

ON THE HOMOTOPY CLASSIFICATION OF 4-MANIFOLDS HAVING THE FUNDAMENTAL GROUP OF AN ASPHERICAL 4-MANIFOLD

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1. Introduction.

In this paper we shall study the homotopy type of closed connected oriented topological 4-manifolds M^4 with fundamental group isomorphic to $\Pi_1(Q)$, where Q is a fixed closed oriented aspherical 4-manifold. A standard example of such a manifold is the connected sum $M = Q \# M'$, where M' is an arbitrary simply-connected closed 4-manifold. In general, we shall always assume that M and Q are provided with CW-structures (up to homotopy) such that $M^{(3)} = M \setminus \overset{\circ}{D}^4$ and $Q^{(3)} = Q \setminus \overset{\circ}{D}^4$ (see for example [16], Lemma 2.9). Here the symbol $X^{(q)}$ denotes the q -skeleton of a CW-complex X as usual.

There are long outstanding conjectures concerning the topological structure of aspherical 4-manifolds (see for example [5]). One of these states that the Whitehead group of $\Pi_1(Q)$ is zero. So we can not assume in our case that homotopy equivalences are automatically simple.

Let $\Lambda = \mathbb{Z}[\Pi_1(Q)]$ be the integral group ring of $\Pi_1(Q)$ and $\text{Out}(\Pi_1(Q))$ the outer automorphism group of $\Pi_1(Q)$, i.e., automorphisms modulo inner automorphisms.

Let $f : M \rightarrow Q$ be the classifying map of the universal covering. For this we shall prove the following result (see Section 3).

Theorem 1.1. *If f is of degree 1, then there is a homotopy equivalence of $M^{(3)}$ with $(Q \# M')^{(3)}$ for some simply-connected closed topological 4-manifold M' .*

As a consequence, $H_2(M; \Lambda)$ is Λ -free. In Section 2 we show that the classifying map $f : M \rightarrow Q$ is of degree 1 if and only if the k -invariant $k_M^3 \in H^3(B\Pi_1; \Pi_2(M))$

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of M vanishes. Observe that $B\Pi_1 = Q$.

For degree one maps we have split exact sequences

$$0 \longrightarrow K_2(f; \Lambda) \longrightarrow H_2(M; \Lambda) \xrightarrow{f_*} H_2(Q; \Lambda) \longrightarrow 0$$

and

$$0 \longrightarrow K_2(f; \mathbb{Z}) \longrightarrow H_2(M; \mathbb{Z}) \xrightarrow{f_*} H_2(Q; \mathbb{Z}) \longrightarrow 0.$$

Note that $H_2(Q; \Lambda) \cong 0$ in our case. The splittings preserve the intersection forms. By the result of Freedman (see [6] and [7]) there is a simply-connected closed topological 4-manifold M' which realizes the intersection form on $K_2(f; \mathbb{Z})$.

Using a result of [1] we are going to prove the main theorem of the present paper.

Theorem 1.2. *Let M^4 be a closed connected oriented topological 4-manifold with $\Pi_1(M) \cong \Pi_1(Q)$, where Q is a fixed closed connected aspherical 4-manifold. Assume that $k_M^3 = 0$. Then $M^{(3)}$ is homotopy equivalent to $(Q\#M')^{(3)}$. If the Λ -intersection form $\mu_M^\Lambda : H_2(M; \Lambda) \times H_2(M; \Lambda) \rightarrow \Lambda$ is extended from the \mathbb{Z} -intersection form $\mu_M^\mathbb{Z} : H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z}$, then M is homotopy equivalent to $Q\#M'$. Moreover, there is an obstruction*

$$\tau(M) \in \text{Wh}(\Pi_1(Q)) / \text{Out}(\Pi_1(Q))$$

for M being simple homotopy equivalent to $Q\#M'$.

Examples for the case when the Whitehead group vanishes are given by flat Riemannian (resp. hyperbolic) manifolds and manifolds M with $\Pi_1(M) \cong \Pi_1(Q)$ poly- \mathbb{Z} -group, i.e., a group having finite composition series whose factors are all infinite cyclic (see [2], [3], [4] and [5]).

