Nishitani, T.
Osaka J. Math.
32 (1995), 41–69

ON LOCALIZATIONS OF A CLASS OF STRONGLY HYPERBOLIC SYSTEMS

TATSUO NISHITANI

(Received May 10, 1993)

1. Introduction

In a previous paper, henceforth quoted as [7], necessary conditions have been obtained for a first order system L to be strongly hyperbolic. In patricular, these conditions assert that, at an involutive characteristic, say z^0, the dimension of $\mathrm{Ker}L(z^0)$ must be equal to the order (or the multiplicity) of z^0 (Corollary 1.4 in [7]). Then the Taylor expansion of L along $\mathrm{Ker}L(z^0)$ starts with a linear term L_{z^0}, called the localization at z^0 (Section 3), which would be the first candidate for approximations of L on $\mathrm{Ker}L(z^0)$.

Unfortunately, in general, the localization is not diagonalizable even if the original system is strongly hyperbolic and the characteristic is involutive, in contrast with constant coefficient case (see Lemma 8 in [8] and Example 4.1 below). Our aim in this paper is then to make more detailed studies on localizations at an involutive characteristic of strongly hyperbolic systems.

Our first result is concerned with the localization L_{z^0} at an involutive characteristic z^0 of order r of a strongly hyperbolic system L. Hence L_{z^0} is a $r\times r$ system. Then we show that every $(r-1)$-th minor of L_{z^0} vanishes of order $s-2$ at every characteristic of order s of L_{z^0} (Theorem 4.1). This means that the localization must satisfy the same necessary condition which is verified by the original strongly hyperbolic system (see Theorem 1.1 in [7]).

The second result is stated as: Let z^0, z^1 be characteristics of the original system L and of the localization L_{z^0} of order r and s respectively. Then, assuming that the characteristic set of L is an involutive C^∞ manifold, every $(r-1)$-th minor of L_{z^0} vanishes of order $s-1$ at z^1 if (z^0,z^1) is "involutive" (Theorem 5.1). In particular, the dimension of $\mathrm{Ker}L_{z^0}(z^1)$ is equal to s and hence $L_{z^0}(z^1)$ is diagonalizable. If the characteristic z^0 is non degenerate (Definition 3.3) and (z^0,z^1) is "involutive" for every characteristic z^1 of L_{z^0}, then the localization L_{z^0} is strongly hyperbolic, more precisely the coefficient matrices of L_{z^0} are simultaneously symmetrizable (Theorem 5.2). We also show that the same results hold under less restrictive assumptions on the characteristics, which though, are not coordinate free (Propositions 6.1 and 6.2). These results, applied to the constant coefficients case, generalize Theorem 1 in [8].

2. Higher order localizations

Let $h(x)$ be a monic polynomial in x_1 of degree m:

$$h(x)=x_1^m+\sum_{j=1}^{m} a_j(x')x_1^{m-j}$$

where $a_j(x')\in C^\infty(U)$, $x'=(x_2,\cdots,x_n)$ and U is an open neighborhood of the origin of $\boldsymbol{R}^{n-1}$. We assume that $h(x)$ is hyperbolic with respect to the x_1 variable, that is the equation $h(x)=0$ in x_1 has only real roots for every $x'\in U$. Let $x^0\in\boldsymbol{R}\times U=\Omega$ be a characteristic of h of order r_0:

$$d^jh(x^0)=0,\ j<r_0,\quad d^{r_0}h(x^0)\neq 0.$$

We define $h_{x^0}(x)$ as

$$h(x^0+\mu x)=\mu^{r_0}(h_{x^0}(x)+O(\mu)),\ \mu\to 0$$

which is a well defined homogeneous polynomial of degree r_0 on $T_{x^0}\Omega/\Lambda_{x^0}(h)$ where $\Lambda_{x^0}(h)$ is the lineality of $h_{x^0}(x)$ defined as

$$\Lambda_{x^0}(h)=\{x\in T_{x^0}\Omega\,|\,h_{x^0}(y+tx)=h_{x^0}(y),\quad \forall t\in\boldsymbol{R},\quad \forall y\in T_{x^0}\Omega\}$$

which is a linear subspace in $T_{x^0}\Omega$ (see [1], [2]). Moreover $h_{x^0}(x)$ is hyperbolic with respect to the x_1 variable (cf. Lemma 1.3.3 in [3]).

