperbolic operators that show that our criterion is sharp.

1. Introduction

In this paper, we consider initial boundary value problems for hyperbolic systems. Such problems read:

$$\begin{cases} Lu := \partial_t u + \displaystyle\sum_{j=1}^{d} A_j \, \partial_{x_j} u = F(t, x), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d_+, \\ Bu(t, y, 0) = g(t, y), & (t, y) \in \mathbb{R}^+ \times \mathbb{R}^{d-1}, \\ u(0, x) = f(x), & x \in \mathbb{R}^d_+, \end{cases} \tag{1}$$

where the spatial domain is the half-space $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d,\ x_d > 0\}$ and the notation $x = (y, x_d)$ is used. The A_j's are $N \times N$ real matrices, and B is a $p \times N$ real matrix. We always assume $d \geq 2$ in what follows.

The well-posedness of (1) can be characterized with the help of a complex valued function Δ, that is known as the Lopatinskiĭ determinant and that depends on the variables (z, η), $z \in \mathbb{C}$ with $\operatorname{Im} z \leq 0$ and $\eta \in \mathbb{R}^{d-1}$. We refer to the original articles [5, 9, 10] as well as to the book [2, Chapter 4] for a detailed description of the theory. The function Δ can be chosen to be positively homogeneous of degree 0 with respect to the variables (z, η). If Δ does not vanish on the closed half-sphere $\{\operatorname{Im} z \leq 0,\ |z|^2 + |\eta|^2 = 1\}$, then (1) is strongly well-posed, meaning that source terms in L^2 give rise to a unique solution u in L^2 that depends continuously on the data. When Δ vanishes in the open half-sphere $\{\operatorname{Im} z < 0,\ |z|^2 + |\eta|^2 = 1\}$, (1) is ill-posed.

In [1], an open class of weakly well-posed problems has been exhibited. This so-called WR class is made of problems for which Δ does not vanish in the open

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half-sphere but vanishes at first order in the so-called hyperbolic region of $\{\mathrm{Im}\, z = 0, |z|^2 + |\eta|^2 = 1\}$. Problems in the WR class arise naturally in shock wave theory in fluid dynamics, see e.g. [2, Chapter 15], and in other various physical contexts. Such problems give rise to amplification and weak stability phenomena for geometric optics expansions (see e.g. the review [7] and the references therein) and to an increase in the speed of propagation (see e.g. [4]).

In this paper, we aim at finding an *easy* way of determining whether a given problem of the form (1) belongs to the WR class. There are two main points in such an analysis. One should first locate the so-called hyperbolic region. Then, after computing the Lopatinskiĭ determinant Δ, one should look for zeros of Δ in the hyperbolic region. If one could easily locate the hyperbolic region[1], half of the job would already be done. We thus raise the problem of trying to locate the hyperbolic region as easily as possible. More precisely, we are interested in finding a criterion that allows to compute easily the hyperbolic region in terms of the so-called forward cone.

Our criterion involves the decomposition of the characteristic polynomial as a product of irreducible factors, see Theorem 2 below. When the irreducible factor associated with the extreme eigenvalues has degree 2, the hyperbolic region coincides with the projected forward cone. This criterion covers some well-known examples such as the linearized Euler equations, the wave equation and the elasticity system. We also give some examples of irreducible hyperbolic polynomials of degree 3 or 4 for which the hyperbolic region does not coincide with the projected forward cone. This shows that our criterion is sharp. The conclusion to be drawn from our results is that, in general, it is difficult to locate the hyperbolic region and consequently to determine whether a problem belongs to the WR class.

Notation. In all this article, $\boldsymbol{\xi}$ denotes a frequency vector in $\mathbb{R}^d$ that is decomposed as $\boldsymbol{\xi} = (\eta, \xi)$ with $\eta \in \mathbb{R}^{d-1}$, $\xi \in \mathbb{R}$. For instance, $(0, 1)$ denotes the last vector of the canonical basis of $\mathbb{R}^d$. The coordinates of $\boldsymbol{\xi} \in \mathbb{R}^d$ are denoted $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_d$. If $(\tau, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^d$, we define $\pi(\tau, \boldsymbol{\xi}) = (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1}$ the vector obtained by deleting the last coordinate of $\boldsymbol{\xi}$. In view of the notation used in the introduction, τ has to be understood as the real part of the complex number z. Since we are interested in the behavior of the Lopatinskiĭ determinant for real z, we shall only use the notation τ in what follows and z will no longer appear.

2. Main results

Let us consider the operator L in (1) and introduce the symbol

$$A(\boldsymbol{\xi}) := \sum_{j=1}^{d} \boldsymbol{\xi}_j A_j, \quad \boldsymbol{\xi} \in \mathbb{R}^d. \tag{2}$$

[1] This region only depends on the hyperbolic operator L and does not depend on the boundary conditions encoded in the matrix B.