The fact about the torsion invariant $\tau(M)$ follows from standard arguments of simple homotopy theory (see for example [15]).

2. The classifying map $f : M \rightarrow Q$.

We begin by choosing an isomorphism $\Pi_1(M) \xrightarrow{\cong} \Pi_1(Q)$. The homotopy equivalence to be constructed depends on this isomorphism. According to it there is a classifying map $f : M \rightarrow Q$. We observe that Q is homotopy equivalent (not simple homotopy equivalent) to a space K obtained from M by adjoining cells of dimension $q \geq 3$. Then we have $\Pi_q(Q, M) \cong \Pi_q(K, M) \cong 0$ for $q \leq 2$ and $H_q(Q, M; \mathcal{B}) \cong 0$ for $q \leq 2$, where \mathcal{B} is an arbitrary local coefficient system.

Lemma 2.1. *If $k_M^3 = 0$, then there is a map $j : Q^{(3)} \rightarrow M$ such that the composition*

$$Q^{(3)} \xrightarrow{j} M \xrightarrow{f} Q$$

is homotopic to the inclusion $Q^{(3)} \subset Q$.

Proof. Let $D \rightarrow B\Pi_1 = Q$ be the 2-stage Postnikov system classified by k_M^3 . Then there is a 3-equivalence $\gamma : M \rightarrow D$ such that the composite map

$$M \xrightarrow{\gamma} D \longrightarrow Q$$

is homotopic to f . There exists a map $j' : Q^{(2)} \rightarrow M$ such that $f \circ j' : Q^{(2)} \rightarrow Q$ is homotopic to the inclusion. Hence we can consider $s = \gamma \circ j'$ as a section in $D \rightarrow Q$ over $Q^{(2)}$. Since $k_M^3 = 0$, the map s extends over $Q^{(3)}$ (in fact, over Q since $\Pi_3(Q) \cong 0$). Because γ is a 3-equivalence, this gives us a map j with the desired property (since $\Pi_q(Q) \cong 0$ for any $q > 1$, $f \circ j$ is homotopic to the inclusion $Q^{(3)} \subset Q$). □

Lemma 2.2. *The map $f : M \rightarrow Q$ is of degree 1 by choosing appropriated orientations of M and Q .*

Proof. First we note that $\Pi_1(Q)$ is necessarily an infinite group (see [5]), hence $H_4(Q; \Lambda) \cong H_4(M; \Lambda) \cong 0$. Of course, $H_q(Q; \Lambda) \cong H_q(\tilde{Q}; \mathbb{Z}) \cong 0$ for any $q > 0$. Moreover, by Poincaré duality it follows that

$$H_3(M; \Lambda) \cong H^1(M; \Lambda) \cong H^1(Q; \Lambda) \cong H_3(Q; \Lambda) \cong 0,$$

hence we have isomorphisms

$$\Lambda \cong H_4(M, M \setminus \overset{\circ}{D}^4; \Lambda) \xrightarrow{\cong} H_3(M \setminus \overset{\circ}{D}^4; \Lambda)$$

and

$$\Lambda \cong H_4(Q, Q \setminus \overset{\circ}{D}^4; \Lambda) \xrightarrow{\cong} H_3(Q \setminus \overset{\circ}{D}^4; \Lambda).$$

Let $f^{(3)} : M \setminus \overset{\circ}{D}^4 \rightarrow Q \setminus \overset{\circ}{D}^4$ denote a cellular approximation of $f : M \rightarrow Q$ restricted to the 3-skeletons. By Lemma 2.1, the composition map $f \circ j$ is homotopic to the inclusion $Q^{(3)} \subset Q$, hence

$$f_*^{(3)} : H_3(M \setminus \overset{\circ}{D}^4; \Lambda) \rightarrow H_3(Q \setminus \overset{\circ}{D}^4; \Lambda)$$

is surjective. It follows that

$$f_*^\Lambda : H_4(M, M \setminus \overset{\circ}{D}^4; \Lambda) \rightarrow H_4(Q, Q \setminus \overset{\circ}{D}^4; \Lambda)$$

