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ENDOMORPHISMS OF HOMOGENEOUS SPACES OF LIE GROUPS

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If H is a closed subgroup of a topological group G it is well-known that there is a bijection

 $\operatorname{Map}_{G}(G/H, G/H) \xrightarrow{\sim} (G/H)^{H}$

which is actually a homeomorphism when the mapping space is equipped with compact-open topology. Homeomorphisms correspond to the subspaces

Homeo_G(G/H) \cong NH/H.

Our main purpose is to prove

Theorem. If G is a Lie group and H is a closed subgroup then NH/H is open in $(G/H)^{H}$.

In [tD, Ch. IV.1] Tammo tom Dieck defines a universal additive invariant U(G) of pointed finite G-CW-complexes for arbitrary topological groups G and computes it for compact Lie groups. As a corollary we obtain that his result is valid for arbitrary Lie groups, too.

Corollary. U(G) is a free abelian group on elements $u(G/H^+)$ where H runs through a complete set of conjugacy classes of closed subgroups H in G for any Lie group G.

The condition

(O)
$$NH/H$$
 is open in $(G/H)^H$

was introduced in a study with Wolfgang Lück [LL] in order to define the equivariant Lefschetz class of a G-endomorphism $f: X \to X$ of a finite G-CW-complex.

The inverse images of the subspaces $NH/H \subset (G/H)^H \subset G/H$ are

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$$NH = \{g \in G | g^{-1}Hg = H\}$$
 and $SH = \{g \in G | g^{-1}Hg \subset H\}$

and we claim that NH is open in SH, when G is a Lie group. It is well-known that NH = SH when H is compact. As H is closed in G, both NH and SH are always closed in G, so that for Lie groups G the fixed point space splits as a topological sum

$$(G/H)^H = NH/H + (G/H)^{> H}$$

The Lie theory we need can be found e.g. in the books Helgason [H, Ch.II] or Kawakubo [K, Ch.3].

Reduction to a discrete subgroup

Let G be a Lie group and H be a closed subgroup. We first claim that it suffices to prove the Theorem for all Lie groups G in the case where H is discrete. Indeed, let H_0 denote the unit component of H. Then H_0 is a closed and open subgroup of H and $H/H_0 = \pi_0(H)$. If $g^{-1}Hg \subset H$ then $g^{-1}H_0g \subset H$ is a connected set which contains e, whence $g^{-1}H_0g \subset H_0$. Then it holds for the Lie algebras that $L(g^{-1}H_0g) \subset L(H_0)$, but as they have the same dimension they must coincide. By connectedness $g^{-1}H_0g = H_0$ and $NH \subset SH \subset N(H_0)$. We can therefore assume that $G = N(H_0)$, i.e. that H_0 is normal in G. Then the normalizer of the discrete subgroup $\pi_0(H) = H/H_0$ of G/H_0 is $N\pi_0(H) = NH/H_0$, $S\pi_0(H) = SH/H_0$ and it clearly suffices to prove the claim for the subgroup $\pi_0(H)$ of G/H_0 .

Lie algebra of the centralizer

Let G be a Lie group and let H be a discrete closed subgroup of G. The centralizer

$$ZH = \{g \in G | ghg^{-1} = h \text{ for } h \in H\}$$

is a closed subgroup of G and is normal in NH for any closed H. When H is moreover discrete, then it holds $(NH)_0 = (ZH)_0$: each $g \in (NH)_0$ can be connected to e by a path g_t in NH. The corresponding conjugations $c_{g_t}: H \to H$ give a homotopy from c_g to $c_e = id_H$. As H is discrete, the homotopy is constant and therefore $c_g = id_H$, i.e. $g \in ZH$. We conclude that LNH = LZH.

Recall that the adjoint representation of G in LG is defined by attaching to an element $g \in G$ the differential Ad(g): $LG \to LG$ of the conjugation $c_g: G \to G$.

Lemma. $LZH = \{X \in LG | X = Ad(h)X \text{ for } h \in H\}.$

Proof. A closed subgroup H of a Lie group G is itself a Lie group with Lie

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algebra

$$LH = \{ X \in LG | exp(tX) \in H \text{ for } t \in \mathbf{R} \},\$$

see [H, Theorem II 2.3] or [K, Theorem 3.36]. Hence

$$LZH = \{X \in LG | exp(tX) \in ZH \text{ for } t \in \mathbf{R} \}$$
$$= \{X \in LG | c_h(exp(tX)) = exp(tX) \text{ for } h \in H, t \in \mathbf{R} \}$$

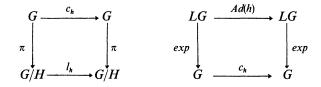
as Ad(h) is the differential of c_h , the last set equals to

$$= \{X \in LG | exp(tX) = exp(tAd(h)X) \text{ for } h \in H, t \in \mathbb{R} \}$$
$$= \{X \in LG | X = Ad(h)X \text{ for } h \in H \}$$

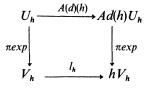
as exp is a diffeomorphism near the origin. This proves the Lemma.