The characteristic polynomial of L is

$$P(\tau, \boldsymbol{\xi}) := \det(\tau I + A(\boldsymbol{\xi})), \quad (\tau, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^d. \tag{3}$$

We make the following assumption of hyperbolicity with constant multiplicity.

ASSUMPTION 1. There exist some real valued analytic functions $\lambda_1, \dots, \lambda_q$ on $\mathbb{R}^d \setminus \{0\}$ and some integers $\alpha_1, \dots, \alpha_q$ such that the characteristic polynomial P defined by (3) satisfies

$$\forall (\tau, \boldsymbol{\xi}) \in \mathbb{R} \times (\mathbb{R}^d \setminus \{0\}), \quad P(\tau, \boldsymbol{\xi}) = \prod_{j=1}^{q} (\tau + \lambda_j(\boldsymbol{\xi}))^{\alpha_j}, \quad \lambda_1(\boldsymbol{\xi}) < \cdots < \lambda_q(\boldsymbol{\xi}).$$

Moreover, the $\lambda_j(\boldsymbol{\xi})$'s are semi-simple eigenvalues of the matrix $A(\boldsymbol{\xi})$ (their geometric and algebraic multiplicity are equal).

For simplicity, we also assume that the boundary $\{x_d = 0\}$ is non-characteristic, that is:

ASSUMPTION 2. The matrix A_d is invertible.

The two main objects used in this paper are the so-called forward cone and hyperbolic region, and are defined as follows, see e.g. [2, Chapters 1 and 4].

DEFINITION 1. • The characteristic variety of the operator L is $\operatorname{Char} L := \{(\tau, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^d / P(\tau, \boldsymbol{\xi}) = 0\}$. The forward cone Γ is the connected component of $(1, 0)$ in the complementary set of $\operatorname{Char} L$.
• The hyperbolic region $\mathscr{H}$ is the set of all $(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} \setminus \{(0, 0)\}$ such that the matrix

$$\mathscr{A}(\tau, \eta) := -A_d^{-1}(\tau I + A(\eta, 0)), \tag{4}$$

is diagonalizable with real eigenvalues.

The nonzero elements of $\pi\Gamma$ always belong to the hyperbolic region $\mathscr{H}$, see [2, Chapter 8]. Recall that $\pi(\tau, \boldsymbol{\xi}) = (\tau, \eta)$. Thanks to the fact that $\mathscr{H}$ is a symmetric cone:

$$(\tau, \eta) \in \mathscr{H}, \ s \in \mathbb{R} \setminus \{0\} \Longrightarrow (s\tau, s\eta) \in \mathscr{H},$$

we have

$$(\pi\Gamma \cup -\pi\Gamma) \setminus \{(0, 0)\} \subset \mathscr{H}. \tag{5}$$

In this paper, we are interested in characterizing the operators L for which the opposite inclusion in (5) holds, meaning that the hyperbolic region $\mathscr{H}$ coincides with the projected forward cone $\pi\Gamma$ and its symmetric set $-\pi\Gamma$. If these regions coincide, then it is easier to determine whether the initial boundary value problem (1) belongs to the WR class since we would already know the region where the zeros of the Lopatinskiĭ determinant should be sought.

Let us first begin with the special (easy) case when the spectrum of A_d is either positive or negative.

Proposition 1. *Let* Assumptions 1 *and* 2 *hold. Then we have either* $\lambda_1(0,1) > 0$ *or* $\lambda_q(0,1) < 0$ *if and only if* $\pi\Gamma = \mathbb{R} \times \mathbb{R}^{d-1}$. *In that case, we have* $\mathscr{H} = \mathbb{R} \times \mathbb{R}^{d-1} \setminus \{(0,0)\}$, *and* $\mathscr{H}$ *is connected.*

We now consider the general case where A_d has positive and negative eigenvalues. Our first main result is the following.

Theorem 1. *Let* Assumptions 1 *and* 2 *hold, and assume that the inequalities* $\lambda_1(0,1) < 0 < \lambda_q(0,1)$ *hold. Then* $\pi\Gamma$ *and* $-\pi\Gamma$ *are two disjoint connected components of* $\mathscr{H}$. *Consequently, we have* $\mathscr{H} = \pi\Gamma \cup -\pi\Gamma$ *if and only if* $\mathscr{H}$ *has exactly two connected components.*

Our purpose is now to give necessary or sufficient conditions on the operator L that ensure that $\mathscr{H}$ has two connected components. Our criterion below involves the decomposition of P as a product of irreducible factors. We therefore recall the following result.