is onto, and hence

$$\mathbb{Z} \cong H_4(M, M \setminus \overset{\circ}{D}^4; \Lambda) \otimes_\Lambda \mathbb{Z} \rightarrow H_4(Q, Q \setminus \overset{\circ}{D}^4; \Lambda) \otimes_\Lambda \mathbb{Z} \cong \mathbb{Z}$$

is onto too, i.e., an isomorphism. But we have an isomorphism

$$H_4(M, M \setminus \overset{\circ}{D}^4; \Lambda) \otimes_\Lambda \mathbb{Z} \cong H_4(M, M \setminus \overset{\circ}{D}^4; \mathbb{Z})$$

in a natural way. Hence the map $f : M \rightarrow Q$ induces an isomorphism

$$H_4(M, M \setminus \overset{\circ}{D}^4; \mathbb{Z}) \xrightarrow{\cong} H_4(Q, Q \setminus \overset{\circ}{D}^4; \mathbb{Z}).$$

Now the lemma follows from the diagram

$$\begin{array}{ccc} H_4(M; \mathbb{Z}) & \xrightarrow{\cong} & H_4(M, M \setminus \overset{\circ}{D}^4; \mathbb{Z}) \\ f_* \downarrow & & \downarrow \cong \\ H_4(Q; \mathbb{Z}) & \xrightarrow{\cong} & H_4(Q, Q \setminus \overset{\circ}{D}^4; \mathbb{Z}). \end{array}$$

The horizontal isomorphisms are given by the local orientations. □

If, conversely, $f : M \rightarrow Q$ is of degree 1, then we have

$$H^3(B\Pi_1; \Pi_2(M)) \cong H^3(Q; \Pi_2(M)) \cong H_1(Q; \Pi_2(M)) \cong \text{Tor}_1^\Lambda(\mathbb{Z}, \Pi_2(M)) \cong 0$$

as $\Pi_2(M)$ is stably Λ -free. In particular, it follows that $k_M^3 = 0$.

Summarizing, we have proved the following

Proposition 2.3. *Let M^4 be a closed orientable 4-manifold with fundamental group isomorphic to that of an orientable aspherical 4-manifold Q . Then the classifying map $f : M \rightarrow Q$ is of degree 1 if and only if $k_M^3 = 0$.*

Let us denote

$$K_q(f; \mathcal{B}) = \text{Ker}(f_* : H_q(M; \mathcal{B}) \rightarrow H_q(Q; \mathcal{B})),$$

where \mathcal{B} is an arbitrary local coefficient system.

It follows that the exact sequence

$$0 \longrightarrow K_q(f; \mathcal{B}) \longrightarrow H_q(M; \mathcal{B}) \longrightarrow H_q(Q; \mathcal{B}) \longrightarrow 0$$

splits. Moreover, the restrictions of the intersection forms

$$\mu_M^\Lambda : H_2(M; \Lambda) \times H_2(M; \Lambda) \rightarrow \Lambda$$

and

$$\mu_M^\mathbb{Z} : H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z}$$

to $K_2(f; \Lambda)$ and $K_2(f; \mathbb{Z})$, respectively, are non-degenerate (see for instance [16]).

In particular, we obtain the following consequence.

Corollary 2.4. $\mu_M^\mathbb{Z} \cong \mu_Q^\mathbb{Z} \oplus \mu_M^\mathbb{Z}|_{K_2(f; \mathbb{Z})}$

Lemma 2.5. $K_2(f; \mathbb{Z})$ is isomorphic to $H_2(M; \Lambda) \otimes_\Lambda \mathbb{Z}$.

Proof. First we note that $K_q(f; \mathcal{B})$ can be identified with $H_{q+1}(Q, M; \mathcal{B})$ for any local coefficient system \mathcal{B} . Then we consider the universal coefficient spectral sequence:

$$\text{Tor}_p^\Lambda(H_{q+1}(Q, M; \Lambda), \mathbb{Z}) \implies H_{p+q+1}(Q, M; \mathbb{Z}).$$

Since $H_q(Q; \Lambda) \cong 0$ for $q \neq 0$ and $H_q(M; \Lambda) \cong 0$ for $q \neq 0, 2$, we have

$$H_{q+1}(Q, M; \Lambda) \cong \begin{cases} 0 & \text{if } q \neq 2 \\ H_2(M; \Lambda) & \text{if } q = 2. \end{cases}$$

