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QUASITORIC MANIFOLDS OVER A PRODUCT OF SIMPLICES

Dedicated to Professor Takao Matumoto on his sixtieth birthday

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Abstract

A quasitoric manifold (resp. a small cover) is a $2n$-dimensional (resp. an n-dimensional) smooth closed manifold with an effective locally standard action of $(S^1)^n$ (resp. $(\mathbb{Z}_2)^n$) whose orbit space is combinatorially an n-dimensional simple convex polytope P. In this paper we study them when P is a product of simplices. A generalized Bott tower over $\mathbb{F}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, is a sequence of projective bundles of the Whitney sum of $\mathbb{F}$-line bundles starting with a point. Each stage of the tower over $\mathbb{F}$, which we call a generalized Bott manifold, provides an example of quasitoric manifolds (when $\mathbb{F} = \mathbb{C}$) and small covers (when $\mathbb{F} = \mathbb{R}$) over a product of simplices. It turns out that every small cover over a product of simplices is equivalent (in the sense of Davis and Januszkiewicz [5]) to a generalized Bott manifold. But this is not the case for quasitoric manifolds and we show that a quasitoric manifold over a product of simplices is equivalent to a generalized Bott manifold if and only if it admits an almost complex structure left invariant under the action. Finally, we show that a quasitoric manifold M over a product of simplices is homeomorphic to a generalized Bott manifold if M has the same cohomology ring as a product of complex projective spaces with $\mathbb{Q}$ coefficients.

1. Introduction

Toric varieties in algebraic geometry and Hamiltonian torus actions on symplectic manifolds exhibit fascinating relations between the geometry of algebraic varieties or smooth manifolds and the combinatorics of their orbit spaces. Considering the success of toric theory, it is natural to generalize them to the topological category, and a monumental development in this direction was obtained by the work of Davis and Januszkiewicz in [5]. They defined a topological generalization of toric variety by the name of "toric manifold", which is a $2n$-dimensional closed manifold M with a locally standard action of n-torus $G = (S^1)^n$ whose orbit space is combinatorially an n-dimensional simple convex polytope P. In this case M is said to be a "toric manifold" over P. They also defined a $\mathbb{Z}_2$-analogue of a "toric manifold" called a small cover, which is an n-dimensional man-

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ifold with an effective action of the $\mathbb{Z}_2$-torus of rank n with an n-dimensional simple polytope as the orbit space.

Unfortunately the term “toric manifolds” is already well-established among algebraic geometers as “non-singular toric variety”. Moreover there are “toric manifolds” (in the sense of Davis and Januszkiewicz) which are not algebraic varieties, for example $\mathbb{C}P^2 \sharp \mathbb{C}P^2$. Because of this reason Buchstaber and Panov introduced the term “quasitoric manifold” as an alias for Davis and Januszkiewicz’s “toric manifold” in [1]. In this paper we adopt Buchstaber and Panov’s “quasitoric manifold” instead of “toric manifold”. We refer the reader to Chapter 5 of [1] for an excellent exposition on quasitoric manifolds including their comparison with (compact non-singular) toric varieties.

This paper is motivated by the work [10] which investigates quasitoric manifold over a cube. A cube is a product of 1-simplices. We take a product of simplices as the simple polytope P and describe quasitoric manifolds and small covers over P in terms of matrices with vectors as entries. A typical example of quasitoric manifolds or small covers over a product of simplices appears in a sequence of projective bundles

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

where B_i for $i = 1, \dots, m$ is the projectivization of the Whitney sum of $n_i + 1$ $\mathbb{F}$-line bundles over B_{i-1} ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). Grossberg–Karshon [7] considered the sequence above when $\mathbb{F} = \mathbb{C}$ and $n_i = 1$ for any i, and they named it a *Bott tower*. Motivated by this, we call the sequence above a *generalized Bott tower* (over $\mathbb{F}$). The j-stage B_j of the tower provides a quasitoric manifold (when $\mathbb{F} = \mathbb{C}$) and a small cover (when $\mathbb{F} = \mathbb{R}$) over $\prod_{i=1}^{j} \Delta^{n_i}$ where Δ^{n_i} is the n_i-simplex. We call each B_j a *generalized Bott manifold* (over $\mathbb{F}$) and especially call it a *Bott manifold* when the tower is a Bott tower. It turns out that any small cover over a product of simplices is equivalent (in particular, homeomorphic) to a generalized Bott manifold (over $\mathbb{R}$) (see Remark 6.5) but this is not the case for quasitoric manifolds. We give a necessary and sufficient condition for a quasitoric manifold over a product of simplices to be equivalent to a generalized Bott manifold (over $\mathbb{C}$) (see Theorem 6.4), where a part of the statement is a particular case of [6, Theorem 6].

This paper is organized as follows. In Section 2 we recall general facts on quasitoric manifolds and small covers over a simple polytope. From Section 3 we restrict our concern to a product of simplices as the simple polytope and treat only quasitoric manifolds because small covers can be treated similarly. In Section 3 we introduce some notation needed for later discussion and associate a matrix with vectors as entries to a quasitoric manifold over a product of simplices. In Section 4 we describe quasitoric manifolds over a product of simplices as the orbit space of a product of odd dimensional spheres by some free torus action. This is done in [7] and [4] when the orbit space is a product of 1-simplices, that is, a cube. The association of the matrix with vectors as entries to a quasitoric manifold over a product of simplices depends on the order of the product of the simplices. We discuss this in Section 5. Generalized Bott towers are introduced in Section 6 and generalized Bott manifolds are characterized among quasitoric manifolds over a product of simplices (Theorem 6.4). In Section 7 we explicitly describe the cohomology ring of a quasitoric manifold over a product of simplices and prove in Section 8 that such a quasitoric manifold is homeomorphic to a generalized Bott manifold if it has the same cohomology ring as a product of complex projective spaces with $\mathbb{Q}$ coefficients.

2. General facts

An n-dimensional convex polytope P is said to be *simple* if precisely n facets (namely codimension-one faces of P) meet at each vertex. Equivalently, P is simple if the dual of the boundary complex ∂P of P is a simplicial complex. It is clear that every simplex is simple and a product of simple convex polytopes is simple. Therefore a product of simplices is simple.

Let $d = 1$ or 2. We denote by S_d an order two group S^0 when $d = 1$ and a circle group S^1 when $d = 2$, and by G_d a group isomorphic to $(S_d)^n$. A dn-dimensional smooth G_d-manifold M_d with a projection $\pi: M_d \to P$ is called a *small cover* (when $d = 1$) and a *quasitoric manifold* (when $d = 2$) over an n-dimensional simple convex polytope P if M_d is locally isomorphic to a faithful real dn-dimensional representation of G_d and each fiber of π is a G_d-orbit. The orbit space M_d/G_d can be identified with P. Two quasitoric manifolds or small covers $\pi: M_d \to P$ and $\pi': M'_d \to P$ are *equivalent* (*in the sense of Davis and Januszkiewicz*) if there is a homeomorphism $f: M_d \to M'_d$ covering the identity on P and an automorphism $\theta: G_d \to G_d$ such that f satisfies θ-equivariance, i.e., $f(gm) = \theta(g)f(m)$ for all $m \in M_d$ and $g \in G_d$. Note that the equivalence is neither weaker nor stronger than G_d-homeomorphism, because any G_d-homeomorphism must satisfy θ-equivariance with $\theta = \mathrm{id}$, but it may not cover the identity on the orbit space.