Proof of the Theorem

Let G be a Lie group and let H be a closed discrete subgroup. The quotient space G/H is then a smooth manifold and the projection $\pi: G \to G/H$ is a smooth covering projection. Then the diagrams



commute for each $h \in H$, where $l_g: G/H \to G/H$ is left translation by g and $c_g: G \to G$ is conjugation by g. As the exponential map exp is a local diffeomorphism at the origin 0, exp(0)=e and similarly π is a local diffeomorphism at the unit element e and $\pi(e)=eH$, the composite πexp is a diffeomorphism of a small enough open disk $U_h \subset LG$ onto its image $V_h \subset G/H$. V_h is an open neighborhood of eH and the diagram



commutes. In paricular $U_h^{Ad(h)}$ is diffeomorphic to V_h^h .

By the Lemma $LZH = \{X \in LG | X = Ad(h)X \text{ for } h \in H\}$. As the spaces in question are finite-dimensional vector spaces, we can choose a finite set $h_1, h_2, \dots, h_n \in H$

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such that $LZH = \{X \in LG | X = Ad(h_i)X \text{ for } i = 1, \dots, n\}$. Let $U = \bigcap_{i=1}^{n} U_{h_i}$ and $V = \bigcap_{i=1}^{n} V_{h_i}$. The map πexp restricts to a diffeomorphism $U \to V$, which induces a diffeomorphism

$$U \cap LZH = \bigcap_{i=1}^{n} U_{h_i}^{Ad(h_i)} \xrightarrow{\sim} \bigcap_{i=1}^{n} V_{h_i}^{h_i} = V^{(h_1, \dots, h_n)}.$$

Choose U and consequently V is so small that $\pi exp(U \cap LZH) = V \cap (ZH/H)$ holds. Then the neighborhood V of the point eH = G/H satisfies

$$V \cap (ZH/H) = V \cap (G/H)^{\{h_1, \dots, h_n\}}$$

But clearly we have $ZH/H \subset NH/H \subset (G/H)^H \subset (G/H)^{\{h_1,\dots,h_n\}}$ so in fact equality

 $V \cap (NH/H) = V \cap (G/H)^H$

holds. Hence $NH/H \subset (G/H)^H$ is open at the point $eH \in NH/H$. Using the left action of NH/H we see that NH/H is open in $(G/H)^H$. This proves the Theorem.

Proof of the Corollary

Recall tom Dieck's definition of the universal additive invariant of a topological group G. An additive invariant consists of an pair (B, b), B an abelian group and b an assignment which associates to each pointed finite G-CW-complex X an element $b(X) \in B$ such that b(X) = b(Y) if X and Y are pointed G-homotopy equivalent and that the condition

$$b(X) = b(A) + b(X/A)$$

holds when A is a pointed subcomplex of X. An additive invariant (U, u) is universal if every other additive invariant factors through it uniquely. A universal additive invariant is uniquely determined by the usual Grothendieck construction, and it is denoted by (U(G), u).

It follows by an easy argument that U(G) is always generated by the classes $u(G/H^+)$ [tD, Proposition IV.1.8]. Although not explicitly stated, the proof that there are no relations between the classes $u(G/H^+)$ uses implicitly the fact that G is a compact Lie group since it is based on the Euler characteristics $\chi(X^H/NH)$, which are guaranteed to exist if G is a compact Lie group since then X^H/NH is a compact ENR but not otherwise (cf. the example given below.)

Let G be a Lie group. Using our Theorem we can alternatively proceed as follows. As noted in [LL, p. 495], the condition (O) implies that a G-CW-complex structure on X induces a relative NH/H-CW-complex structure on the pair $(X^H, X^{>H})$. The quotient $(X^H/NH, X^{>H}/NH)$ is then an ordinary relative CW-complex (possibly non-Hausdorff) whose cells correspond to the G-cells of X of type G/H. If n(X, H, i) is the number of such *i*-cells, it follows that the numbers

 $n(X,H) = \sum_{i \ge 0} n(X,H,i)$ are G-homotopy invariants of X as

$$n(X,H) = \chi(X^H/NH, X^{>H}/NH).$$

Then $(\mathbf{Z}, n(\mathbf{X}, H))$ is an additive invariant such that

$$n(G/H^+, H) = 1, n(G/K^+, H) = 0$$
 for K not conjugate to H.

This proves the Corollary.

Example

We conclude with an example taken from Fuchsian groups. Let $G = PSL(2, \mathbb{R})$ considered as the group of Möbius transformations

$$g(z) = \frac{az+b}{cz+d}, ad-bc=1, a, b, c, d \in \mathbf{R}$$

of the complex plane. Let H be the discrete subgroup of translations

$$h(z)=z+n, n\in \mathbb{Z}.$$

Then it is easy to check that the normalizer NH of H in G consists of translations

$$n(z) = z + b, \quad b \in \mathbf{R},$$

whereas SH equals the affine transformations

$$s(z) = mz + b, \quad m = 1, 2, \cdots, b \in \mathbf{R}.$$

In particular NH/H is a circle S^1 and $SH/H = \mathbf{N} \times S^1$.

Taking X = G/H gives an example of a finite G-CW-complex (a zero-cell) with X^H/NH a countable discrete set and therefore of infinite Euler characteristic.

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