Therefore it follows that

$$\begin{aligned} K_2(f; \mathbb{Z}) &\cong H_3(Q, M; \mathbb{Z}) \cong \text{Tor}_0^\Lambda(H_3(Q, M; \Lambda), \mathbb{Z}) \\ &\cong H_3(Q, M; \Lambda) \otimes_\Lambda \mathbb{Z} \cong H_2(M; \Lambda) \otimes_\Lambda \mathbb{Z} \end{aligned}$$

as claimed. □

Lemma 2.6. The Λ -module $H_2(M; \Lambda) \cong K_2(f; \Lambda)$ is stably Λ -free. Moreover, $H_2(M; \Lambda)$ has a preferred s -base.

Proof. This is the assertion of Lemma 2.3 (c) in [16]. Indeed, we have $K_q(f; \Lambda) \cong 0$ for $q \neq 2$, i.e. $H_q(f; \Lambda) \cong H_q(Q, M; \Lambda) \cong 0$ for $q \neq 3$. Moreover, we have

$$H^4(f; \mathcal{B}) \cong K^3(f; \mathcal{B}) \cong K_1(f; \mathcal{B}) \cong 0$$

for any local coefficient system \mathcal{B} . Therefore the hypothesis of Lemma 2.3 (c) in [16] are verified. This completes the proof. \square

REMARK. The s -base of the Λ -module $K_2(f; \Lambda) \cong H_2(M; \Lambda)$ is determined by the CW-structure considered in M .

Since

$$H_q(Q, M; \Lambda) \cong \begin{cases} 0 & \text{if } q \neq 3 \\ H_2(M; \Lambda) & \text{if } q = 3 \end{cases}$$

and $H_2(M; \Lambda)$ is stably Λ -free, we obtain $H_3(Q, M; \Lambda) \otimes_{\Lambda} \mathbb{Z} \cong H_3(Q, M; \mathbb{Z})$ as proved in Lemma 2.5. In other words,

$$K_2(f; \mathbb{Z}) \cong H_2(M; \Lambda) \otimes_{\Lambda} \mathbb{Z} \cong \Pi_2(M) \otimes_{\Lambda} \mathbb{Z}$$

is \mathbb{Z} -free, of rank r say, i.e., the restriction $\mu_M^{\mathbb{Z}}|_{K_2(f; \mathbb{Z})}$ is an unimodular symmetric non-degenerate form. By the fundamental result of Freedman (see [6] and [7]) there is a simply-connected closed topological 4-manifold M' such that

$$\mu_{M'}^{\mathbb{Z}} = \mu_M^{\mathbb{Z}}|_{K_2(f; \mathbb{Z})}.$$

Lemma 2.7. *There exists a map*

$$\psi : (Q \# M') \setminus \overset{\circ}{D}^4 \rightarrow M$$

which induces isomorphisms on Π_1 and on $H_2(\cdot; \mathbb{Z})$.

Proof. First we observe that $(Q \# M') \setminus \overset{\circ}{D}^4$ is homotopy equivalent to the wedge $(Q \setminus \overset{\circ}{D}^4) \vee (M' \setminus \overset{\circ}{D}^4)$. Now $M' \setminus \overset{\circ}{D}^4$ is homotopy equivalent to $\vee_r \mathbb{S}^2$ and by the above isomorphism

$$K_2(f; \mathbb{Z}) \cong \Pi_2(M) \otimes_{\Lambda} \mathbb{Z}$$

we can represent a basis of $H_2(M; \mathbb{Z})$ by a map $\varphi : \vee_r \mathbb{S}^2 \rightarrow M$.

Let us define

$$\psi = j \vee \varphi : (Q \# M') \setminus \overset{\circ}{D}^4 \rightarrow M.$$

Obviously, the induced homomorphism

$$\psi_* : \Pi_1((Q \# M') \setminus \overset{\circ}{D}^4) \rightarrow \Pi_1(M)$$

is bijective.