In the following we denote by μ_0, μ_1 two small parameters with $0<\mu_0\leq\mu_1\ll 1$.

Lemma 2.1. *Let $x^1\in T_{x^0}\Omega/\Lambda_{x^0}(h)$ be a characteristic of order r_1 of h_{x^0} and let $y\in\Lambda_{x^0}(h)$. Then we have*

$$h(x^0+\mu_0(x^1+y)+\mu_0\mu_1x)=\mu_0^{r_0}\mu_1^{r_1}\ (h_1(y,x,\mu_0/\mu_1)+\mu_1g_1(y,x,\mu_1,\mu_0/\mu_1))$$

where $h(y,x,s)$ is a polynomial in (y,x,s), homogeneous of degree r_1 in (x,s) which is hyperbolic with respect to the x_1 variable and $g_1(y,x,\mu_1,s)$ is C^∞ in $|\mu_1|+|\mu_0\mu_1x|+|\mu_0y|<\varepsilon$, $|s|<2$ with sufficiently small $\varepsilon>0$.

Proof. It is clear that we can write

$$h(x^0+\mu_0x)=\mu_0^{r_0}(h_{x^0}(x)+\mu_0g_0(x,\mu_0))$$

where $g_0(x,\mu_0)$ is C^∞ in $|\mu_0|+|\mu_0x|<\varepsilon$ with small ε. By Rouché's theorem and the hyperbolicity of h it follows that

$$h_{x^0}(x^1+x)+\mu_0g_0(x^1+y+x,\mu_0)=0$$

has r_1 real zeros converging to zero with $(x', \mu_0) \to (0,0)$. Applying Lemma 1.3.3 in [3] we obtain

$$h(x^0 + \mu_0(x^1 + y + x)) = \mu_0^{r_0}(h_1(y, x, \mu_0) + \tilde{g}_0(y, x, \mu_0)) \tag{2.1}$$

where $h_1(y, x, \mu_0)$ is a polynomial in (y, x, μ_0), homogeneous in (x, μ_0) of degree r_1 which is hyperbolic with respect to the x_1 variable and $\tilde{g}_0(y, x, \mu_0)$ is C^∞ in $|\mu_0| + |\mu_0 x| + |\mu_0 y| < \varepsilon$ with small ε of the form

$$\tilde{g}_0(y, x, \mu_0) = \sum_{|\alpha| + j = r_1 + 1} x^\alpha \mu_0^j G_{\alpha j}(y, x, \mu_0).$$

Here note that

$$\begin{aligned} \tilde{g}_0(y, \mu_1 x, \mu_0) &= \mu_1^{r_1 + 1} \sum_{|\alpha| + j = r_1 + 1} x^\alpha (\mu_0/\mu_1)^j G_{\alpha j}(y, \mu_1 x, \mu_1(\mu_0/\mu_1)) \\ &= \mu_1^{r_1 + 1} \tilde{g}_1(y, x, \mu_1, \mu_0/\mu_1). \end{aligned}$$

It is clear that g_1 is C^∞ in $|\mu_1| + |\mu_0 \mu_1 x| + |\mu_0 y| < \varepsilon$, $|\mu_0/\mu_1| < 2$ with small ε. Then replacing x by $\mu_1 x$ in (2.1) we get the desired result. □

We are interested in the case when either $\mu_0 = \mu_1$ or $\mu_0 = O(\mu_1^{m+1})$. In the former case we set

$$h_{\{x^0, x^1\}}(y, x) = h_1(y, x, 1), \; g_1(y, x, \mu) = \mu \tilde{g}_1(y, x, \mu, 1)$$

so that

$$h(x^0 + \mu(x^1 + y) + \mu^2 x) = \mu^{r_0 + r_1}(h_{\{x^0, x^1\}}(y, x) + g_1(y, x, \mu)) \tag{2.2}$$

where g_1 is C^∞ in $|\mu| + |\mu^2 x| + |\mu y| < \varepsilon$ with small ε and $g_1(y, x, 0) = 0$. In the latter case we set

$$h_{(x^0, x^1)}(y, x) = h_1(y, x, 0),$$

$$g_1(y, x, \mu_1, \mu_0/\mu_1) = \mu_1 \tilde{g}_1(y, x, \mu_1, \mu_0/\mu_1) + h_1(y, x, \mu_0/\mu_1) - h_1(y, x, 0)$$

