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WEAKLY POLYMATROIDAL IDEALS WITH APPLICATIONS TO VERTEX COVER IDEALS

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Abstract

In this paper we extend the concept of weakly polymatroidal ideals to monomial ideals which are not necessarily generated in one degree, and show that any ideal in this class has linear quotients. As an application we study some vertex cover ideals of weighted hypergraphs.

Introduction

Hibi and Kokubo in [21] introduced weakly polymatroidal ideals as a generalization of polymatroidal ideals. They considered ideals which are generated in the same degree. In this paper we extend their definition to ideals which are not necessarily generated in one degree. We show that these ideals have linear quotients, and consequently are componentwise linear.

Let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K and $I \subset R$ be a monomial ideal. Recall that I has *linear quotients*, if there exists a system of minimal generators $f_1, f_2, \dots, f_m$ of I such that the colon ideal $(f_1, \dots, f_{i-1})\colon f_i$ is generated by a subset of $\{x_1, \dots, x_n\}$ for all i. Ideals with linear quotients were introduced by Herzog and Takayama in [19].

For a homogeneous ideal $I \subset R$ and $d \geq 1$, we denote by (I_d) the ideal generated by all forms of degree d in I. The ideal I is called *componentwise linear* if for each d, (I_d) has a linear resolution. Componentwise linear ideals were introduced by Herzog and Hibi in [10] and they proved that the Stanley–Reisner ideal I_Δ is componentwise linear if and only if $I_{\Delta^\vee}$ is sequentially Cohen–Macaulay (see [10], [18]).

In general it is hard to prove that an ideal is componentwise linear. A criteria for an ideal being componentwise linear are given in [5]. On the other hand, if an ideal has linear quotients, then it is componentwise linear, as shown in [22]. If, in addition, I is a monomial ideal, then I even has componentwise linear quotients, see [23].

Due to these facts one would expect that a weakly polymatroidal ideal is componentwise weakly polymatroidal. However, Example 1.8 shows that this is in general not the case. Nevertheless, if all generators of the weakly polymatroidal ideal are of same degree, then all components of the ideal are weakly polymatroidal. This is a consequence of The-

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orem 1.6 where we show more generally that if I is a weakly polymatroidal ideal and $\mathfrak{m}$ is the graded maximal ideal of polynomial ring, then $\mathfrak{m}I$ is again weakly polymatroidal.

In Section 2 we study classes of ideals which are weakly polymatroidal. The ideals we study are vertex cover ideals of weighted hypergraphs, introduced in [16]. Let V be a finite set and E be a finite collection of nonempty subsets of V. Then $\mathcal{H} = (V, E)$ is called a *hypergraph* and the elements of E are called the edges of $\mathcal{H}$. A *vertex cover* of $\mathcal{H}$ is a subset of V which meets every edge of $\mathcal{H}$. Any vertex cover of V can be considered as a (0, 1) vector $c = (c_1, \ldots, c_n)$, where $\sum_{A \in E(\mathcal{H})} c_i \geq 1$ for all $A \in E(\mathcal{H})$. Given a hypergraph $\mathcal{H}$ and an integer valued function $\omega : E(\mathcal{H}) \to \mathbb{N}$, $A \to \omega_A$, the pair $(\mathcal{H}, \omega)$ is called a *weighted hypergraph*. For $k \in \mathbb{N}$, a *k-cover* is defined as a vector $c \in \mathbb{N}^n$ that satisfies the condition $\sum_{i \in A} c_i \geq k\omega_A$ for any $A \in E(\mathcal{H})$. For a nonempty subset $A = \{a_1, \ldots, a_r\}$ of V let $P_A = (x_{a_1}, \ldots, x_{a_r})$. For a weighted hypergraph $(\mathcal{H}, \omega)$, the ideal $A_k(\mathcal{H}, \omega) = \bigcap_{A \in E(\mathcal{H})} P_A^{k\omega_A}$ is called the *ideal of k-covers* of $(\mathcal{H}, \omega)$. The graded vertex cover algebra $A(\mathcal{H}, \omega)$ is defined as $\bigoplus_{k \geq 0} A_k(\mathcal{H}, \omega)$. If $\omega_A = 1$ for any $A \in E(\mathcal{H})$, then $A(\mathcal{H}, \omega)$ is denoted by $A(\mathcal{H})$.