Let us consider the following diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & K_2(f; \mathbb{Z}) & \rightarrow & H_2(M; \mathbb{Z}) & \xrightarrow{f_*} & H_2(Q; \mathbb{Z}) & \rightarrow & 0 \\
 & & & & \uparrow j_* \vee \varphi_* & & \cong \uparrow i_* & & \\
 & & & & H_2((Q \setminus \overset{\circ}{D}^4) \vee (\vee_r \mathbb{S}^2); \mathbb{Z}) & \xrightarrow{c_*} & H_2(Q \setminus \overset{\circ}{D}^4; \mathbb{Z}), & &
 \end{array}$$

where $c : (Q \setminus \overset{\circ}{D}^4) \vee (\vee_r \mathbb{S}^2) \rightarrow Q \setminus \overset{\circ}{D}^4$ is the projection map. By construction we have $\text{Ker } c_* \cong \text{Ker } f_*$, hence $\text{Ker}(j_* \vee \varphi_*) \cong 0$. Moreover,

$$\varphi_* : H_2(M' \setminus \overset{\circ}{D}^4; \mathbb{Z}) \xrightarrow{\cong} K_2(f; \mathbb{Z}),$$

hence φ_* is surjective. This completes the proof of the lemma. □

Let us denote $M_1 = Q \# M'$. In Section 3 we will show that the map ψ is a homotopy equivalence from $M_1^{(3)}$ to $M^{(3)}$. So it induces an isomorphism

$$\psi_* : \Pi_2(M_1^{(3)}) \cong H_2(M_1; \Lambda) \rightarrow \Pi_2(M^{(3)}) \cong H_2(M; \Lambda).$$

We can complete the proof of the first statement in Theorem 1.2 under this hypothesis by using a method described in [1]. By construction,

$$\psi_* : H_2(M_1^{(3)}; \mathbb{Z}) \cong H_2(M_1; \mathbb{Z}) \rightarrow H_2(M^{(3)}; \mathbb{Z}) \cong H_2(M; \mathbb{Z})$$

is an isomorphism of the \mathbb{Z} -intersection forms. Obviously, the Λ -intersection form

$$\mu_{M_1}^\Lambda : H_2(M_1; \Lambda) \times H_2(M_1; \Lambda) \rightarrow \Lambda$$

is extended from $\mu_{M_1}^\mathbb{Z} : H_2(M_1; \mathbb{Z}) \times H_2(M_1; \mathbb{Z}) \rightarrow \mathbb{Z}$. By hypothesis, this holds also in M . Therefore, $\psi_* : H_2(M_1; \Lambda) \rightarrow H_2(M; \Lambda)$ is an isomorphism of the Λ -intersection forms. We can now apply the following result proved in [1] only for free fundamental groups (further information on closed 4-manifolds with free fundamental group or with infinite cyclic first homology can be found in [9], [11–13], [14] and [17]). Let X and Y be closed connected oriented 4-dimensional Poincaré spaces (in particular, 4-manifolds) with $\Pi_1(X) \cong \Pi_1(Y) \cong *_p \mathbb{Z}$ (free product of p factors \mathbb{Z} , $p \geq 1$). Then X is homotopy equivalent to Y if and only if the intersection pairings $(H_2(X; \Lambda), \mu_X^\Lambda)$ and $(H_2(Y; \Lambda), \mu_Y^\Lambda)$ are isomorphic, where Λ denotes here the group ring of $*_p \mathbb{Z}$. But one can verify that the proof of this result is based on the following facts: Π_1 is a finitely presentable torsion free infinite group, Π_2 is Λ -free (and whence on the use of the special Künneth formula), and the first k -invariant vanishes. These are all verified in our case. Observe that the fundamental group of an aspherical manifold is in fact torsion free since a $K(\mathbb{Z}_n, 1)$ can not be finite dimensional when $n > 1$ (see for example

[5]). Thus the result on the homotopy type holds also for fundamental groups $\Pi_1(Q)$ since $B\Pi_1 = Q$ and, in particular, $H_q(B\Pi_1) \cong 0$ for any $q \geq 5$. Moreover, $\Pi_1(Q)$ is a finitely presentable PD_4 -group of type FF, i.e., the augmentation Λ -module \mathbb{Z} has a finite resolution consisting of finitely generated free Λ -modules, where $\Lambda = \mathbb{Z}[\Pi_1(Q)]$ (see [10], Theorem 5). Thus the proof of the first statement in Theorem 1.2 has been completed.