Let $\pi: M_d \to P$ be a small cover or a quasitoric manifold and let $\mathcal{F}$ be the set of facets of P. If $F \in \mathcal{F}$, then the isotropy subgroup of a point $x \in \pi^{-1}(\operatorname{int} F)$ is independent of the choice of x, and is a rank-one subgroup $G_d(F)$ of G_d. The group $\mathrm{Hom}(S_d, G_d)$ of homomorphisms from S_d to G_d is isomorphic to $(R_d)^n$ where R_d is $\mathbb{Z}/2$ when $d = 1$ and $\mathbb{Z}$ when $d = 2$. Each rank-one subgroup of G_d corresponds uniquely (up to sign) to a primitive vector of $\mathrm{Hom}(S_d, G_d)$ which generates a rank-one direct summand of $\mathrm{Hom}(S_d, G_d)$. Therefore every M_d defines what is called the *characteristic function* of M_d

$$\lambda: \mathcal{F} \to \mathrm{Hom}(S_d, G_d)$$

such that the image of $F \in \mathcal{F}$ is a primitive vector of $\mathrm{Hom}(S_d, G_d)$ corresponding to the rank-one subgroup $G_d(F)$. When $d = 1$, such a primitive vector is unique for each F, but sign ambiguity arises when $d = 2$. This sign ambiguity can be resolved if an omni-orientation (see [1]) is assigned to a quasitoric manifold M_d, in particular if M_d admits an almost complex structure left invariant under the action (see Lemma 1.5 and 1.10

of [9]). In any case, the characteristic function λ of M_d must satisfy the following condition, see [5].

CONDITION 2.1. If n facets $F_1, \dots, F_n$ of P intersect at a vertex, then their images $\lambda(F_1), \dots, \lambda(F_n)$ must form a basis of $\mathrm{Hom}(S_d, G_d)$.

Conversely, for a function $\lambda\colon \mathcal{F} \to \mathrm{Hom}(S_d, G_d)$ satisfying Condition 2.1, there exists a unique (up to equivalence) small cover (when $d = 1$) and quasitoric manifold (when $d = 2$) with λ as the characteristic function, see [5] or [2] for details. Therefore in order to classify all small covers or quasitoric manifolds over a simple convex polytope P, it is necessary and sufficient to understand the functions λ satisfying Condition 2.1.

Let $F_1, \dots, F_k$ be the all facets of P and let $\omega_1, \dots, \omega_k$ be the indeterminates corresponding to the facets. Then it is shown in [5] that the equivariant cohomology ring $H^*_{G_d}(M_d; R_d)$ is the face ring (or the Stanley–Reisner ring) of P with R_d coefficient as graded rings, that is,

$$H^*_{G_d}(M_d; R_d) = R_d[\omega_1, \dots, \omega_k]/I, \tag{2.1}$$

where the degree of ω_i is d for each i and I is the homogeneous ideal of the polynomial ring $R_d[\omega_1, \dots, \omega_k]$ generated by all square-free monomials of the form $\omega_{i_1} \cdots \omega_{i_s}$ such that the intersection of the corresponding facets $F_{i_1}, \dots, F_{i_s}$ is empty.

We choose a basis of $\mathrm{Hom}(S_d, G_d)$ and identify $\mathrm{Hom}(S_d, G_d)$ with $(R_d)^n$. We form a $k \times n$ matrix whose i-th row is $\lambda(F_i) \in (R_d)^n$, i.e.,

$$(\lambda_{ij}) = \begin{pmatrix} \lambda(F_1) \\ \vdots \\ \lambda(F_k) \end{pmatrix}. \tag{2.2}$$

Let $\lambda_j = \lambda_{1j}\omega_1 + \cdots + \lambda_{kj}\omega_k$, and let J be the ideal of $R_d[\omega_1, \dots, \omega_k]$ generated by λ_j for $j = 1, \dots, n$. Then we have

$$H^*(M_d; R_d) = R_d[\omega_1, \dots, \omega_k]/(I + J). \tag{2.3}$$

REMARK 2.2. In general it would be natural to use a *column* vector to express $\lambda(F_i)$ (see [1]), but then, as noticed in [10], we need to take a transpose of a matrix at some point to adjust our description to the notation used in [4] and [7]. Therefore we will use a *row* vector to express $\lambda(F_i)$ in this paper.

As is seen above, most of the arguments for quasitoric manifolds work for small covers with S^1 and $\mathbb{Z}$ replaced by S^0 and $\mathbb{Z}/2$ respectively. In fact, the study of small covers is a bit simpler than that of quasitoric manifolds in our case. So we shall treat only quasitoric manifolds throughout this paper. The main difference between quasitoric manifolds and small covers in our arguments is stated in Remark 6.5, so that the arguments after Section 7 are unnecessary for small covers.

3. Vector matrices

From now on, we take

$$P = \prod_{i=1}^{m} \Delta^{n_i}, \quad \text{with} \quad \sum_{i=1}^{m} n_i = n,$$

where Δ^{n_i} is the n_i-simplex for $i = 1, \dots, m$. Let $\{v_0^i, \dots, v_{n_i}^i\}$ be the set of vertices of the simplex Δ^{n_i}. Then each vertex of P is the product of vertices of Δ^{n_i}'s for $i = 1, \dots, m$, hence the set of vertices of P is

$$\left\{ v_{j_1 \cdots j_m} = v_{j_1}^1 \times \cdots \times v_{j_m}^m \;\middle|\; 0 \le j_i \le n_i \right\}.$$

Each facet of P is the product of a codimension-one face of one of Δ^{n_i}'s and the remaining simplices. Therefore the set of facets of P is

$$\mathcal{F} = \{F_{k_i}^i \mid 0 \le k_i \le n_i,\ i = 1, \dots, m\}$$

where $F_{k_i}^i = \Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times f_{k_i}^i \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_m}$, and $f_{k_i}^i$ is the codimension-one face of the simplex Δ^{n_i} which is opposite to the vertex $v_{k_i}^i$. Hence there are $\sum_{i=1}^m (n_i + 1) = n + m$ facets in P. Since P is simple, exactly n facets meet at each vertex. Indeed, at each vertex $v_{j_1 \cdots j_m}$ of P all n facets in $\mathcal{F} - \{F_{j_i}^i \mid i = 1, \dots, m\}$ intersect, in particular, the n facets in the set

$$\mathcal{F} - \{F_0^i \mid i = 0, \dots, m\} = \{F_1^1, \dots, F_{n_1}^1, \dots, F_1^m, \dots, F_{n_m}^m\}$$

intersect at the vertex $v_{0 \cdots 0}$.

Let $\lambda \colon \mathcal{F} \to \mathrm{Hom}(S^1, (S^1)^n)$ be the characteristic function of a quasitoric manifold over P. By Condition 2.1, n vectors

$$\lambda(F_1^1), \dots, \lambda(F_{n_1}^1), \dots, \lambda(F_1^m), \dots, \lambda(F_{n_m}^m) \tag{3.1}$$

form a basis of $\mathrm{Hom}(S^1, (S^1)^n)$ and we identify $\mathrm{Hom}(S^1, (S^1)^n)$ with $\mathbb{Z}^n$ through this basis. Then the vectors in (3.1) correspond to the standard basis elements

$$\mathbf{e}_1 = (1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, \dots, 0, 1)$$

in the given order. For the remaining m facets F_0^i, we set

$$\lambda(F_0^i) = \mathbf{a}_i \in \mathbb{Z}^n \quad \text{for} \quad i = 1, \dots, m.$$

In this way, to the characteristic function λ of a quasitoric manifold over P we have a corresponding $m \times n$ matrix

$$A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}, \quad \text{where} \quad \mathbf{a}_i \in \mathbb{Z}^n.$$