The fundamental question we want to address in this paper is the following: for which weighted hypergraph $(\mathcal{H}, \omega)$ are all its ideals of k-covers componentwise linear? We call a hypergraph with this property *uniformly linear*. To classify all uniformly linear hypergraphs is quite hopeless. It is known by a result of Francisco and Van Tuyl that the ideal 1-covers of any chordal graph is componentwise linear, see [8]. However it seems to be unknown whether chordal graphs are uniformly linear.

In this paper we consider the following classes of uniformly linear weighted hypergraphs. Actually, we show in all these cases that for any k the ideal of k-covers is either weakly polymatroidal or componentwise weakly polymatroidal.

(1) G is a Cohen–Macaulay bipartite graph (see Theorem 2.2).

(2) Let $\mathcal{H}$ be a hypergraph on the vertex set $[n]$ satisfying one of the following conditions:

(a) $\mathcal{H}$ has only two facets (see Theorem 2.3).

(b) $E(\mathcal{H}) = \{K, J_1, \ldots, J_s\}$ and there exists an integer t such that $\bigcap_{i=1}^{s} J_i = J_i \cap J_j = [t]$ for all $i \neq j$ and $K \cap J_i = \emptyset$ for $i = 1, \ldots, s$ (see Theorem 2.4).

(c) $E(\mathcal{H}) = \{K, J_1, \ldots, J_s\}$ with $J_i \cup J_j = [n]$ for all $i \neq j$ (see Theorem 2.5).

Herzog and Hibi in [9] showed that the ideal of k-covers of a Cohen–Macaulay bipartite graph has linear quotients. The result in (1) is inspired by their work. Francisco and Van Tuyl in [7] among other results proved that the ideals of k-covers of hypergraphs in (a), and (c) in the case that $K = \emptyset$ are componentwise linear.

1. Weakly polymatroidal ideals

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over the field K. For any monomial ideal I, let $G(I)$ be the minimal set of generators of I and $m(u)$ be the greatest integer i for which x_i divides u. For $u = x_1^{a_1} \cdots x_n^{a_n}$, we denote a_i by $\deg_{x_i}(u)$. For a and b in N^n we have $a >_{\text{lex}} b$ if and only if the left-most nonzero entry in $a - b$ is positive.

DEFINITION 1.1. A monomial ideal I is called weakly polymatroidal if for every two monomials $u = x_1^{a_1} \cdots x_n^{a_n} >_{\text{lex}} v = x_1^{b_1} \cdots x_n^{b_n}$ in $G(I)$ such that $a_1 = b_1, \ldots, a_{t-1} = b_{t-1}$ and $a_t > b_t$, there exists $j > t$ such that $x_t(v/x_j) \in I$.

The monomial ideal I is called *stable* if, for any monomial u in I and $i < m(u)$ one has $x_i(u/x_{m(u)}) \in I$. From the above definition one can see that each stable ideal is weakly polymatroidal. In [1] *weakly stable* ideals are defined as a generalization of stable ideals and it was shown that every weakly stable ideal generated in degree d has a linear resolution. Also, in that paper an explicit formula for the Betti numbers of such ideals were given. A squarefree monomial ideal I is called weakly stable if for every squarefree monomials $u \in I$ and $u' = u/m(u)$ the following condition $(*)$ holds:

$(*)$ For every integer $l \notin \text{Supp}(u)$ such that $l < m(u')$, there exists an integer $i \in \text{Supp}(u)$ with $i > l$ such that $x_l(u/x_i) \in G(I)$.

From the definition of weakly stable ideals, it is easy to see that each weakly stable ideal which is generated in one degree is weakly polymatroidal. The following example shows that the converse of the above statement is not true.

EXAMPLE 1.2. The ideal $I = (x_1x_3x_5, x_1x_3x_6, x_1x_4x_6, x_2x_4x_6)$ is weakly polymatroidal, but there exists no permutation of variables such that I is weakly stable.

First we prove the following theorem for weakly polymatroidal ideals.