Proposition 2 ([2]). *Let* Assumption 1 *hold. Then the characteristic polynomial* P *splits as*

$$P(\tau, \boldsymbol{\xi}) = \prod_{j=1}^{J} P_j(\tau, \boldsymbol{\xi})^{\beta_j}, \tag{6}$$

where the polynomials P_j *are normalized by* $P_j(1,0) = 1$ *and satisfy the following properties*:

- *each* P_j *is a homogeneous polynomial of* $(\tau, \boldsymbol{\xi})$,
- *the* P_j*'s are irreducible in* $\mathbb{R}[\tau, \boldsymbol{\xi}]$ *and pairwise distinct,*
- *for* $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{0\}$, *the roots of each* $P_j(\cdot, \boldsymbol{\xi})$ *are real and simple,*
- *for* $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{0\}$, *the roots of* $P_j(\cdot, \boldsymbol{\xi})$ *and* $P_k(\cdot, \boldsymbol{\xi})$ *are pairwise distinct if* $j \neq k$.

Up to reordering the P_j's, we can always assume that $-\lambda_1(\boldsymbol{\xi})$ is a root of $P_1(\cdot, \boldsymbol{\xi})$ for all $\boldsymbol{\xi}$. This convention is used from now on. Our criterion is the following.

Theorem 2. *Let* Assumptions 1 *and* 2 *hold, and assume that the inequalities* $\lambda_1(0, 1) < 0 < \lambda_q(0, 1)$ *hold. Assume furthermore that in the decomposition* (6), P_1 *has degree* 2. *Then* $\mathscr{H} = \pi\Gamma \cup -\pi\Gamma$.

As we shall see with some explicit examples below, the criterion of Theorem 2 is *optimal*. More precisely, there are examples of strictly hyperbolic operators L with P_1 of degree 3 or 4 such that the corresponding hyperbolic region $\mathscr{H}$ has more than two connected components. In [1, p. 1080], the authors expected[2] that for hyperbolic operators with constant multiplicity, $\mathscr{H}$ would coincide with $\pi\Gamma \cup -\pi\Gamma$. Our examples show that this is unfortunately not true. As a matter of fact, we believe that in space dimension 2, as soon as the degree of P_1 is greater than or equal to 3, it may happen that $\mathscr{H}$ has more than two connected components.

We shall also give an example where P_1 has degree 3 and where we still have $\mathscr{H} = \pi\Gamma \cup -\pi\Gamma$. Therefore the criterion of Theorem 2 is not a necessary and sufficient condition. However Theorem 2 predicts that the only general case where the hyperbolic region is easily computable corresponds to a polynomial P_1 of degree 2. This situation occurs for the linearized Euler equations, as well as for the wave equation or the elasticity system (these were the examples treated in [1]). When the degree of P_1 is greater than or equal to 3, the hyperbolic region can have many connected components, and it may become difficult to check the WR condition for the system (1) since the hyperbolic region must then be determined by computing the spectrum of $\mathscr{A}$, which may be difficult, especially when the size N of the system is large.

3. Proof of the main results

We recall that the forward cone Γ coincides with the set $\{(\tau, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^d / \tau + \lambda_1(\boldsymbol{\xi}) > 0\}$, see e.g. [2, Chapter 1]. In a similar way, there holds $-\Gamma = \{(\tau, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^d / \tau + \lambda_q(\boldsymbol{\xi}) < 0\}$. We also recall that Γ and $-\Gamma$ are open and convex.

3.1. Proof of Proposition 1.

Let us first assume that $\lambda_1(0, 1) > 0$, or in other words that A_d has only positive eigenvalues. Let $(\tau_0, \eta_0) \in \mathbb{R} \times \mathbb{R}^{d-1}$. We introduce the function

$$g : \xi \in \mathbb{R} \longmapsto \tau_0 + \lambda_1(\eta_0, \xi).$$

The homogeneity and continuity properties of λ_1 show that $g(\xi) \sim \xi\lambda_1(0, 1)$ as ξ tends to $+\infty$. Consequently there exists a real number ξ_0 such that $g(\xi_0) > 0$, and we get $(\tau_0, \eta_0) \in \pi\Gamma$. A similar proof with ξ tending to $-\infty$ works if we consider the case $\lambda_q(0, 1) < 0$. This argument gives the equality $\pi\Gamma = \mathbb{R} \times \mathbb{R}^{d-1}$ if we assume $\lambda_1(0, 1) > 0$ or $\lambda_q(0, 1) < 0$.

[2]And so did the author of this article until he found the examples given at the end of this article!