Each row vector $\mathbf{a}_i$ can be written as

$$\begin{aligned} \mathbf{a}_i &= (\mathbf{a}_i^1, \dots, \mathbf{a}_i^j, \dots, \mathbf{a}_i^m) \\ &= ([a_{i1}^1, \dots, a_{in_1}^1], \dots, [a_{i1}^j, \dots, a_{in_j}^j], \dots, [a_{i1}^m, \dots, a_{in_m}^m]) \end{aligned}$$

where $\mathbf{a}_i^j = [a_{i1}^j, \dots, a_{in_j}^j] \in \mathbb{Z}^{n_j}$ for $j = 1, \dots, m$. Therefore we may write

$$\begin{aligned} A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} &= \begin{pmatrix} \mathbf{a}_1^1 & \cdots & \mathbf{a}_1^m \\ \vdots & \cdots & \vdots \\ \mathbf{a}_m^1 & \cdots & \mathbf{a}_m^m \end{pmatrix} \\ &= \begin{pmatrix} a_{11}^1 & \cdots & a_{1n_1}^1 & \cdots & a_{11}^m & \cdots & a_{1n_m}^m \\ \vdots & & & & & & \vdots \\ a_{m1}^1 & \cdots & a_{mn_1}^1 & \cdots & a_{m1}^m & \cdots & a_{mn_m}^m \end{pmatrix} \end{aligned} \tag{3.2}$$

with $\mathbf{a}_i^j \in \mathbb{Z}^{n_j}$ for all $i = 1, \dots, m$. In other words, the $m \times n$ matrix A can be viewed as an $m \times m$ matrix whose entries in the j-th column are vectors in $\mathbb{Z}^{n_j}$. From now on, we shall view the matrix A this way and call it a *vector matrix*.

Since the characteristic function λ satisfies Condition 2.1, we need to translate this into a condition on the corresponding matrix A. For this let us fix some more notation. For given $1 \le k_j \le n_j$ with $j = 1, \dots, m$, let $A_{k_1 \cdots k_m}$ be the $m \times m$ submatrix of A whose j-th column is the k_j-th column of the $m \times n_j$ matrix

$$\begin{pmatrix} \mathbf{a}_1^j \\ \vdots \\ \mathbf{a}_m^j \end{pmatrix} = \left(\begin{array}{cc|c|cc} a_{11}^j & \cdots & a_{1k_j}^j & \cdots & a_{1n_j}^j \\ \vdots & & \vdots & & \vdots \\ a_{m1}^j & \cdots & a_{mk_j}^j & \cdots & a_{mn_j}^j \end{array}\right).$$

Thus

$$A_{k_1 \cdots k_m} = \begin{pmatrix} a_{1k_1}^1 & \cdots & a_{1k_m}^m \\ \vdots & & \vdots \\ a_{mk_1}^1 & \cdots & a_{mk_m}^m \end{pmatrix}.$$

EXAMPLE 3.1. Let $P = \Delta^2 \times \Delta^1$ be a triangular cylinder. Let $\{v_0^1, v_1^1, v_2^1\}$ be the vertices of Δ^2 and $\{v_0^2, v_1^2\}$ the vertices of Δ^1. Then

$$\{v_{00}, v_{10}, v_{20}, v_{01}, v_{11}, v_{21}\}$$

is the vertex set of P where $v_{ij} = v_i^1 \times v_j^2$. The set of facets of P is

$$\{F_0^1, F_1^1, F_2^1, F_0^2, F_1^2\}$$

where $F_i^1 = f_i^1 \times \Delta^1$ for $i = 0, 1, 2$ are the side rectangles and $F_j^2 = \Delta^2 \times f_j^2$ for $j = 0, 1$ are the top and bottom triangles. The characteristic function $\lambda\colon \mathcal{F} \to \mathbb{Z}^3$ is assigned as follows:

$$\lambda(F_0^1) = \mathbf{a}_1, \quad \lambda(F_1^1) = \mathbf{e}_1, \quad \lambda(F_2^1) = \mathbf{e}_2,$$
$$\lambda(F_0^2) = \mathbf{a}_2, \quad \lambda(F_1^2) = \mathbf{e}_3.$$

The corresponding 2×3 matrix A is

$$\begin{aligned} A &= \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{a}_1^1 & \mathbf{a}_1^2 \\ \mathbf{a}_2^1 & \mathbf{a}_2^2 \end{pmatrix} \quad \text{as a} \quad 2 \times 2 \quad \text{vector matrix} \\ &= \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{11}^2 \\ a_{21}^1 & a_{22}^1 & a_{21}^2 \end{pmatrix}. \end{aligned}$$

Thus the 2×2 submatrices A_{11} and A_{21} are as follows:

$$A_{11} = \begin{pmatrix} a_{11}^1 & a_{11}^2 \\ a_{21}^1 & a_{21}^2 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} a_{12}^1 & a_{11}^2 \\ a_{22}^1 & a_{21}^2 \end{pmatrix}.$$

Condition 2.1 at a vertex, say v_{21}, can be translated as follows: since the facets F_0^1, F_1^1 and F_0^2 intersect at v_{21}

$$\begin{aligned} \det \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} &= \det \begin{pmatrix} 1 & 0 & 0 \\ a_{11}^1 & a_{12}^1 & a_{11}^2 \\ a_{21}^1 & a_{22}^1 & a_{21}^2 \end{pmatrix} \\ &= \det A_{21} = \pm 1. \end{aligned}$$

Similarly Condition 2.1 at v_{01} is equivalent to $a_{21}^2 = \pm 1$, and that at v_{20} is equivalent to $a_{12}^1 = \pm 1$. These conditions are equivalent to the condition that all principal minors of A_{21} (including the determinant of A_{21} itself) are ± 1. Similarly Condition 2.1 at other vertices is equivalent to all principal minors of A_{11} being ± 1.

The last statement in Example 3.1 holds in general. A *principal minor* of an $m \times m$ vector matrix A of the form (3.2) means a principal minor of an $m \times m$ matrix $A_{j_1 \cdots j_m}$ for some $1 \le j_1 \le n_1, \dots, 1 \le j_m \le n_m$ where the determinant of $A_{j_1 \cdots j_m}$ itself is understood to be a principal minor of $A_{j_1 \cdots j_m}$.

Lemma 3.2. *Let $P = \prod_{i=1}^m \Delta^{n_i}$. If an $m \times m$ vector matrix A of the form* (3.2) *is associated with the characteristic function λ of a quasitoric manifold over P, then* Condition 2.1 *for λ at all vertices of P is equivalent to all principal minors of A being* ± 1.

Proof. The basic idea of the proof is same as in Example 3.1. Indeed, at a vertex $v_{j_1 \cdots j_m}$ of P all n facets in $\mathcal{F}' = \mathcal{F} - \{F^i_{j_i} \mid i = 1, \dots, m\}$ intersect. Hence Condition 2.1 at $v_{j_1 \cdots j_m}$ is equivalent to the determinant of the $n \times n$ matrix having $\lambda(F)$ as its row vectors for all $F \in \mathcal{F}'$ being ± 1. But this determinant is nothing but a principal minor of the $m \times m$ matrix $A_{j_1 \cdots j_m}$ up to sign. Therefore the lemma follows. □

REMARK 3.3. It follows from the lemma above that each component a^i_{ij} in the diagonal entry vector $\mathbf{a}^i_i = (a^i_{i1}, \dots, a^i_{in_i})$ of the matrix A, see (3.2), is ± 1 for $j = 1, \dots, n_i$. The characteristic function λ is defined up to sign and if we change the sign of a vector $\lambda(F^j_k)$ in (3.1) (say $\lambda(F^j_k) = \mathbf{e}_l$), then the column vector corresponding to $\lambda(F^j_k)$ (the l-th column) changes the sign; so we can always arrange $a^i_{i,j} = 1$ for $i = 1, \dots, m$ and $j = 1, \dots, n_i$, i.e., $\mathbf{a}^i_i = (1, \dots, 1)$ by an appropriate choice of signs of the vectors in (3.1). In the following we always take $\mathbf{a}^i_i = (1, \dots, 1)$ for $i = 1, \dots, m$ for the matrix A associated with a quasitoric manifold unless otherwise stated.