Theorem 1.3. *Any weakly polymatroidal ideal I has linear quotients.*

Proof. Let $G(I) = \{u_1, \ldots, u_m\}$, where $u_1 > u_2 > \cdots > u_m$ in the lexicographical order with respect to $x_1 > x_2 > \cdots > x_n$. We show that I has linear quotients with respect to $u_1, \ldots, u_m$. Let u_i and u_j be in $G(I)$ and $u_i >_{\text{lex}} u_j$. We can assume that $u_i = x_1^{a_1} \cdots x_n^{a_n}$ and $u_j = x_1^{b_1} \cdots x_n^{b_n}$ and for some t, $1 \le t \le n$ we have $a_1 = b_1, \ldots, a_{t-1} = b_{t-1}$ and $a_t > b_t$. Therefore there exists $l > t$ such that $x_t(u_j/x_l) \in I$. Thus the set $A = \{u_k \colon x_t(u_j/x_l) \in (u_k)\}$ is nonempty. Let $u_s \in A$ be the unique element such that for any $u_k \in A$ $(k \ne s)$, we have either $\deg(u_k) > \deg(u_s)$ or $\deg(u_k) = \deg(u_s)$ and $u_k <_{\text{lex}} u_s$. Assume that $x_t(u_j/x_l) = u_s h$ for some $h \in R$. If $x_t \mid h$, then $u_j = u_s h'$ for some $h' \in R$, which is a contradiction by the assumption that $u_j \in G(I)$. So we have $x_t^{b_t+1} \mid u_s$.

We claim that $u_s >_{\text{lex}} u_j$. By contradiction assume that $u_s <_{\text{lex}} u_j$. Let $u_s = x_1^{c_1} \cdots x_n^{c_n}$, $c_1 = b_1, \ldots, c_{r-1} = b_{r-1}$ and $c_r < b_r$ for some $1 \le r \le n$. Since $x_t^{b_t+1} \mid u_s$, one has $r < t$. Then from the definition of weakly polymatroidal, one has $w = u_s x_r / x_k \in I$ for some $k > r$. Since $r < l$, $x_r \mid h$ and so $x_k h / x_r \in R$. From $w(x_k h/x_r) = x_t(u_j/x_l)$, $w >_{\text{lex}} u_s$ and $\deg(w) = \deg(u_s)$, we have $w \notin G(I)$. Let $w = u_{s'} h'$ for some $1 \le s' \le m$

and $1 \neq h' \in R$. Then $\deg(u_{s'}) < \deg(w) = \deg(u_s)$ which is a contradiction, since $u_{s'} \in A$. Therefore one has $u_s h \in (u_1, \ldots, u_{j-1})$. Since $(u_s h : u_j) = x_t$, the proof is complete. □

From [23, Theorem 2.7] we have the following

Corollary 1.4. *Any weakly polymatroidal ideal has componentwise linear quotients.*

REMARK 1.5. Let $I = \langle u_1, u_2, \ldots, u_r \rangle$ be a weakly polymatroidal ideal with respect to the ordering $u_1 >_{\text{lex}} u_2 >_{\text{lex}}> \cdots >_{\text{lex}} u_r$ on $G(I)$. With the notation in the proof of above theorem for $u_i >_{\text{lex}} u_j$ there exists $l > t$, $s < j$ and $h \in R$ such that $x_t(u_j/x_l) = u_s h$. Therefore, for each $k < r$, the ideal $I' = \langle u_1, \ldots, u_k \rangle$ is again weakly polymatroidal.

It is known that the product of any two polymatroidal ideals is again polymatroidal, see [4, Theorem 5.3]. This is not the case for weakly polymatroidal ideals. The ideal $I = (x_1^2x_2, x_1^2x_3, x_1x_3^2, x_2x_3^2, x_1x_3x_4)$ is weakly polymatroidal, but I^2 does not even have linear resolution. However, we have

Theorem 1.6. *Let I be a weakly polymatroidal ideal and $\mathfrak{m}$ be the maximal ideal of R. Then $\mathfrak{m}I$ is again weakly polymatroidal.*

Proof. Let $G(I) = \{u_1, \ldots, u_m\}$, where $u_1 > u_2 > \cdots > u_m$ in the lexicographical order with respect to $x_1 > x_2 > \cdots > x_n$. Let w_1 and w_2 be elements of $G(\mathfrak{m}I)$ and $w_1 >_{\text{lex}} w_2$. Consider the sets $A = \{u \in G(I)\colon w_1 = x_i u \text{ for some } 1 \leq i \leq n\}$ and $B = \{v \in G(I)\colon w_2 = x_j v \text{ for some } 1 \leq j \leq n\}$. Let $u \in A$ and $v \in B$ be the greatest elements in A and B with respect to lex order, respectively. Assume that $w_1 = x_i u = x_1^{a_1} \cdots x_n^{a_n}$ and $w_2 = x_j v = x_1^{b_1} \cdots x_n^{b_n}$, $a_i = b_i$ for $i < t$ and $a_t > b_t$. One has $u = x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_j^{b_j - 1} \cdots x_n^{b_n}$. First we consider the case $t < i$. We have $u >_{\text{lex}} v$. If $t > j$, then j is the smallest index with $\deg_{x_j}(u) > \deg_{x_j}(v)$. Then $x_j(v/x_l) \in I$ for some $l > j$ and we have $x_j v = x_l w$ for some $v <_{\text{lex}} w \in I$. Since $x_l w$ is in $G(\mathfrak{m}I)$, one has $w \in G(I)$ which is a contradiction by the way of choosing v. So $t \leq j$ and t is the smallest index with $\deg_{x_t}(u) > \deg_{x_t}(v)$. Therefore $x_t(v/x_l) \in I$ for some $l > t$ and so $x_t(x_j v/x_l) = x_j(x_t v/x_l)$ is in $\mathfrak{m}I$.