3. Proof of Theorem 1.2.

We assume in this section that $k_M^3 = 0$. Let $j : Q^{(3)} \rightarrow M$ be as in Section 2. In that section we have chosen a basis in $K_2(f; \mathbb{Z}) \cong H_2(M; \mathbb{Z}) \cong H_2(M; \Lambda) \otimes_{\Lambda} \mathbb{Z}$ which defines a map

$$\varphi : \vee_1^r \mathbb{S}^2 \rightarrow M.$$

Let (e_1, \dots, e_r) be this basis, and let (e_1^*, \dots, e_r^*) be its dual in $\text{Hom}_{\mathbb{Z}}(H_2(M; \mathbb{Z}), \mathbb{Z})$. We can represent each element e_i^* by a map

$$e_i^* : M \rightarrow K(\mathbb{Z}, 2) = \mathbb{C}P^{\infty},$$

so we obtain a map

$$\prod_{i=1}^r e_i^* : M \rightarrow \prod_1^r \mathbb{C}P^{\infty}$$

which induces a map on the 2-skeleton

$$g : M^{(2)} \rightarrow \left(\prod_1^r \mathbb{C}P^{\infty} \right)^{(2)} = \vee_1^r \mathbb{S}^2.$$

By construction, the composite map

$$(M')^{(2)} = \vee_1^r \mathbb{S}^2 \xrightarrow{\varphi} M^{(2)} \xrightarrow{g} \vee_1^r \mathbb{S}^2 = (M')^{(2)}$$

is homotopic to the identity.

REMARK. It is not difficult to show that g extends to a degree 1 map from M to M' .

Let us consider the map

$$h = (f \times g)|_{M^{(2)}} : M^{(2)} \rightarrow (Q \times M')^{(2)} = Q^{(2)} \vee (M')^{(2)}.$$

Lemma 3.1. *The map h extends to a map $M^{(3)} \rightarrow M_1^{(3)}$, again denoted by h . Moreover, the induced homomorphism $h_* : \Pi_2(M^{(3)}) \rightarrow \Pi_2(M_1^{(3)})$ is bijective (here $M_1 = Q \# M'$ as usual).*

Proof. The obstruction for extending h over $M^{(3)}$ belongs to

$$H^3(M; \Pi_2(M_1^{(3)})) \cong H_1(M; \Pi_2(M_1^{(3)})) \cong 0$$

since $\Pi_2(M_1^{(3)}) \cong \Pi_2(M') \otimes_{\mathbb{Z}} \Lambda$ is Λ -free.

The composition

$$M_1^{(3)} \xrightarrow{j \vee \varphi} M^{(3)} \xrightarrow{h} M_1^{(3)}$$

induces an isomorphism on Π_2 , hence the homomorphism

$$h_* : \Pi_2(M^{(3)}) \rightarrow \Pi_2(M_1^{(3)})$$

is surjective. Because $\Pi_2(M_1^{(3)})$ is Λ -free, we have

$$\Pi_2(M^{(3)}) \cong \Pi_2(M_1^{(3)}) \oplus \text{Ker } h_*.$$

Observe that $\text{Ker } h_*$ is stably Λ -free (since the Λ -module $\Pi_2(M^{(3)}) \cong K_2(f; \Lambda)$ is stably Λ -free). But tensoring with $\otimes_{\Lambda} \mathbb{Z}$ gives isomorphisms

$$\Pi_2(M^{(3)}) \otimes_{\Lambda} \mathbb{Z} \cong \oplus_1 \mathbb{Z} \cong \Pi_2(M_1^{(3)}) \otimes_{\Lambda} \mathbb{Z},$$

hence $\text{Ker } h_* \otimes_{\Lambda} \mathbb{Z} \cong 0$. By Kaplansky's lemma (see for example [8] and [10]) we get $\text{Ker } h_* \cong 0$. □

Corollary 3.2. *The map $\psi = j \vee \varphi : M_1^{(3)} \rightarrow M^{(3)}$ is a homotopy equivalence.*