4. Quotient construction

It is known that any quasitoric manifold over a simple polytope is realized as the orbit space of the moment-angle manifold of the polytope by some free torus action, see [1] and [2]. When the polytope is $\prod_{i=1}^m \Delta^{n_i}$, the moment-angle manifold is the product $\prod_{i=1}^m S^{2n_i+1}$ of odd dimensional spheres. In this section we shall describe the free torus action on it explicitly. We remark that the case where $n_i = 1$ for all i (i.e., the polytope is an m-cube) is treated in [7] and [4].

Lemma 4.1. *If $C = (c_{ij})$ is a unimodular matrix of size m, then the system of equations*

$$z_1^{c_{i1}} \cdots z_m^{c_{im}} = 1, \quad \text{for} \quad i = 1, \dots, m$$

has a unique solution $z_1 = \cdots = z_m = 1$ in $S^1 \subset \mathbb{C}$.

Proof. Write $z_j = \exp(2\pi\theta_j\sqrt{-1})$ with $\theta_j \in \mathbb{R}$ for $j = 1, \ldots, m$. Then the equations in the lemma are equivalent to

$$c_{i1}\theta_1 + \cdots + c_{im}\theta_m = k_i \quad \text{for} \quad i = 1, \ldots, m$$

for some $k_i \in \mathbb{Z}$. Since C is unimodular and k_i's are integers, θ_j's are also integers, which means $z_j = 1$ for $j = 1, \ldots, m$. $\square$

Let A be an $m \times m$ vector matrix in (3.2). We construct a quasitoric manifold $M(A)$ with A as its corresponding matrix. Consider the subspace $X = \prod_{i=1}^m S^{2n_i+1}$ of $\prod_{i=1}^m \mathbb{C}^{n_i+1}$, which is the moment-angle manifold of $\prod_{i=1}^m \Delta^{n_i}$. Let $K = (S^1)^m$ and define an action of K on X by

$$\begin{aligned}(4.1)\qquad &(g_1, \ldots, g_m) \cdot ((z_0^1, \ldots, z_{n_1}^1), \ldots, (z_0^m, \ldots, z_{n_m}^m)) \\ &= \Big(\big(g_1 z_0^1, \big(g_1^{a_{11}^1} \cdots g_m^{a_{m1}^1}\big)z_1^1, \ldots, \big(g_1^{a_{1n_1}^1} \cdots g_m^{a_{mn_1}^1}\big)z_{n_1}^1\big), \ldots, \\ &\quad \big(g_m z_0^m, \big(g_1^{a_{11}^m} \cdots g_m^{a_{m1}^m}\big)z_1^m, \ldots, \big(g_1^{a_{1n_m}^m} \cdots g_m^{a_{mn_m}^m}\big)z_{n_m}^m\big)\Big)\end{aligned}$$

where $(g_1, \ldots, g_m) \in K$ and $(z_0^i, \ldots, z_{n_i}^i) \in S^{2n_i+1} \subset \mathbb{C}^{n_i+1}$ for $i = 1, \ldots, m$.

Lemma 4.2. *The action of K on X defined in* (4.1) *is free if all principal minors of A are equal to* ± 1.

Proof. To prove that the action is free we have to show that the equation

$$\begin{aligned}(4.2)\qquad &(g_1, \ldots, g_m) \cdot ((z_0^1, \ldots, z_{n_1}^1), \ldots, (z_0^m, \ldots, z_{n_m}^m)) \\ &= ((z_0^1, \ldots, z_{n_1}^1), \ldots, (z_0^m, \ldots, z_{n_m}^m))\end{aligned}$$

implies $g_1 = \cdots = g_m = 1$. Since $(z_0^i, \ldots, z_{n_i}^i) \in S^{2n_i+1}$, at least one component, say $z_{j_i}^i$, is nonzero for every $i = 1, \ldots, m$. If $z_0^i = 0$ for all $i = 1, \ldots, m$, then equation (4.2) implies that $g_1^{a_{1j_i}^i} \cdots g_m^{a_{mj_i}^i} = 1$ for all $i = 1, \ldots, m$. Since $\det A_{j_1 \cdots j_m} = \pm 1$ from the hypothesis, Lemma 4.1 implies that $g_1 = \cdots = g_m = 1$. Now suppose $z_0^i \neq 0$ for some $i = 1, \ldots, m$. For simplicity let us assume that there is some $0 \leq s \leq m$ such that $z_0^1 = \cdots = z_0^s = 0$ and $z_0^i \neq 0$ for all $i = s+1, \ldots, m$. Then equation (4.2) implies that $g_1 = \cdots = g_s = 1$ and $g_{s+1}^{a_{(s+1)j_i}^i} \cdots g_m^{a_{mj_i}^i} = 1$ for all $i = s+1, \ldots, m$. Since all principal minors of $A_{j_1 \cdots j_m}$ are ± 1, Lemma 4.1 implies that $g_{s+1} = \cdots = g_m = 1$, which proves the lemma. $\square$

Since the action K on X is free, the orbit space X/K is a smooth manifold of dimension $2n$. Let $M(A)$ be the orbit space X/K with the action of $G=(S^1)^n$ defined by

$$(t_1,\dots,t_n)\cdot[(z_0^1,\dots,z_{n_1}^1),\dots,(z_0^m,\dots,z_{n_m}^m)] \\ =[(z_0^1,t_1z_1^1,\dots,t_{n_1}z_{n_1}^1),\dots,(z_0^m,t_{n-n_m+1}z_1^m,\dots,t_nz_{n_m}^m)]. \tag{4.3}$$

Then we have the following proposition.

Proposition 4.3. *$M(A)$ is a quasitoric manifold over $\prod_{i=1}^m\Delta^{n_i}$ with A as its associated matrix.*

Proof. We think of q-simplex Δ^q as

$$\Delta^q=\left\{(x_0,\dots,x_q)\in\mathbb{R}^{q+1}\;\middle|\;x_0\geq 0,\dots,x_q\geq 0,\ \sum_{i=0}^q x_i=1\right\}.$$

Then $P=\prod_{i=1}^m\Delta^{n_i}$ sits in $\prod_{i=1}^m\mathbb{R}^{n_i+1}$. It is easy to see that $M(A)$ with the action of $G=(S^1)^n$ is a quasitoric manifold over P with the projection $\pi\colon M(A)\to P$ defined by

$$\pi([(z_0^1,\dots,z_{n_1}^1),\dots,(z_0^m,\dots,z_{n_m}^m)])=((|z_0^1|,\dots,|z_{n_1}^1|),\dots,(|z_0^m|,\dots,|z_{n_m}^m|)).$$