Let $t \geq i$. If $j = i$, then the result is clear. If $j < i$, then $u >_{\text{lex}} v$, so j is the first index such that $a_j = \deg_{x_j}(u) > \deg_{x_j}(v) = b_j - 1$. Therefore $x_j(v/x_l)$ is in I for some $l > j$ and we have $x_j v = x_l w$ for some $w \in G(I)$, $w >_{\text{lex}} v$, which is a contradiction.

Therefore, one can assume that $j > i$. If $i < t$, then $v >_{\text{lex}} u$, since $b_i = \deg_{x_i}(v) > \deg_{x_i}(u) = a_i - 1$. Then there exists $l > i$ such that $x_i(u/x_l) \in I$. So $ux_i = wx_l$ for some $w \in G(I)$, where $w >_{\text{lex}} u$ which is a contradiction. Let $t = i$. Then $x_t v = x_t(x_j v/x_j) \in I$ and $j > t$, which completes the proof. □

As an immediate consequence of the above theorem we have

Corollary 1.7. *Let I be a weakly polymatroidal ideal generated by monomials in one degree. Then I is componentwise weakly polymatroidal.*

Proof. Assume that the minimal generators of I are of degree d. Then for any $j \geq 0$ we have $(I_{d+j}) = \mathfrak{m}^j I$, where $\mathfrak{m}$ is the maximal ideal of R. Therefore by the above theorem (I_{d+j}) is weakly polymatroidal. □

A weakly polymatroidal ideal for which the minimal generators are not of the same degree, is not necessarily componentwise weakly polymatroidal. The following example shows this fact.

EXAMPLE 1.8. The ideal $I = (x_1x_3, x_2x_3, x_1x_4x_5, x_2x_4x_5) = (x_1, x_2) \cap (x_3, x_4) \cap (x_3, x_5)$ is weakly polymatroidal. But there exists no permutation of variables such that (I_3) is weakly polymatroidal.

2. Some applications

As the first application we consider the ideal of k-covers of a Cohen–Macaulay bipartite graph G. Let P be a finite poset associated to G, see [11, Theorem 3.4], and let $J(P)$ be the distributive lattice consisting of all poset ideals of P, ordered by inclusion. Recall that a subset $I \subset P$ is a poset ideal of P if for all $x \in I$ and $y \in P$ such that $y < x$, one has $y \in I$. Let $P = \{p_1, \dots, p_n\}$ and $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring in $2n$ variables over a field K. To each poset ideal I of P we associate the monomial $u_I = \prod_{p_i \in I} x_i \prod_{p_i \in P \setminus I} y_i$. The squarefree monomial ideal of S generated by all monomials u_I with $I \in J(P)$ is denoted by H_P. The ideal H_P is in fact the ideal of 1-covers of G. Herzog and Hibi in [11, Theorem 3.4] proved that a bipartite graph G with vertex partition $X \cup Y$ is Cohen–Macaulay if and only if there exists a labeling on the vertices $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ such that:
(i) x_iy_i are edges for $i = 1, \dots, n$;
(ii) if x_iy_j is an edge, then $i \leq j$;
(iii) if x_iy_j and x_jy_k are edges, then x_iy_k is an edge.
In [9] it was shown that the powers of H_P have linear quotients. In the following we show that the powers of H_P are even weakly polymatroidal.

The following lemma was proved in [20, p. 99]. For the convenience of the reader we give a proof of it.