Proof. It suffices to show that the induced homomorphism

$$\psi_* : \Pi_3(M_1^{(3)}) \rightarrow \Pi_3(M^{(3)})$$

is bijective. For this, it is convenient to recall the Whitehead certain exact sequence [18] for a CW-complex X :

$$H_4(X; \Lambda) \longrightarrow \Gamma(\Pi_2(X)) \longrightarrow \Pi_3(X) \longrightarrow H_3(X; \Lambda) \longrightarrow 0.$$

Here $\Gamma(A)$ is the quadratic functor on the category of abelian groups, and $\Pi_3(X) \rightarrow H_3(X; \Lambda)$ is the Hurewicz homomorphism. This sequence is natural, hence the map ψ induces a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\Pi_2(M_1^{(3)})) & \longrightarrow & \Pi_3(M_1^{(3)}) & \longrightarrow & H_3(M_1^{(3)}; \Lambda) \longrightarrow 0 \\ & & \downarrow \Gamma(\psi_*) & & \downarrow \psi_* & & \downarrow \psi_* \\ 0 & \longrightarrow & \Gamma(\Pi_2(M^{(3)})) & \longrightarrow & \Pi_3(M^{(3)}) & \longrightarrow & H_3(M^{(3)}; \Lambda) \longrightarrow 0. \end{array}$$

It follows from Lemma 3.1 that $\Gamma(\psi_*)$ is an isomorphism. The claim is proved once we have $\psi_* : H_3(M_1^{(3)}; \Lambda) \xrightarrow[\cong]{} H_3(M^{(3)}; \Lambda)$. If we denote by

$$c : M_1^{(3)} \simeq Q^{(3)} \vee (M')^{(3)} \rightarrow Q^{(3)}$$

the projection map, then we can immediately see that $f \circ \psi \simeq c$. Observe that c is the restriction of the collapsing map $c : M_1 \rightarrow Q$. Now the result follows from the commutative diagram

$$\begin{array}{ccc} \Lambda \cong H_4(M, M^{(3)}; \Lambda) & \xrightarrow[\cong]{} & H_3(M^{(3)}; \Lambda) \\ f_* \downarrow \cong & & \cong \downarrow f_* \\ \Lambda \cong H_4(Q, Q^{(3)}; \Lambda) & \xrightarrow[\cong]{} & H_3(Q^{(3)}; \Lambda) \\ c_* \uparrow \cong & & \cong \uparrow c_* \\ \Lambda \cong H_4(M_1, M_1^{(3)}; \Lambda) & \xrightarrow[\cong]{} & H_3(M_1^{(3)}; \Lambda). \end{array} \quad \square$$

4. The torsion invariant.

Let us fix the manifolds M^4 and Q^4 with $\Pi_1(M) \cong \Pi_1(Q)$, where Q is aspherical and $H_2(M; \Lambda)$ is Λ -flat. Recall that M and Q are provided with CW-structures such that $M^{(3)} = M \setminus \overset{\circ}{D}^4$ and $Q^{(3)} = Q \setminus \overset{\circ}{D}^4$. Let $f : M \rightarrow Q$ be a classifying map.

We have proved in Section 2 that $H_3(f; \Lambda)$ is Λ -free. Moreover, $H_q(f; \Lambda) \cong 0$ for any $q \neq 3$. Under these conditions one can define the torsion

$$\tau(f) \in \text{Wh}(\Pi_1(Q))$$

of the map f . Namely, it is the torsion of the cellular complex of the pair $((M \times I) \cup_f Q, M \times 0)$, where $(M \times I) \cup_f Q$ denotes the mapping cylinder of f , that is the quotient space

$$(M \times I) \cup_f Q = \frac{(M \times I) \cup Q}{\{(x, 1) \equiv f(x)\}}$$

$I = [0, 1]$. The torsion $\tau(f)$ is defined upon the choice of a Λ -basis of

$$H_3(f; \Lambda) \cong H_2(M; \Lambda).$$

Hence we shall denote it by $\tau_e(f)$, where $e = (e_1, e_2, \dots, e_r)$ indicates a Λ -basis of $H_2(M; \Lambda)$ (see [15]).