The facets F_j^i of P are given by $x_j^i=0$ for some $1\leq i\leq m$ and $0\leq j\leq n_i$, where x_j^i denotes the $(j+1)$-st coordinate of the i-th factor $\mathbb{R}^{n_i+1}$. The isotropy subgroup of a point in $\pi^{-1}(\operatorname{int}F_j^i)$ is a circle subgroup. One can check that it is the $\left(\sum_{k=1}^{i-1}n_k+j\right)$-th factor of $G=(S^1)^n$ when $j\geq 1$ and the circle subgroup

$$\left\{\left(\left(g^{a_{i1}^1},\dots,g^{a_{in_1}^1}\right),\dots,\left(g^{a_{i1}^m},\dots,g^{a_{in_m}^m}\right)\right)\;\middle|\;g\in S^1\right\}$$

when $j=0$. This shows that if we denote the characteristic function of $M(A)$ by λ, then

$$\lambda(F_1^1),\dots,\lambda(F_{n_1}^1),\dots,\lambda(F_1^m),\dots,\lambda(F_{n_m}^m)$$

are the standard basis elements of $\mathbb{Z}^n$ in the given order and

$$\lambda(F_0^i)=((a_{i1}^1,\dots,a_{in_1}^1),\dots,(a_{i1}^m,\dots,a_{in_m}^m))\in\mathbb{Z}^n\quad\text{for}\quad i=1,\dots,m,$$

which is the i-th row of our matrix A, proving the lemma. □

5. Conjugation of vector matrices

The correspondence between a quasitoric manifold over $P=\prod_{i=1}^m\Delta^{n_i}$ and an $m\times m$ vector matrix A depends on the order of the simplices Δ^{n_i}'s in the product formula of P. Namely, if we consider $P = \prod_{i=1}^{m} \Delta^{n_{\sigma(i)}}$ for some permutation σ of $\{1, \dots, m\}$, then the corresponding $m \times m$ vector matrix A_σ will be different from A. In fact it is not difficult to see that if E_σ is the $m \times m$ permutation matrix of σ obtained from the identity matrix by permuting the i-th row and column to $\sigma(i)$-th row and column respectively for all $i = 1, \dots, m$, then $A_\sigma = E_\sigma A E_\sigma^{-1}$. One should be cautious that, as an $m \times m$ vector matrix, the entries in the j-th column of A_σ are vectors in $\mathbb{Z}^{n_{\sigma(j)}}$ while the j-th column of A are vectors in $\mathbb{Z}^{n_j}$.

As an example let us consider P as in Example 3.1. If we consider $P = \Delta^1 \times \Delta^2$ instead of $\Delta^2 \times \Delta^1$ then the corresponding 2×2 vector matrix A_σ is given by

$$\begin{aligned} A_\sigma &= \begin{pmatrix} \mathbf{a}_2^2 & \mathbf{a}_2^1 \\ \mathbf{a}_1^2 & \mathbf{a}_1^1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}. \end{aligned}$$

The entries of the first column above are vectors in $\mathbb{Z}$ and the ones in the second column are in $\mathbb{Z}^2$.

We say that two $m \times m$ vector matrices A and B are *conjugate* if there exists an $m \times m$ permutation matrix E_σ such that $B = E_\sigma A E_\sigma^{-1}$. In this case, the quasitoric manifolds $M(A)$ and $M(B)$ defined in Proposition 4.3 are equivariantly diffeomorphic.

Let A be an $m \times m$ vector matrix of the form (3.2). A *proper principal minor* (resp. *determinant*) of A means that a proper principal minor (resp. determinant) of $A_{j_1 \cdots j_m}$ for some $1 \leq j_1 \leq n_1, \dots, 1 \leq j_m \leq n_m$. The set of proper principal minors or determinants is invariant under the conjugation relation.

Lemma 5.1. *Let A be an $m \times m$ vector matrix of the form* (3.2) *such that all the proper principal minors of A are* 1. *If all the determinants of A are* 1, *then A is conjugate to a unipotent upper triangular vector matrix of the following form*:

$$\begin{pmatrix} \mathbf{1} & \mathbf{b}_1^2 & \mathbf{b}_1^3 & \cdots & \mathbf{b}_1^m \\ \mathbf{0} & \mathbf{1} & \mathbf{b}_2^3 & \cdots & \mathbf{b}_2^m \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{1} & \mathbf{b}_{m-1}^m \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{5.1}$$

where $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$ *of appropriate sizes. If all the determinants of A*

are ± 1 *and at least one of them is* -1, *then* A *is conjugate to a vector matrix of the following form*:

$$\begin{pmatrix} \mathbf{1} & \mathbf{b}^2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{b}^3 & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{1} & \mathbf{b}^m \\ \mathbf{b}^1 & \cdots & \cdots & \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{5.2}$$

where $\mathbf{b}^i$ *is non-zero for any* i *and* $\prod_{i=1}^m b_i$, *where* b_i *is any non-zero component of* $\mathbf{b}^i$, *is* $(-1)^m 2$. (*Therefore, the non-zero components in* $\mathbf{b}^i$ *are all same for each* i *and they are* ± 1 *or* ± 2.)

Proof. The lemma is proved in [10] when A is an ordinary $m \times m$ matrix except the last statement on the components of $\mathbf{b}^i$, and the proof for an $m \times m$ vector matrix is quite similar. So we refer the reader to the cited paper and shall prove only the statement on the components of $\mathbf{b}^i$.

Let B be the vector matrix of the form (5.2). The determinants of A are ± 1 and at least one of them is -1 by assumption while any determinant of B is of the form $1 + (-1)^{m+1} \prod_{i=1}^m b_i$ where b_i is a component of $\mathbf{b}^i$. Since the set of determinants of A agrees with that of B as remarked above, it follows that there is a non-zero b_i for each i and $\prod_{i=1}^m b_i = (-1)^m 2$ whenever each b_i is non-zero. This implies the statement on b_i's in the lemma. □

6. Generalized Bott towers

A quasitoric manifold over a product of simplices also appears in iterated projective bundles. For a complex vector bundle E, we denote the total space of its projectivization by $P(E)$.

DEFINITION 6.1. We call a sequence

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}, \tag{6.1}$$

where $B_j = P(\mathbb{C} \oplus \xi_j)$ and ξ_j is the Whitney sum of complex line bundles over B_{j-1}, a *generalized Bott tower* and each B_j for $j = 1, \dots, m$ a *generalized Bott manifold.*

Each B_j admits an effective action of $G_j = (S^1)^{\sum_{i=1}^j \dim \xi_i}$ defined as follows. Assume by induction that B_{j-1} admits an effective action of G_{j-1}. Then it lifts to an action on ξ_j since $H^1(B_{j-1}) = 0$ although the lifting is not unique, see [8]. On the other hand since ξ_j is the Whitney sum of complex line bundles, it admits an action of $(S^1)^{\dim \xi_j}$ by scalar multiplication on fibers. These two actions commute and define an action of G_j on ξ_j, which induces an effective action of G_j on B_j. Without much difficulty it can be shown that B_j with the action of G_j is a quasitoric manifold over $\prod_{i=1}^{j} \Delta^{\dim \xi_i}$. Furthermore each B_j is a nonsingular toric variety (i.e., a toric manifold).

Proposition 6.2. *Let M be a quasitoric manifold over $P = \prod_{i=1}^{m} \Delta^{n_i}$, and let A be an $m \times m$ vector matrix associated with M. Then M is equivalent to a generalized Bott manifold if A is conjugate to an $m \times m$ upper triangular vector matrix of the form* (5.1).

REMARK 6.3. We will see later that the "only if" statement in the proposition above also holds, see Lemma 5.1 and Theorem 6.4.

Proof of Proposition 6.2. We may assume that $M = M(A)$ and A is of the form (5.1). We recall the quotient construction in Section 3. Let $X_j = \prod_{i=1}^{j} S^{2n_i+1}$ for $j = 1, \dots, m$, so X_m agrees with X in Section 3. The group $K = (S^1)^m$ is acting on X as in (4.1) and $X/K = M(A)$. We set $B_j = X_j/K$, so $B_m = M(A)$. In the following we claim that the sequence

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}$$

induced from the natural projections from X_j on X_{j-1} for $j = m, \dots, 2, 1$ is a generalized Bott tower.