Let us now assume that we have $\pi\Gamma = \mathbb{R} \times \mathbb{R}^{d-1}$. Since $(0, 0) \in \pi\Gamma$, there exists $\xi_0 \in \mathbb{R} \setminus \{0\}$ such that $\lambda_1(0, \xi_0) > 0$. We have

$$\lambda_1(0, \xi) = \begin{cases} \xi\lambda_1(0, 1), & \text{if} \quad \xi \geq 0, \\ -|\xi|\lambda_q(0, 1), & \text{if} \quad \xi \leq 0. \end{cases}$$

This shows that we have either $\lambda_1(0, 1) > 0$ (when $\xi_0 > 0$) or $\lambda_q(0, 1) < 0$ (when $\xi_0 < 0$). The proof of Proposition 1 is complete.

In the case of Proposition 1, the hyperbolic region $\mathscr{H}$ is connected and $\pi\Gamma = -\pi\Gamma$, even though Γ and $-\Gamma$ are disjoint.

3.2. Proof of Theorem 1. We now assume that the inequalities $\lambda_1(0, 1) < 0 < \lambda_q(0, 1)$ hold. Then the proof of Proposition 1 above shows that $(0, 0) \notin \pi\Gamma \cup -\pi\Gamma$. More precisely, $\pi\Gamma$ does not contain any element of the form $(\tau, 0)$ with $\tau \leq 0$, and $-\pi\Gamma$ does not contain any element of the form $(\tau, 0)$ with $\tau \geq 0$.

Let us first begin with the following result.

Lemma 1. *The sets $\pi\Gamma$ and $-\pi\Gamma$ are open, convex, and their intersection is empty.*

Proof of Lemma 1. It is clear that π is linear and surjective, so π maps convex open sets onto convex open sets. Since Γ and $-\Gamma$ are convex open sets, we can already conclude that $\pi\Gamma$ and $-\pi\Gamma$ are convex open sets. It remains to show that their intersection is empty.

Let us assume that $(\underline{\tau}, \underline{\eta}) \in \pi\Gamma \cap -\pi\Gamma$. In particular, we necessarily have $\underline{\eta} \neq 0$ (otherwise $\underline{\tau} > 0$ and $\underline{\tau} < 0$). Moreover there exist some real numbers ξ_1, ξ_q such that

$$\underline{\tau} + \lambda_1(\underline{\eta}, \xi_1) > 0, \quad \underline{\tau} + \lambda_q(\underline{\eta}, \xi_q) < 0.$$

For all $j = 1, \ldots, q$, we introduce the real analytic function:

$$f_j : \xi \in \mathbb{R} \mapsto \underline{\tau} + \lambda_j(\underline{\eta}, \xi).$$

Following an argument used in the proof of Proposition 1, we know that $f_1(\xi)$ tends to $-\infty$ as ξ tends to $\pm\infty$. In the same way, $f_q(\xi)$ tends to $+\infty$ as ξ tends to $\pm\infty$. The intermediate value Theorem shows that both f_1 and f_q vanish at least twice: there exist $\underline{\xi}_1 < \overline{\xi}_1$ and $\underline{\xi}_q < \overline{\xi}_q$ such that

$$f_1\left(\underline{\xi}_1\right) = f_1(\overline{\xi}_1) = f_q\left(\underline{\xi}_q\right) = f_q(\overline{\xi}_q) = 0.$$

If $2 \leq j \leq q - 1$, we have

$$f_j(\xi_1) = \underline{\tau} + \lambda_j(\underline{\eta}, \xi_1) > \underline{\tau} + \lambda_1(\underline{\eta}, \xi_1) > 0, \quad f_j(\xi_q) < 0.$$

This shows that there exists a real number $\underline{\xi}_j$ such that $f_j(\underline{\xi}_j) = 0$. It is not difficult to check that the $q+2$ real numbers $\underline{\xi}_1, \dots, \underline{\xi}_q, \overline{\xi}_1, \overline{\xi}_q$ are pairwise distinct.

The polynomial $P(\underline{\tau}, \underline{\eta}, \cdot) = f_1^{\alpha_1} \cdots f_q^{\alpha_q}$ has degree N, and we have shown that $\underline{\xi}_1, \dots, \underline{\xi}_q$ are roots of multiplicity at least equal to $\alpha_1, \dots, \alpha_q$ and $\overline{\xi}_1, \overline{\xi}_q$ are roots of multiplicity at least equal to α_1, α_q. We thus obtain

$$N \geq 2\alpha_1 + \alpha_2 + \cdots + \alpha_{q-1} + 2\alpha_q = N + \alpha_1 + \alpha_q,$$

which is a contradiction. The proof of Lemma 1 is complete. □

Our goal now is to show that $\pi\Gamma$ is the connected component of $(1, 0)$ in $\mathscr{H}$, and similarly that $-\pi\Gamma$ is the connected component of $(-1, 0)$ in $\mathscr{H}$.