Lemma 2.1. *Let G be a Cohen–Macaulay bipartite graph as above. Then each element of the set of minimal generators of the ideal of k-covers of G, $\bigcap_{A \in E(G)} P_A^k$, can be written as $u_{B_1} \cdots u_{B_k}$, where $B_i \in J(P)$ and $B_k \subseteq \cdots \subseteq B_1$.*

Proof. By [16, Theorem 5.1] we know that the vertex cover algebra of G is standard graded, therefore it follows that $\bigcap_{A\in E(G)} P_A^k = \left(\bigcap_{A\in E(G)} P_A\right)^k = (H_P)^k$. Let $u_{A_1}, \dots, u_{A_k} \in H_P$. From [14, Theorem 2.2] we have $u_{A_i\cup A_j}$ and $u_{A_i\cap A_j}$ are elements of H_P for any i and j. Then we can write

$$u_{A_i}u_{A_j} = u_{A_i\cup A_j}u_{A_i\cap A_j}.$$

By induction we show that $u_{A_1}\cdots u_{A_k}$ can be written as $u_{A'_1}\cdots u_{A'_k}$ such that $A'_i \subseteq A'_1$ for all $2 \le i \le k$. Let $u_{A_1}\cdots u_{A_{k-1}} = u_{A'_1}\cdots u_{A'_{k-1}}$ such that $A'_i \subseteq A'_1$ for all $2 \le i \le k-1$. Then

$$\begin{aligned} u_{A_1}\cdots u_{A_k} &= \left(u_{A'_1}u_{A_k}\right)\left(u_{A'_2}\cdots u_{A'_{k-1}}\right) \\ &= \left(u_{A'_1\cup A_k}u_{A'_1\cap A_k}\right)\left(u_{A'_2}\cdots u_{A'_{k-1}}\right). \end{aligned}$$

We have $A'_1 \cap A_k \subseteq A'_1 \cup A_k$ and $A'_i \subseteq A'_1 \cup A_k$ for all $2 \le i \le k-1$ and the assertion holds. Now we show that one can write $u_{A_1}\cdots u_{A_k} = u_{B_1}\cdots u_{B_k}$ such that $B_k \subseteq \cdots \subseteq B_1$. By the above statements we can assume that $u_{A_1}\cdots u_{A_k} = u_{A'_1}\cdots u_{A'_k}$ such that $A'_i \subseteq A'_1$ for all $2 \le i \le k$. By induction assume that $u_{A'_2}\cdots u_{A'_k} = u_{B_2}\cdots u_{B_k}$ such that $B_k \subseteq \cdots \subseteq B_2$. Since $\bigcup_{i=2}^k B_i = \bigcup_{i=2}^k A'_i \subseteq A'_1$, setting $B_1 = A'_1$ we have $B_k \subseteq \cdots \subseteq B_1$ and $u_{A_1}\cdots u_{A_k} = u_{B_1}\cdots u_{B_k}$. □

Theorem 2.2. *Let G be a Cohen–Macaulay bipartite graph as above. Then the ideal of k-covers of G, $\bigcap_{A\in E(G)} P_A^k$, is weakly polymatroidal.*

Proof. Consider the ordering on the variables corresponding to the vertices of G such that $x_1 > x_2 > \cdots > x_n > y_1 > \cdots > y_n$. Let $u_{A_1}\cdots u_{A_k} >_{\text{lex}} v_{B_1}\cdots v_{B_k}$ be two elements in the minimal generating set of $\bigcap_{A\in E(G)} P_A^k$ such that $A_k \subseteq \cdots \subseteq A_1$ and $B_k \subseteq \cdots \subseteq B_1$ (see Lemma 2.1). Then there exists $t \ge 1$, the smallest integer such that $i = \deg_{x_t}(u_{A_1}\cdots u_{A_k}) > \deg_{x_t}(v_{B_1}\cdots v_{B_k})$.

It is easy to see that for any element $u_{A'_1}\cdots u_{A'_k} \in (H_P)^k$ with $A'_k \subseteq \cdots \subseteq A'_1$ we have $A'_i = \{l\colon \deg_{x_l}(u_{A'_1}\cdots u_{A'_k}) \ge i\}$. Thus $t \in A_i$ and $t \notin B_i$ and so $y_t \mid u_{B_i}$. For any $j < t$ such that $j \in A_i$, we have $j \in B_i$. Otherwise $\deg_{x_j}(v_{B_1}\cdots v_{B_k}) < i$ and since $\deg_{x_j}(u_{A_1}\cdots u_{A_k}) \ge i$ we have $\deg_{x_j}(u_{A_1}\cdots u_{A_k}) > \deg_{x_j}(v_{B_1}\cdots v_{B_k})$ which is a contradiction by assumption that t is the smallest integer with this property.