Since A is of the form (5.1), the last $(m-j)$ factors of $K = (S^1)^m$ are acting on X_j trivially, so the action of K on X_j reduces to an action of the product K_j of the first j factors of $K = (S^1)^m$. This means that $X_j/K = X_j/K_j$. Moreover, the last factor of K_j is acting on the last factor S^{2n_j+1} of X_j as scalar multiplication and trivially on the other factors of X_j. Therefore the map $\pi_j \colon B_j = X_j/K_j \to B_{j-1} = X_{j-1}/K_{j-1}$ is a fibration with $\mathbb{C}P^{n_j} = S^{2n_j+1}/S^1$ as a fiber and this is actually the projectivization of a complex vector bundle ξ_j over B_{j-1}. In fact, the bundle ξ_j is obtained as follows. Let V_j be $\mathbb{C}^{n_j+1}$ with the linear K_{j-1}-action defined by

$$\begin{aligned}
&(g_1, \dots, g_{j-1}) \cdot (z_0^j, \dots, z_{n_j}^j) \\
&= \left(z_0^j, \left(g_1^{b_{11}^j} \cdots g_{j-1}^{b_{j-1\ 1}^j}\right) z_1^j, \dots, \left(g_1^{b_{1n_j}^j} \cdots g_{j-1}^{b_{j-1\ n_j}^j}\right) z_{n_j}^j\right)
\end{aligned}$$

where $\mathbf{b}_i^j = (b_{i1}^j, \dots, b_{in_j}^j)$ is a vector in (5.1) for $i = 1, \dots, j-1$. Since the action of K_{j-1} on X_{j-1} is free, the projection

$$(X_{j-1} \times V_j)/K_{j-1} \to X_{j-1}/K_{j-1} = B_{j-1}$$

becomes a vector bundle, where the action of K_{j-1} on $X_{j-1} \times V_j$ is a diagonal one. This is the desired bundle ξ_j and since V_j decomposes into sum of complex one dimensional

K-modules, the bundle ξ_j decomposes into the Whitney sum of complex line bundles accordingly. $\square$

One can describe the bundles ξ_j in the proof of the proposition above more explicitly. For that let us fix some notation. For a vector bundle η and a vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ let $\eta^{\mathbf{a}}$ denote the bundle $\eta^{a_1} \oplus \cdots \oplus \eta^{a_n}$. For vector bundles $\eta_1, \dots, \eta_k$ over a space and vectors $\mathbf{a}_1 = (a_{11}, \dots, a_{1n}), \dots, \mathbf{a}_k = (a_{k1}, \dots, a_{kn})$ let

$$\begin{aligned} \bigodot_{i=1}^{k} \eta_i^{\mathbf{a}_i} &= \eta_1^{\mathbf{a}_1} \odot \cdots \odot \eta_k^{\mathbf{a}_k} \\ &= (\eta_1^{a_{11}} \otimes \cdots \otimes \eta_k^{a_{k1}}) \oplus \cdots \oplus (\eta_1^{a_{1n}} \otimes \cdots \otimes \eta_k^{a_{km}}) \end{aligned}$$

where the last expression denotes the Whitney sum of componentwise tensor products.

Let ξ_1^2 denote the canonical line bundle over B_1 and let $\xi_1^3 = \pi_2^*(\xi_1^2)$ the pull-back bundle of the canonical line bundle over B_1 to B_2 via the projection $\pi_2 \colon B_2 \to B_1$. In general, let ξ_{j-1}^j be the canonical line bundle over B_{j-1}, and we inductively define

$$\xi_{j-k}^j = \pi_j^* \circ \cdots \circ \pi_{j-k+1}^* \big(\xi_{j-k}^{j-k+1}\big) \quad \text{for} \quad k = 2, \dots, j-1.$$

Then one can see that $\xi_j = \bigodot_{i=1}^{j-1} (\xi_i^j)^{\mathbf{b}_i^j}$.

A generalized Bott manifold is not only a quasitoric manifold over a product of simplices but also a complex manifold on which the action preserves the complex structure, in particular, it has an almost complex structure left invariant under the action. The following theorem shows that the converse holds. We remark that the equivalence (1) $\Leftrightarrow$ (3) is a particular case of [6, Theorem 6].

Theorem 6.4. *Let M be a quasitoric manifold over $P = \prod_{i=1}^m \Delta^{n_i}$, and let A be the $m \times m$ vector matrix associated with M which has* **1** *as the diagonal entries. Then the following are equivalent*:

(1) *M is equivalent to a generalized Bott manifold.*

(2) *M is equivalent to a quasitoric manifold which admits an invariant almost complex structure under the action.*

(3) *All the principal minors of A are* 1.

Proof. The implication (1) $\Rightarrow$ (2) is obvious and the implication (3) $\Rightarrow$ (1) follows from Proposition 6.2 and Lemma 5.1, so it suffices to prove the implication (2) $\Rightarrow$ (3).

We may assume that M itself admits an invariant almost complex structure. As is noted in the paragraph before Condition 2.1 we can define a sign-unambiguous characteristic function λ of M. Let Λ be the matrix associated with λ. To each cubical face of P, the submanifold of M over it inherits an invariant almost complex structure, so it follows from [10, Theorem 3.4] that all principal minors of the restriction of $-\Lambda$ to each cubical face of P are equal to 1. Therefore $A = -\Lambda$ and this proves (3). $\square$

REMARK 6.5. A difference between quasitoric manifolds and small covers appears here. Namely, not every quasitoric manifold over a product of simplices is equivalent to a generalized Bott manifold as is seen from Theorem 6.4, while it follows from the real version of Proposition 6.2 and the $\mathbb{Z}/2$ version of the former part of Lemma 5.1 that every small cover over a product of simplices turns out to be equivalent to a generalized Bott manifold (over $\mathbb{R}$).

7. Cohomology ring

The connected sum $\mathbb{C}P^2 \sharp \mathbb{C}P^2$ is a quasitoric manifold over a square but not homeomorphic to a Bott manifold (or Hirzebruch surface) over a square. In the rest of this paper, we shall give a sufficient condition in terms of cohomology ring for a quasitoric manifold over a product of simplices to be homeomorphic to a generalized Bott manifold (Theorem 8.1). This section is a preliminary section for the purpose.

Lemma 7.1. *Let M be a quasitoric manifold over $\prod_{i=1}^m \Delta^{n_i}$ and let A be the vector matrix of the form* (3.2) *associated with M. Then*

$$H^*(M) = \mathbb{Z}[y_1, \dots, y_m]/L \tag{7.1}$$

where the ideal L is generated by the following m expressions:

$$y_k \cdot \prod_{j=1}^{n_k} \left(\sum_{i=1}^{m} a^k_{ij} y_i \right) \quad \textit{for} \quad k = 1, \dots, m. \tag{7.2}$$

Proof. We will use the result (2.3). In our case, the matrix in (2.2) is of the form

$$(\lambda_{ij}) = \begin{pmatrix} A \\ I_n \end{pmatrix} \tag{7.3}$$

where I_n is the $n \times n$ identity matrix. Let

$$\omega^1_0, \dots, \omega^1_{n_1}, \dots, \omega^m_0, \dots, \omega^m_{n_m}$$

be the indeterminates corresponding to the facets

$$F^1_0, \dots, F^1_{n_1}, \dots, F^m_0, \dots, F^m_{n_m}$$

in the given order. Then by (2.3) we have

$$H^*(M) \cong \mathbb{Z}[\omega^1_0, \dots, \omega^1_{n_1}, \dots, \omega^m_0, \dots, \omega^m_{n_m}]/(I + J) \tag{7.4}$$

where I is the ideal generated by the monomials

$$\omega^i_0 \cdots \omega^i_{n_i} \quad \text{for} \quad i = 1, \dots, m$$

because the intersection of facets $F^i_0, \dots, F^i_{n_i}$ is empty for $i = 1, \dots, m$, and J is the ideal generated by

$$\begin{aligned}\lambda_j = \lambda_{1j}\omega^1_0 + \cdots + \lambda_{mj}\omega^m_0 \\ + \lambda_{(m+1)j}\omega^1_1 + \cdots + \lambda_{(m+n_1)j}\omega^1_{n_1} \\ + \cdots \\ + \lambda_{\left(m+\sum_{i=1}^{m-1} n_i+1\right)j}\omega^m_1 + \cdots + \lambda_{(m+n)j}\omega^m_{n_m}\end{aligned}$$

for $j = 1, \dots, m+n$ because the order of the row vectors in (7.3) is

$$\lambda(F^1_0), \dots, \lambda(F^m_0), \lambda(F^1_1), \dots, \lambda(F^1_{n_1}), \dots, \lambda(F^m_1), \dots, \lambda(F^m_{n_m}).$$