We first use the fact that $\mathscr{H}$ is open. This is indeed a consequence of the block structure condition proved in [8] for the matrix $\mathscr{A}$ defined in (4). We thus know that the connected components of $\mathscr{H}$ are also open. Let Ω denote the connected component of $(1, 0)$ in $\mathscr{H}$ and let us assume that $\pi\Gamma \neq \Omega$, or in other words that $\pi\Gamma$ is strictly included in Ω. We thus consider an element $(\tau_0, \eta_0) \in \Omega \setminus \pi\Gamma$. The set Ω is open and connected. It is therefore pathwise connected. Let us consider a continuous path $\{(\tau_s, \eta_s), s \in [0, 1]\}$ in Ω that joins (τ_0, η_0) and $(1, 0)$. Consider now $\underline{s} := \inf\{s \in [0, 1], (\tau_s, \eta_s) \in \pi\Gamma\}$. It is standard to show that $(\tau_{\underline{s}}, \eta_{\underline{s}})$ belongs to the boundary of $\pi\Gamma$.

Up to now, we have constructed an element $(\underline{\tau}, \underline{\eta}) \in \mathscr{H} \cap \partial(\pi\Gamma)$. In particular, we have $\underline{\eta} \neq 0$. We are going to show that $(\underline{\tau}, \underline{\eta})$ is a glancing mode for which the matrix $\mathscr{A}(\underline{\tau}, \underline{\eta})$ is not diagonalizable. We know that $(\underline{\tau}, \underline{\eta})$ is the limit of a sequence $((\tau_n, \eta_n))_{n\in\mathbb{N}}$ of elements of $\pi\Gamma$. For all integer n there exists a real number ξ_n such that $\tau_n + \lambda_1(\eta_n, \xi_n) > 0$. The sequence $((\tau_n, \eta_n))_{n\in\mathbb{N}}$ is bounded because it converges so the sequence $(\xi_n)_{n\in\mathbb{N}}$ is necessarily bounded. Up to extracting a subsequence, $(\xi_n)_{n\in\mathbb{N}}$ converges towards a real number $\underline{\xi}$, and we obtain

$$\underline{\tau} + \lambda_1(\underline{\eta}, \underline{\xi}) \geq 0.$$

The quantity $\underline{\tau} + \lambda_1(\underline{\eta}, \underline{\xi})$ cannot be positive because $(\underline{\tau}, \underline{\eta})$ belongs to the boundary of $\pi\Gamma$ and not to $\pi\Gamma$. We thus have $\underline{\tau} + \lambda_1(\underline{\eta}, \underline{\xi}) = 0$. Moreover, we have $\partial_\xi \lambda_1(\underline{\eta}, \underline{\xi}) = 0$ for otherwise we could find a real number ξ close to $\underline{\xi}$ such that $\underline{\tau} + \lambda_1(\underline{\eta}, \xi) > 0$. We have thus obtained

$$\underline{\tau} + \lambda_1(\underline{\eta}, \underline{\xi}) = \partial_\xi \lambda_1(\underline{\eta}, \underline{\xi}) = 0,$$

which means that $(\underline{\tau}, \underline{\eta})$ is a so-called glancing mode. For such frequencies, the matrix $\mathscr{A}$ is not diagonalizable, see [8]. For the sake of completeness, we briefly recall the proof of this claim. First of all, we have $\mathrm{Ker}(\mathscr{A}(\underline{\tau}, \underline{\eta}) - \underline{\xi} I) = \mathrm{Ker}(\underline{\tau} I + A(\underline{\eta}, \underline{\xi}))$ so the geometric multiplicity of the eigenvalue $\underline{\xi}$ equals α_1. Moreover, we can write

$$\underline{\tau} + \lambda_1(\underline{\eta}, \xi) = (\xi - \underline{\xi})^2 \vartheta_1(\xi),$$

where ϑ_1 is smooth in a neighborhood of $\underline{\xi}$. A simple calculation then gives

$$\det(\mathscr{A}(\underline{\tau}, \underline{\eta}) - \xi I) = (\xi - \underline{\xi})^{2\alpha_1} \vartheta_2(\xi),$$

where ϑ_2 is smooth in a neighborhood of $\underline{\xi}$. This shows that the algebraic multiplicity of the eigenvalue $\underline{\xi}$ equals at least $2\alpha_1$, so $\mathscr{A}(\underline{\tau}, \underline{\eta})$ is not diagonalizable. This is a contradiction because $(\underline{\tau}, \underline{\eta})$ has been assumed to belong to $\mathscr{H}$.