Let $L = \{l < t\colon l \in A_i\}$. Then as was shown $L \subseteq B_i$. For any l such that $x_l y_t \in E(G)$, the labeling on the vertices of G implies that $l < t$. Moreover $l \in A_i$, since $\mathrm{Supp}(u_{A_i})$ is a minimal vertex cover of G which does not contain y_t. Therefore $\{1 \le l \le n\colon x_l y_t \in E(G)\} \subseteq L$. We have u_{A_i} and u_{B_i} are in H_P, i.e. $\mathrm{Supp}(u_{A_i})$ and $\mathrm{Supp}(u_{B_i})$ are minimal vertex covers of G. Then from $\{1 \le l \le n\colon x_l y_t \in E(G)\} \subseteq B_i$, we have $\{x_l\colon l \in B_i\} \cup \{y_l\colon l \in B_i^c, l \ne t\} \cup \{x_t\}$ is again a minimal vertex cover of G, equivalently $w = u_{B_i\cup\{t\}}$ is in H_P. Then $(v_{B_1}\cdots v_{B_k})x_t/y_t = v_{B_1}\cdots v_{B_{i-1}}v_{B_i\cup\{t\}}v_{B_{i+1}}\cdots v_{B_k}$ is in $\bigcap_{A\in E(G)} P_A^k$. Moreover, we have $(v_{B_1}\cdots v_{B_k})x_t/y_t >_{\text{lex}} v_{B_1}\cdots v_{B_k}$, which completes the proof. □

The result of the above theorem seems to be true for any weighted Cohen–Macaulay bipartite graph as we checked in many examples.

Next we consider some classes of weighted hypergraphs for which all ideals of k-covers are either weakly polymatroidal or componentwise weakly polymatroidal. In [7, Corollary 3.2] it is shown that any ideal $I = P_J^a \cap P_K^b$ is componentwise linear and [7, Remark 3.3] shows that they are not necessarily polymatroidal. Here we show that these ideals are weakly polymatroidal.

Theorem 2.3. *Let J and K be subsets of $[n]$. Let $I = P_J^a \cap P_K^b \subset K[x_1, \ldots, x_n]$. Then I is weakly polymatroidal.*

Proof. Let $N_1 = J \cap K$, $N_2 = J \setminus (J \cap K)$ and $N_3 = K \setminus (J \cap K)$. Let $x_{l,1}, \ldots, x_{l,n_l}$ be the variables correspond to the integers in N_l for $l = 1, 2, 3$. Consider the ordering on the variables of R such that $x_{1,1} > \cdots > x_{1,n_1} > \cdots > x_{3,1} > \cdots > x_{3,n_3}$. Let u and v be two monomials in $G(I)$ such that $u >_{\text{lex}} v$. Assume that $u = u_1u_2u_3$ and $v = v_1v_2v_3$, where $u_l = x_{l,1}^{e_{l,1}} \cdots x_{l,n_l}^{e_{l,n_l}}$ and $v_l = x_{l,1}^{e'_{l,1}} \cdots x_{l,n_l}^{e'_{l,n_l}}$, for $l = 1, 2, 3$. First let $x_{1,i}$ be the first index such that $e_{1,i} > e'_{1,i}$. There exists $x_{l,j}$ in $\operatorname{Supp}(v)$ with $x_{1,i} > x_{l,j}$, since $v \nmid u$. The element $h = x_{1,i}(v/x_{l,j})$ is in I. Let $x_{l,i}$ be the first index such that $e_{l,i} > e'_{l,i}$ for $l = 2$ or 3. Since $u_1 = v_1$, we have $\deg(u_2) = a - \deg(u_1) = a - \deg(v_1) = \deg(v_2)$. Then there exists $x_{l,j}$ in $\operatorname{Supp}(v)$ with $j > i$. The element $h = x_{l,i}(v/x_{l,j})$ has desired properties. □

Theorem 2.4. *Let $K, J_1, \ldots, J_s$ be subsets of $[n]$ such that $\bigcap_{i=1}^s J_i = J_i \cap J_j = [t]$ for all $i \neq j$ and $K \cap J_i = \emptyset$. Let $I = P_K^{a_0} \cap P_{J_1}^{a_1} \cap \cdots \cap P_{J_s}^{a_s} \subset K[x_1, \ldots, x_n]$. Then I is weakly polymatroidal.*

Proof. Let $P_K = (y_1, \ldots, y_l)$ and $P_{J_i} = (z_1, \ldots, z_t, x_{i,1}, \ldots, x_{i,b_i})$ for all i. Consider the following ordering on the variables of R.

$$y_1 > \cdots > y_l > z_1 > \cdots > z_t > x_{1,1} > \cdots > x_{1,b_1} > \cdots > x_{s,1} > \cdots > x_{s,b_s}.$$