If $j = \left(\sum_{i=1}^{k-1} n_i\right) + l$ and $1 \le l \le n_k$, then

$$\lambda_j = a^k_{1l}\omega^1_0 + a^k_{2l}\omega^2_0 + \cdots + a^k_{ml}\omega^m_0 + \omega^k_l.$$

Since $\lambda_j = 0$ in $H^*(M)$, we have that

(7.5) $$\omega^k_l = -(a^k_{1l}\omega^1_0 + a^k_{2l}\omega^2_0 + \cdots + a^k_{ml}\omega^m_0).$$

Set $y_k = \omega^k_0$ for $k = 1, \dots, m$. Then $\omega^k_0 \cdots \omega^k_{n_1} = 0$ in the cohomology ring implies that

$$y_k \prod_{l=1}^{n_k}(a^k_{1l}y_1 + a^k_{2l}y_2 + \cdots + a^k_{ml}y_m) = 0.$$

This proves the relation in the lemma. □

Lemma 7.2. *Let M and $y_1, \dots, y_m$ be as above. Let $x = \sum_{j=1}^m b_j y_j$ be an element of $H^*(M)$ such that $b_j \neq 0$ for some j. Then $x^{n_j} \neq 0$ in $H^*(M)$.*

Proof. Suppose $x^{n_j} = 0$ on the contrary. Then $\left(\sum_{j=1}^m b_j y_j\right)^{n_j}$ must be in the ideal L in (7.2). However, $y_j^{n_j+1}$ is the least power of y_j which appears as a term in a polynomial of L while $\left(\sum_{j=1}^m b_j y_j\right)^{n_j}$ contains a non-zero scalar multiple of $y_j^{n_j}$ because $b_j \neq 0$ by assumption. This is a contradiction. □

Lemma 7.3. *Let $M(j)$ be a facial submanifold of M over $\prod_{i\neq j}^m \Delta^{n_i}$. Then $H^*(M(j))$ is equal to* (7.1) *with $y_j = 0$ plugged in.*

Proof. Let $y_1, \dots, y_m$ be the generators of $H^*(M)$ in Lemma 7.1. We may assume that $M(j)$ is over $\prod_{i\neq j}^m \Delta^{n_i} \times \{v\}$ where v is a vertex of Δ^{n_j} and also that y_j is the dual of the characteristic submanifold M_j over $\prod_{i\neq j}^m \Delta^{n_i} \times \Delta^{n_j-1}(v)$ where $\Delta^{n_j-1}(v)$ is the facet of Δ^{n_j} not containing v. Since $M(j)$ and M_j have no intersection, the restriction of y_j to $M(j)$ vanishes.

We know that

$$H^*(M) = \mathbb{Z}[y_1, \dots, y_m]/(g_1, \dots, g_m), \tag{7.6}$$

where g_k is the polynomial in (7.2). Since y_j maps to zero in $H^*(M(j))$ and g_j contains y_j as a factor, we have a natural surjective map

$$\mathbb{Z}[y_1, \dots, \widehat{y_j}, \dots, y_m]/(g_1', \dots, \widehat{g_j'}, \dots, g_m') \to H^*(M(j)),$$

where g_k' denotes g_k with $y_j = 0$ plugged in and $\widehat{\ }$ denotes the term there is dropped. The degree of g_k' for $k \neq j$ is $n_k + 1$ and g_k' contains the term $y_k^{n_k+1}$. Therefore, the ranks of the both sides above agree, so that the map is an isomorphism. This proves the lemma. $\square$

Lemma 7.4. *Let N be the smallest number among n_i's. If the vector matrix associated with M is of the form* (5.2) *in* Lemma 5.1, *then there is no non-zero element in $H^2(M)$ whose $(N+1)$-st power vanishes.*

Proof. Let y be an element of $H^2(M)$ whose $(N+1)$-st power vanishes. Since N is smallest among n_i's, y can be expressed as a linear combination of the canonical generators y_i's with $n_i = N$ by Lemma 7.2, say $y = \sum_{n_i=N} a_i y_i$ with $a_i \in \mathbb{Z}$. All relations in $H^*(M)$ of cohomological degree $2(N+1)$ are generated by $y_i^{k_i+1}(y_i + b_i y_{i-1})^{n_i-k_i}$'s with $n_i = N$ over $\mathbb{Z}$, where y_{i-1} with $i = 1$ is understood to be y_m, b_i is the non-zero component of the vector $\mathbf{b}_i$ in Lemma 5.1 and k_i is the number of zero components of $\mathbf{b}_i$. Note that $k_i < N$ when $n_i = N$ since $\mathbf{b}_i$ is non-zero. It follows that we obtain a polynomial identity

$$\left(\sum_{n_i=N} a_i y_i\right)^{N+1} = \sum_{n_i=N} a_i^{N+1} y_i^{k_i+1}(y_i + b_i y_{i-1})^{N-k_i}. \tag{7.7}$$

CASE 1. The case where $N = 1$. In this case $k_i = 0$ for i with $n_i = N = 1$. Suppose that a_i is non-zero for some i with $n_i = 1$. Comparing the coefficients of y_i^2 and $y_i y_{i-1}$ at both sides of the identity (7.7) with an observation that the right-hand side of (7.7) contains a $y_i y_{i-1}$-term, we see that $n_{i-1} = 1$ and $2a_i a_{i-1} = a_i^2 b_i$. Since a_i and b_i are both non-zero, this shows that a_{i-1} is also non-zero and $2a_{i-1} = a_i b_i$. Since $n_{i-1} = 1$ and a_{i-1} is non-zero, the same argument can be applied to $i - 1$ instead of i. Repeating this argument, we see that $n_i = 1$ and $2a_{i-1} = a_i b_i$ for any i. It follows that $\prod_{i=1}^m b_i = 2^m$ which contradicts the fact that $\prod_{i=1}^m b_i = (-1)^m 2$ in Lemma 5.1.

CASE 2. The case where $N \geq 2$. When we expand the right hand side of the identity (7.7), no monomial in more than two variables appears. Since $N \geq 2$, this implies that at most two coefficients among a_i's are non-zero. Since all b_i's are non-zero, it easily follows from (7.7) that the case where only one coefficient among a_i's is non-zero does not occur.

Suppose that there are exactly two non-zero coefficients, say a_i and a_j. Then only two variables appear at the left hand side. Unless $m = 2$ and $n_1 = n_2 = N$, at least three variables appear at the right hand side of (7.7) which is a contradiction. If $m = 2$ and $n_1 = n_2 = N$, then the identity (7.7) is

$$(a_1 y_1 + a_2 y_2)^{N+1} = a_1^{N+1} y_1^{k_1+1} (y_1 + b_1 y_2)^{N-k_1} + a_2^{N+1} y_2^{k_2+1} (y_2 + b_2 y_1)^{N-k_2}.$$

Replacing y_2 by $-b_2 y_1$ above, we obtain an identity

$$|a_1 - a_2 b_2|^{N+1} = |a_1|^{N+1}$$

where we used the fact $b_1 b_2 = 2$ in Lemma 5.1. Since $a_2 b_2 \neq 0$, it follows from the identity above that $2a_1 = a_2 b_2$. Similarly, replacing y_1 by $-b_1 y_2$ above, we obtain $2a_2 = a_1 b_1$. These two identities imply that $b_1 b_2 = 4$ which contradicts to $b_1 b_2 = 2$.