We have therefore proved that $\pi\Gamma$ is the connected component of $(1, 0)$ in $\mathscr{H}$, and in a similar way $-\pi\Gamma$ is the connected component of $(-1, 0)$ in $\mathscr{H}$. In particular, $\mathscr{H}$ has at least two connected components, and the proof of Theorem 1 follows.

3.3. Proof of Theorem 2. Let us first observe that P_1 can not have degree 1. Otherwise, λ_1 would be a linear function of $\boldsymbol{\xi}$. In particular, we would have $\lambda_1(-\boldsymbol{\xi}) = -\lambda_1(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$. But we also have $\lambda_1(-\boldsymbol{\xi}) = -\lambda_q(\boldsymbol{\xi})$ so q would equal 1, and this is incompatible with the assumption that A_d has two distinct eigenvalues.

Let us assume from now on that P_1 has degree 2. Because P_1 is homogeneous of degree 2, we have $P_1(\lambda_1(\boldsymbol{\xi}), -\boldsymbol{\xi}) = 0$ for all $\boldsymbol{\xi}$. Applying to $-\boldsymbol{\xi}$, we get $P_1(-\lambda_q(\boldsymbol{\xi}), \boldsymbol{\xi}) = 0$ for all $\boldsymbol{\xi}$, so we obtain

$$P_1(\tau, \boldsymbol{\xi}) = (\tau + \lambda_1(\boldsymbol{\xi}))(\tau + \lambda_q(\boldsymbol{\xi})).$$

Consider now $(\tau, \eta) \in \mathscr{H}$. In particular, the characteristic polynomial of the matrix $\mathscr{A}(\tau, \eta)$ has real roots. We compute

$$\det(\xi I - \mathscr{A}(\tau, \eta)) = (\det A_d^{-1}) P(\tau, \eta, \xi),$$

so $P(\tau, \eta, \cdot)$ has only real roots. Using the decomposition (6), $P_1(\tau, \eta, \cdot)$ has only real roots. Let $\xi_1 \in \mathbb{R}$ satisfy $P_1(\tau, \eta, \xi_1) = 0$. We have either $\tau + \lambda_1(\eta, \xi_1) = 0$ or $\tau + \lambda_q(\eta, \xi_1) = 0$. In the first case, we necessarily have $\partial_\xi \lambda_1(\eta, \xi_1) \neq 0$, otherwise (τ, η) would be a glancing mode and, as we have seen at the end of the proof of Theorem 1, $\mathscr{A}(\tau, \eta)$ would not be diagonalizable. Consequently, there exists ξ close to ξ_1 such that $\tau + \lambda_1(\eta, \xi) > 0$, and $(\tau, \eta) \in \pi\Gamma$. In the second case, similar arguments lead to $(\tau, \eta) \in -\pi\Gamma$. We have thus obtained $\mathscr{H} \subset \pi\Gamma \cup -\pi\Gamma$. Together with the inclusion (5), the proof of Theorem 2 is complete.

4. Some examples

The linearized isentropic Euler equations. We consider the isentropic Euler equations and linearize them around a state that corresponds to a positive density $\underline{\rho} > 0$, and a velocity $\mathbf{u} = \underline{u} e_d$, with e_d the last vector of the canonical basis of $\mathbb{R}^d$. We assume that the fluid is incoming and subsonic, that is

$$0 < \underline{u} < \underline{c},$$

with $\underline{c}$ the sound speed corresponding to the density $\underline{\rho}$. In space dimension d, the characteristic polynomial of the corresponding linear operator L splits as follows:

$$P(\tau, \eta, \xi) = ((\tau + \underline{u}\xi)^2 - \underline{c}^2(|\eta|^2 + \xi^2))(\tau + \underline{u}\xi)^{d-1}.$$

The matrix A_d has one negative eigenvalue and d positive eigenvalues (counted with their multiplicity). The hyperbolic region $\mathscr{H}$ is given by

$$\mathscr{H} = \{(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} / \tau^2 > (\underline{c}^2 - \underline{u}^2)|\eta|^2\},$$

and has two connected components, which is consistent with the result of Theorem 2.

The wave equation. Even though the analysis above is done for first-order systems, we feel free, as in [1], to apply the results in the case of higher order hyperbolic equations or systems. For instance, in the case of the wave equation, the characteristic polynomial is

$$P(\tau, \boldsymbol{\xi}) = \tau^2 - |\boldsymbol{\xi}|^2.$$

The hyperbolic region $\mathscr{H}$ is given by

$$\mathscr{H} = \{(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} / \tau^2 > |\eta|^2\},$$

and has two connected components, which is consistent with the result of Theorem 2 because P has degree 2.