Any monomial u in I can be written as $f = w_1w_2u_1 \cdots u_s$, where $w_1 \in K[y_1, \ldots, y_l]$, $w_2 \in K[z_1, \ldots, z_t]$, and $u_i \in K[x_{i,1}, \ldots, x_{i,b_i}]$ for $i = 1, \ldots, s$. For any monomial $f = w_1w_2u_1 \cdots u_s \in G(I)$ we have $\deg(w_1) = a_0$, $\deg(u_i) = a_i - l$, where $\deg(w_2) = l$. Let $f = w_1w_2u_1 \cdots u_s$ and $g = w'_1w'_2u'_1 \cdots u'_s$ be two monomials in $G(I)$ such that $f >_{\text{lex}} g$. We will denote the exponent of any variable x in f, by $f(x)$. Let x be the first variable such that $f(x) > g(x)$. The following cases may be considered:

CASE (a). Let $x = y_i$ for some $1 \leq i \leq l$. Since $\deg(w_1) = \deg(w'_1)$, then for some $j > i$ we have $y_j \in \operatorname{Supp}(w'_1)$. Then let $h = x(g/y_j)$. We have $h = w''_1w''_2u''_1 \cdots u''_s$, where $\deg(w''_1) = \deg(w'_1)$, $\deg(w''_2) = \deg(w'_2)$ and $\deg(u''_i) = \deg(u'_i)$ for all i. Thus $h \in I$.

CASE (b). Let $x = z_t$ for some $1 \leq t \leq k$. Since $f \nmid g$, there exists a variable $x < y \in \operatorname{Supp}(g)$, where $y = z_j$ for some $j > t$ or $y = x_{i,j}$ for some i, j. Then let

$h = x(g/y)$. We have $h = w_1''w_2''u_1''\cdots u_s''$. If $y = z_j$, then $\deg(w_1'') = \deg(w_1')$, $\deg(w_2'') = \deg(w_2')$ and $\deg(u_i'') = \deg(u_i')$ for all i. If $y = x_{i,j}$, then $\deg(w_1'') = \deg(w_1')$, $\deg(w_2'') = \deg(w_2') + 1$ and $\deg(u_i'') \geq \deg(u_i') - 1$ for all i. Thus $h \in I$.

CASE (c). Let $x = x_{l,i}$ for some $1 \leq t \leq k$. Since $w_1 = w_1'$ and $w_2 = w_2'$, we have $\deg(u_j) = \deg(u_j')$ for any j. Therefore there exists a $j' > j$ such that $g(x_{l,j'}) > 0$. Then $h = x(g/x_{l,j'})$ is in I, since $h = w_1''w_2''u_1''\cdots u_s''$ and $\deg(w_1'') = \deg(w_1')$, $\deg(w_2'') = \deg(w_2')$ and $\deg(u_i'') = \deg(u_i')$ for all i. $\square$

Example 1.8 shows that the ideals considered in the above theorem are not necessarily componentwise weakly polymatroidal.

Theorem 2.5. *Let $J_1, \ldots, J_s, K$ be subsets of $[n]$ such that $J_i \cup J_j = [n]$ for all $i \neq j$ and $K \subset [n]$. Let $I = P_{J_1}^{a_1} \cap \cdots \cap P_{J_s}^{a_s} \cap P_K^b \subset K[x_1, \ldots, x_n]$. Then (I_d) is weakly polymatroidal for any d.*

Proof. Rename the variables to $z_1, \ldots, z_k, z_1', \ldots, z_{k'}', x_{1,1}, \ldots, x_{1,b_1}, y_{1,1}, \ldots, y_{1,c_1}, \ldots, x_{s,1}, \ldots, x_{s,b_s}, y_{s,1}, \ldots, y_{s,c_s}$ such that variables z_j $(1 \leq j \leq k)$ correspond to the integers in $\left(\bigcap_{i=1}^s J_i\right) \cap K$, the variables z_j' $(1 \leq j \leq k')$ correspond to the integers in $\left(\bigcap_{i=1}^s J_i\right) \setminus K$, the variables $x_{i,j}$ correspond to the integers in $[n]$ missing from $J_i \cup K$ and the variables $y_{i,j}$ correspond to the integers in K missing from J_i. Since $J_i \cup J_j = [n]$, any $r \in [n]$ is missing at most one of the J_i. For any l, $1 \leq l \leq s$ the only variables which do not appear in J_l are $x_{l,1}, \ldots, x_{l,b_l}, y_{l,1}, \ldots, y_{l,c_l}$.