This completes the proof of the lemma. □

8. Cohomologically product quasitoric manifolds

We say that a quasitoric manifold M over $\prod_{i=1}^m \Delta^{n_i}$ is *cohomologically product* over $\mathbb{Q}$ if there are elements $x_1, \dots, x_m$ in $H^2(M; \mathbb{Q})$ such that

$$H^*(M; \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_m]/(x_1^{n_1+1}, \dots, x_m^{n_m+1}). \tag{8.1}$$

The purpose of this section is to prove the following.

Theorem 8.1. *If a quasitoric manifold M over $\prod_{i=1}^m \Delta^{n_i}$ is cohomologically product over $\mathbb{Q}$, then the vector matrix associated with M is conjugate to a unipotent upper triangular vector matrix, so that M is homeomorphic to a generalized Bott manifold.*

REMARK 8.2. We prove in [3] that if a generalized Bott manifold is cohomologically trivial over $\mathbb{Z}$, then it is diffeomorphic to a product of complex projective spaces. This together with Theorem 8.1 implies that if a quasitoric manifold over a product of simplices is cohomologically trivial over $\mathbb{Z}$, then it is homeomorphic to a product of complex projective spaces.

In the following M is assumed to be cohomologically product over $\mathbb{Q}$. We have another set of generators $\{y_1, \dots, y_m\}$ in Lemma 7.1. Since both $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_m\}$ are sets of generators of $H^2(M; \mathbb{Q})$, one can write

$$y_j = \sum_{i=1}^{m} c_{ji} x_i \quad \text{for} \quad j = 1, \dots, m \quad \text{and} \quad c_{ji} \in \mathbb{Q}, \tag{8.2}$$

where the coefficient matrix $C = (c_{ji})$ has non-zero determinant.

Lemma 8.3. *By an appropriate change of indices in x_i's and y_j's, we may assume that $c_{jj} \neq 0$ for any $j = 1, \dots, m$.*

Proof. We may assume that $n_1 \geq n_2 \geq \cdots \geq n_m$ by an appropriate change of indices. Let $S = \{N_1, \dots, N_k\}$ be the set of all distinct elements of $n_1, \dots, n_m$ such that $N_1 > \cdots > N_k$. We can view $\{n_1, \dots, n_m\}$ as a function $\mu\colon \{1, \dots, m\} \to \mathbb{N}$ such that $\mu(j) = n_j$. Then S is the image of μ. Let $J_l = \mu^{-1}(N_l)$ for $l = 1, \dots, k$. We write

$$x_i = \sum_{j=1}^{m} d_{ij} y_j \quad \text{for} \quad i = 1, \dots, m \quad \text{and} \quad d_{ij} \in \mathbb{Q}. \tag{8.3}$$

Since $x_i^{n_i+1} = 0$, $d_{ij} = 0$ if $n_i < n_j$ by Lemma 7.2. This shows that $D = (d_{ij})$ is a block upper triangular matrix because we assume $n_1 \geq n_2 \geq \cdots \geq n_m$. The matrix C in (8.2) is the inverse of the matrix D, so C is also a block upper triangular matrix and of the same type as D, i.e.,

$$C = \begin{pmatrix} C_{J_1} & & & * \\ & C_{J_2} & & \\ & & \ddots & \\ 0 & & & C_{J_k} \end{pmatrix}$$

where C_{J_l} $(l = 1, \dots, k)$ is a square matrix formed from c_{ij} with $i, j \in J_l$. Since $\det C \neq 0$, we have $\det C_{J_l} \neq 0$ for any l. By definition of determinant $\det C_{J_l} = \sum_\sigma \operatorname{sgn} \sigma \prod_{j \in J_l} c_{j\sigma(j)}$ where the sum is taken over all permutations σ on J_l. Therefore there must exist a permutation σ on J_l such that $\prod_{j \in J_l} c_{j\sigma(j)} \neq 0$. This implies the lemma. $\square$

Lemma 8.4. *The facial submanifold $M(j)$ of M over $\prod_{i \neq j}^{m} \Delta^{n_i}$ is also cohomologically product over $\mathbb{Q}$ for any j.*

Proof. Since $H^*(M(j))$ is $H^*(M)$ with $y_j = 0$ plugged by Lemma 7.3, it follows from (8.2) that

$$H^*(M(j);\mathbb{Q}) = \mathbb{Q}[x_1,\dots,x_m] \Big/ \left(x_1^{n_1+1},\dots,x_m^{n_m+1},\sum_{i=1}^m c_{ji}x_i\right).$$

Here $c_{jj} \neq 0$ by Lemma 8.3, so that one can eliminate the variable x_j using the relation $\sum_{i=1}^m c_{ji}x_i = 0$. Therefore a natural map

$$\mathbb{Q}[x_1,\dots,\widehat{x_j},\dots,x_m]\big/\big(x_1^{n_1+1},\dots,\widehat{x_j^{n_r+1}},\dots,x_m^{n_m+1}\big) \to H^*(M(j);\mathbb{Q})$$

is surjective. Since the dimensions at the both sides above are same, this map is actually an isomorphism, proving the lemma. □

Now we shall prove Theorem 8.1 by induction on the number m of factors in $\prod_{i=1}^m \Delta^{n_i}$. Suppose that M is cohomologically product over $\mathbb{Q}$. Then any facial submanifold $M(j)$ is cohomologically product over $\mathbb{Q}$ by Lemma 8.4. Therefore by induction assumption all the proper principal minors of the vector matrix A associated with M are 1. It follows that the vector matrix A is conjugate to a unipotent upper triangular vector matrix or to a matrix of the form (5.2) in Lemma 5.1. But the latter does not occur because since M is cohomologically product over $\mathbb{Q}$, $H^2(M)$ must contain a non-zero element whose $(N+1)$-st power vanishes, where N is the smallest number among n_j's, but this fact contradicts Lemma 7.4. This proves Theorem 8.1.

References

[1] V.M. Buchstaber and T.E. Panov: Torus Actions and Their Applications in Topology and Combinatorics, University Lecture Series **24**, Amer. Math. Soc., Providence, RI, 2002.

[2] V.M. Buchstaber, T.E. Panov and N. Ray: *Spaces of polytopes and cobordism of quasitoric manifolds*, Mosc. Math. J. **7** (2007), 219–242, arXiv:math/0609346v2.

[3] S. Choi, M. Masuda and D.Y. Suh: *Topological classification of generalized Bott towers*, Trans. Amer. Math. Soc. **362** (2010), 1097–1112, arXiv:0807.4334.

[4] Y. Civan and N. Ray: *Homotopy decompositions and K-theory of Bott towers*, K-Theory **34** (2005), 1–33.

[5] M.W. Davis and T. Januszkiewicz: *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), 417–451.

[6] N.È. Dobrinskaya: *Classification problem for quasitoric manifolds over a given simple polytope*, Funct. Anal. and Appl. **35** (2001), 83–89.

[7] M. Grossberg and Y. Karshon: *Bott towers, complete integrability, and the extended character of representations*, Duke Math. J. **76** (1994), 23–58.

[8] A. Hattori and T. Yoshida: *Lifting compact group actions in fiber bundles*, Japan. J. Math. (N.S.) **2** (1976), 13–25.

[9] M. Masuda: *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J. (2) **51** (1999), 237–265.

[10] M. Masuda and T.E. Panov: *Semifree circle actions, Bott towers, and quasitoric manifolds*, Sbornik Math. **199** (2008), 1201–1223, arXiv:math.AT/0607094.

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