The elasticity system. The linear elasticity system reads

$$\partial_{tt}^2 z - \operatorname{div}(\alpha(\nabla z + \nabla z^T) + (\beta - \alpha)(\operatorname{div} z)I) = 0, \quad z \in \mathbb{R}^d,$$

with $\alpha > 0$, $\beta > 0$ the Lamé coefficients. The characteristic polynomial splits as

$$P(\tau, \boldsymbol{\xi}) = (\tau^2 - c_s^2|\boldsymbol{\xi}|^2)^{d-1}(\tau^2 - c_p^2|\boldsymbol{\xi}|^2),$$

with $c_s^2 = \alpha$ and $c_p^2 = \alpha + \beta$. As expected from Theorem 2, the hyperbolic region has two connected components, and it is given (see [1, p. 1091]) by

$$\mathscr{H} = \{(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} / \tau^2 > c_p^2|\eta|^2\}.$$

We refer to [1] for examples of WR problems in the case of the wave equation or the elasticity system.

From now on, we only consider the case $d = 2$, so $\eta \in \mathbb{R}$. In that case, we recall the result of [3, 6] that any homogeneous hyperbolic polynomial may be represented under the form (3) with suitable real symmetric matrices A_1, A_2. (This is Lax' conjecture.) We therefore work directly with the characteristic polynomial P and forget about the matrices A_j's.

Examples with an irreducible polynomial of degree 3. The following example is taken from [11, p. 426]:

$$P(\tau, \eta, \xi) = \tau^3 - 3(\eta^2 + \xi^2)\tau + \xi^3. \tag{7}$$

Using Cardano's rule (Lemma 12.1 in [11]), it is straightforward to check that for $(\eta, \xi) \neq 0$, $P(\cdot, \eta, \xi)$ has three simple real roots. It is also straightforward to check that P is irreducible in $\mathbb{R}[\tau, \eta, \xi]$ so the only factor in (6) is P itself with multiplicity 1. The hyperbolic region corresponds to the $(\tau, \eta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that $P(\tau, \eta, \cdot)$ has three simple roots. Cardano's rule shows that $\mathscr{H} = \{(\tau, \eta) \in \mathbb{R}^2 / \tau^2 > 3\eta^2\}$, so $\mathscr{H}$ has two connected components. This is a case where the irreducible factor associated with extreme eigenvalues has degree 3 but $\mathscr{H}$ still has two connected component, so $\mathscr{H} = \pi\Gamma \cup -\pi\Gamma$. The example of the polynomial (7) shows that the condition in Theorem 2 is only sufficient and is not necessary. However, it does not seem possible to improve the criterion in Theorem 2 as shown with the two examples below.

Let us now introduce the polynomial

$$P(\tau, \eta, \xi) = \tau^3 - 3(\eta^2 + \xi^2)\tau + \eta^3 + \xi^3. \tag{8}$$

Again, it is straightforward to check that P is irreducible in $\mathbb{R}[\tau, \eta, \xi]$. Cardano's rule shows that for $(\eta, \xi) \neq 0$, $P(\cdot, \eta, \xi)$ has three simple real roots. The hyperbolic region $\mathscr{H}$ is the set of (τ, η) such that $P(\tau, \eta, \cdot)$ has three simple real roots. This is equivalent to asking that the polynomial $P(\tau, \eta, \cdot + \tau)$ has three simple real roots. We compute

$$P(\tau, \eta, \Xi + \tau) = \Xi^3 - 3\tau^2 \Xi + (\eta^3 - 3\tau\eta^2 - \tau^3).$$

Applying Cardano's rule, we get

$$\mathscr{H} = \{(\tau, \eta) \in \mathbb{R}^2 / 4\tau^6 > (\eta^3 - 3\tau\eta^2 - \tau^3)^2\}.$$

The region $\mathscr{H}$ is depicted in black in Fig. 1, where we can see that $\mathscr{H}$ has four connected components. We now give a quick argument that shows why $\mathscr{H}$ has four connected components[3]. We introduce the homogeneous polynomial:

$$Q(\tau, \eta) := 4\tau^6 - (\eta^3 - 3\tau\eta^2 - \tau^3)^2 = (\tau^3 - 3\eta^2\tau + \eta^3)(3\tau^3 + 3\eta^2\tau - \eta^3).$$

[3] I warmly thank the referee for his/her indications on this point.

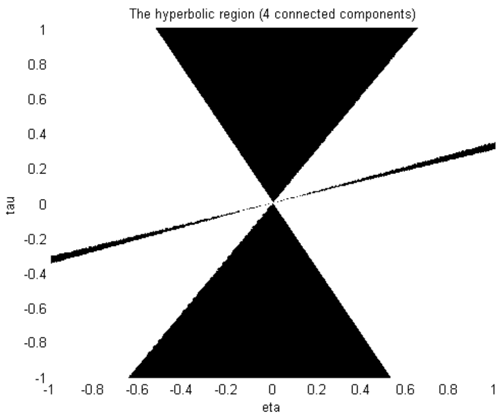

Fig. 1. The hyperbolic region (in black) with four connected components for the third degree polynomial P in (8).