Let us consider the particular case of the collapsing map $g : Q \# M' \rightarrow Q$, where M' is simply-connected. Then we have the following (non surprising) result.

Lemma 4.1. *The torsion $\tau_e(g)$ vanishes for any Λ -basis e of $H_3(g; \Lambda)$.*

Proof. Let us consider the following diagram of inclusions:

$$\begin{array}{ccc} X_2 = (Q \setminus \overset{\circ}{D}^4) \times I \cup_{g_1} Q & \longrightarrow & X = (Q \# M') \times I \cup_g Q \\ \uparrow & & \uparrow \\ X_1 = (Q \setminus \overset{\circ}{D}^4) \times 0 & \longrightarrow & X_3 = (Q \# M') \times 0, \end{array}$$

where $g_1 = g|_{Q \setminus \overset{\circ}{D}^4}$. Note that the torsions of each pair are defined because

$$H_2(X_2, X_1; \Lambda) \cong \Lambda \quad \text{and} \quad H_2(X_3, X_1; \Lambda) \cong H_2(M; \Lambda).$$

Moreover, $\tau_e(X, X_3) = \tau_e(g)$ by definition and formula

$$(*) \quad \tau(X, X_2) + \tau(X_2, X_1) = \tau_e(g) + \tau(X_3, X_1)$$

holds. Since $g_1 : Q \setminus \overset{\circ}{D}^4 \rightarrow Q$ is the inclusion, it follows that $\tau(X_2, X_1) = 0$ for any choice of a generator in $H_2(X_2, X_1; \Lambda) \cong \Lambda$. Because $X \setminus X_2 = (M' \setminus \overset{\circ}{D}^4) \times [0, 1[$ and $X_3 \setminus X_1 \simeq M'$ are simply-connected, Lemma 7.3 of [15] implies that $\tau(X, X_2)$ and $\tau(X_3, X_1)$ are both zero for any basis of $H_2(X, X_2; \Lambda)$ and $H_2(X_3, X_1; \Lambda)$, respectively. Note that the other condition in Lemma 7.3 of [15], concerning the universal covering space, is also satisfied. The result $\tau_e(g) = 0$ then follows from formula (*). \square

Let now $\psi : Q \# M' \rightarrow M$ be the homotopy equivalence constructed above. In particular, the composition map

$$Q \# M' \xrightarrow{\psi} M \xrightarrow{f} Q$$

is homotopic to g . Hence, by applying the standard formulae:

$$0 = \tau_e(g) = \tau(\psi) + \tau_e(f),$$

we obtain $\tau(\psi) = -\tau_e(f)$ for any basis e of $H_3(f; \Lambda) \cong H_2(M; \Lambda)$. Any other classifying map $f' : M \rightarrow Q$ can be written as $\alpha \circ f \circ \beta$, up to homotopy, where $\alpha : Q \rightarrow Q$ and $\beta : M \rightarrow M$ are homotopy equivalences. Let $\psi' : Q \# M' \rightarrow M$ be the resulting homotopy equivalence. Moreover, we can assume that

$$\beta_* = \text{id} : \Pi_1(M) \rightarrow \Pi_1(M)$$

because the effect on Π_1 can be transmitted to α . Now we observe that the effect of β on $\tau(f')$ is restricted to $\beta_* : H_2(M; \Lambda) \xrightarrow{\cong} H_2(M; \Lambda)$, i.e., to a change of basis in

$H_2(M; \Lambda)$. But this has no influence on $\tau(\psi)$. On the other hand, any α changes $\tau(\psi)$. In fact, we have

$$\tau(\psi') = \tau(\psi) + \tau(\alpha).$$

Now we note that the group of homotopy classes of (orientation-preserving) homotopy self-equivalences of Q is isomorphic to $\text{Out}(\Pi_1(Q))$. Hence we can define

$$\tau(M) = [\tau(\psi)] \in \text{Wh}(\Pi_1(Q)) / \text{Out}(\Pi_1(Q)).$$

The following completes our main result.

Theorem 4.2. *The map $\psi : Q \# M' \rightarrow M$ is a simple homotopy equivalence if and only if $\tau(M) = 0$.*

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