Consider the ordering $z_1 > \cdots > z_k > z_1' > \cdots > z_{k'}' > y_{1,1} > \cdots > y_{1,c_1} > \cdots > y_{s,1} > \cdots > y_{s,c_s} > x_{1,1} > \cdots > x_{1,b_1} > \cdots > x_{s,1} > \cdots > x_{s,b_s}$ on the variables of R. For any $f \in I$ we can write $f = w_1w_2v_1\cdots v_su_1\cdots u_s$, where $w_1 \in K[z_1, \ldots, z_k]$, $w_2 \in K[z_1', \ldots, z_k']$, $u_l \in K[x_{l,1}, \ldots, x_{l,b_l}]$ and $v_l \in K[y_{l,1}, \ldots, y_{l,c_l}]$. Therefore $f \in (I_d)$ if and only if $\deg(f) = d$ and

(1) $\deg(u_i) + \deg(v_i) \leq d - a_i$ for $i = 1, \ldots, k$,

(2) $\deg(w_2) + \deg(u_1) + \deg(u_2) + \cdots + \deg(u_s) \leq d - b$.

Let $f = w_1w_2u_1\cdots u_sv_1\cdots v_s$ and $g = w_1'w_2'u_1'\cdots u_s'v_1'\cdots v_s'$ be two monomials in $G(I_d)$ such that $f >_{\text{lex}} g$. The exponent of any variable x in f, is denoted by $f(x)$. Let x be the first variable such that $f(x) > g(x)$. We are going to find a variable $y < x$ such that $h = x(g/y) \in I_d$. The following cases may be considered:

CASE (a). Let $x = z_t$ for some $1 \leq t \leq k$. Since $\deg(f) = \deg(g)$, there exists a variable $y \in \mathrm{Supp}(g)$ such that $y < x$. Since we do not have any condition on $\deg(w_1)$, the monomial $h = x(g/y)$ admits conditions (1) and (2) which implies that $h \in I_d$.

CASE (b). Let $x = z_t'$ for some $1 \leq t \leq k'$. If there exists a variable $y \in \mathrm{Supp}(g)$, where $y = z_k'$ for some $k > t$ or $y = x_{l,i}$ for some l, i, then $h = x(g/y)$ admits conditions (1) and (2). Otherwise $\deg(u_1') = \cdots = \deg(u_s') = 0$ and $\deg(w_2') < \deg(w_2)$. Then for any variable k in $\mathrm{Supp}(g)$ with $k < z_t'$, the element $h = x(g/k)$ has desired properties (there exists such k, otherwise $g \mid f$).

CASE (c). Let $x = y_{l,i}$. If there exists a variable $k \in \mathrm{Supp}(g)$, where $k = y_{l,j}$ for $j > i$ or $k = x_{l,j}$ for some j, then $h = x(g/k)$ has desired properties. Otherwise, $\deg(v_l') < \deg(v_l)$ and $\deg(u_l') = 0$. Since $\deg(f) = \deg(g)$, for some $l' > l$ we have $u_{l'} \neq 0$ or $v_{l'} \neq 0$. Therefore there exists a variable $k \in \mathrm{Supp}(g)$, $k = x_{l',i'}$ or $k = y_{l',i'}$. Then the monomial $h = x(g/k)$ has desired properties.

CASE (d). Let $x = x_{l,i}$. If $x_{l,k} \in \mathrm{Supp}(g)$ for some $k > i$, then $h = x(g/x_{l,k})$ has desired properties. Otherwise we have $\deg(u_l') < \deg(u_l)$. Since $\deg(v_l') = \deg(v_l)$, we have $\deg(u_l') + \deg(v_l') < d - a_l$. Then there exists a variable $x_{l',j} \in \mathrm{Supp}(u_{l+1}' \cdots u_s')$. The element $h = x(g/x_{l',j})$ has desired properties. □

The above theorem improves [7, Theorem 3.1].

REMARK 2.6. The $I = P_{J_1}^{a_1} \cap \cdots \cap P_{J_s}^{a_s} \cap P_K^b \cap P_{K'}^{b'} \subset K[x_1, \dots, x_n]$, where $J_1, \dots, J_s$ are subsets of $[n]$ such that $J_i \cup J_j = [n]$ for all $i \neq j$ and $K, K' \subset [n]$. Such ideals are not necessarily componentwise linear. For example the ideal $I = (x_1, x_2) \cap (x_3, x_4) \cap (x_2, x_3) \cap (x_1, x_4)$ is an ideal as described above, but is not componentwise linear. Hence Theorem 2.5 can not be extended to the case that we add to the edges $J_1, \dots, J_s$ more than one random edge.

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