The roots of the polynomial $X^3 - 3X + 1$ are computed by using Cardano's method, and we obtain

$$X^3 - 3X + 1 = (X - a_1)(X - a_2)(X - a_3), \quad a_k := 2\cos\left(\frac{2^k\pi}{9}\right), \quad k = 1, 2, 3.$$

Using Cardano's method again, we see that the polynomial $3X^3 + 3X - 1$ has only one real root b that is given by the following formula:

$$b := \frac{1}{3}\left(\frac{3(3+\sqrt{21})}{2}\right)^{1/3} - \left(\frac{2}{3(3+\sqrt{21})}\right)^{1/3}.$$

There holds

$$a_3 < 0 < b < a_2 < a_1.$$

We can thus factorize the polynomial Q and obtain

$$Q(\tau, \eta) = (\tau - a_3\eta)(\tau - b\eta)(\tau - a_2\eta)(\tau - a_1\eta)\tilde{Q}(\tau, \eta),$$

where $\tilde{Q}$ is a homogeneous polynomial of degree 2 that is positive when $(\tau, \eta) \neq (0, 0)$. Consequently, we have

$$\begin{aligned}\mathscr{H} &= \{(\tau, \eta) \in \mathbb{R}^2/(\tau - a_3\eta)(\tau - b\eta)(\tau - a_2\eta)(\tau - a_1\eta) > 0\} \\ &= \{(\tau, \eta) \in \mathbb{R}^2/(\tau - a_3\eta)(\tau - a_1\eta) > 0\} \\ &\quad \cup \{(\tau, \eta) \in \mathbb{R}^2/(\tau - b\eta)(\tau - a_2\eta) < 0\},\end{aligned}$$

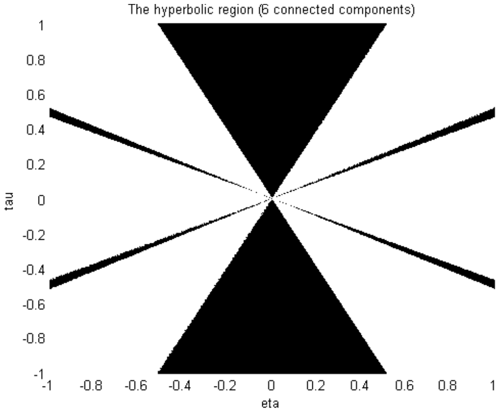

Fig. 2. The hyperbolic region (in black) with six connected components for the fourth degree polynomial P in (9).

and $\mathscr{H}$ has four connected components.

The example (8) shows that when the irreducible factor associated with extreme eigenvalues has degree 3, the equality $\mathscr{H} = \pi\Gamma \cup -\pi\Gamma$ may not hold anymore. We give another example of this fact below with an irreducible factor of degree 4 for which the computations are even easier.

An example with an irreducible polynomial of degree 4. The following result is elementary.

Lemma 2. *Let $a, b \in \mathbb{R}$. Then the polynomial $X^4 - 2aX^2 + b$ has four simple real roots if and only if $a > 0$, $b > 0$ and $a^2 > b$.*

Let us then define the polynomial

$$P(\tau, \eta, \xi) = \tau^4 - 4(\eta^2 + \xi^2)\tau^2 + \eta^4 + \xi^4. \tag{9}$$

Lemma 2 shows that for $(\eta, \xi) \neq 0$, $P(\,\cdot\,, \eta, \xi)$ has four simple real roots. We now consider P as a polynomial in ξ. We can apply Lemma 2 again: $P(\tau, \eta, \,\cdot\,)$ has four simple real roots if and only if the following inequalities hold:

$$\tau^2 > 0, \quad \tau^4 - 4\tau^2\eta^2 + \eta^4 > 0, \quad 3\tau^4 + 4\tau^2\eta^2 - \eta^4 > 0.$$

We thus get

$$\mathscr{H} = \left\{ (\tau, \eta) \in \mathbb{R}^2 \Big/ \frac{\sqrt{7} - 2}{3}\eta^2 < \tau^2 < (2 - \sqrt{3})\eta^2 \text{ or } (2 + \sqrt{3})\eta^2 < \tau^2 \right\}.$$

The hyperbolic region $\mathscr{H}$ is depicted in black in Fig. 2. It has six connected components, so $\mathscr{H} \neq \pi\Gamma \cup -\pi\Gamma$.

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