so that

$$h(x^0 + \mu_0(x^1 + y) + \mu_0 \mu_1 x) = \mu_0^{r_0} \mu_1^{r_1}(h_{(x^0, x^1)}(y, x) + g_1(y, x, \mu_1, \mu_0/\mu_1)) \tag{2.3}$$

where $g_1(y, x, \mu_1, \mu_0/\mu_1)$ is C^∞ in $|\mu_1| + |\mu_0 \mu_1 x| + |\mu_0 y| < \varepsilon$ with small $\varepsilon > 0$ and $g_1(y, x, 0, 0) = 0$. Note that by definition we have

$$h_{(x^0, x^1)}(y, x + w) = h_{(x^0, x^1)}(y, x), \; h_{\{x^0, x^1\}}(y, x + w) = h_{\{x^0, x^1\}}(y, x) \tag{2.4}$$

for every $w \in \Lambda_{x^0}(h)$, that is they depend only on the class of x in $T_{x^0}\Omega/\Lambda_{x^0}(h)$.
The following lemma is easily verified.

Lemma 2.2. *$h_{(x^0,x^1)}(y,x)$ is independent of $y \in \Lambda_{x^0}(h)$ and we have*

$$h_{(x^0,x^1)}(x) = (h_{x^0})_{x^1}(x).$$

In particular, $h_{(x^0,x^1)}(x)$ is independent of the choice of parameters μ_j provided that $\mu_0 = O(\mu_1^{m+1})$.

Note that Lemma 2.1 shows that

$$h_{\{x^0,x^1\}}(y, \lambda x) = h_1(y, \lambda x, 1) = \lambda^{r_1} h_1(y, x, 1/\lambda)$$

which implies that

$$\lim_{\lambda \to \infty} \lambda^{-r_1} h_{\{x^0,x^1\}}(y, \lambda x) = h_1(y, x, 0) = h_{(x^0,x^1)}(x), \tag{2.5}$$

that is $h_{(x^0,x^1)}(x)$ is the principal part of $h_{\{x^0,x^1\}}(y,x)$ with respect to x. Denoting by $\Lambda_{(x^0,x^1)}(h)$ the lineality of $h_{(x^0,x^1)}$, it follows from (2.4) that

$$\Lambda_{x^0}(h) \subset \Lambda_{(x^0,x^1)}(h). \tag{2.6}$$

Lemma 2.3. *We have*

$$h_{\{x^0,x^1\}}(y, x+w) = h_{\{x^0,x^1\}}(y, x), \quad \forall w \in \Lambda_{(x^0,x^1)}(h).$$

Proof. Since $h_{(x^0,x^1)}(x)$ is the principal part of $h_{\{x^0,x^1\}}(y,x)$ with respect to x and $h_{\{x^0,x^1\}}(y,x)$ is hyperbolic with respect to the x_1 variable, the assertion follows from Corollary 12.4.8 in [2]. $\square$

Set

$$H_l(x^0; x) = \sum_{|\beta|=l} h^{(\beta)}(x^0) x^\beta / \beta!$$

where $h^{(\beta)}(x^0) = \partial^\beta h(x^0)/\partial x^\beta$. Then

Lemma 2.4. *Let x^0, $x^1 \in T_{x^0}\Omega/\Lambda_{x^0}(h)$ be characteristics of h, h_{x^0} of order r and s respectively. Assume that $\Lambda_{(x^0,x^1)}(h)$ is given by $x_a = 0$ where $x = (x_a, x_b)$ is a partition of the variable x. Then we have*

$$H_l^{(\alpha)}(x^0; x^1 + y) = 0, \quad \forall y \in \Lambda_{x^0}(h), \quad l + |\alpha| = r + s$$

unless $\alpha = (\alpha_a, 0)$.

Proof. By definition we see easily that

$$h_{\{x^0,x^1\}}(y,x)=\sum_{l+|\beta|=r+s} H_l^{(\beta)}(x^0;x^1+y)x^\beta/\beta!.$$

Since Lemma 2.3 shows that $h_{\{x^0,x^1\}}(y,x)$ is a polynomial in (y,x_a), we obtain the desired result. □

We now study how $h_{\{x^0,x^1\}}(y,x)$ depends on $y\in\Lambda_{x^0}(h)$ assuming that

$$\Sigma=\{x\in\Omega\,|\,d^jh(x)=0, j< r, d^rh(x)\neq 0\}$$

is a C^∞ manifold through x^0. For $y\in\Sigma$ and $x\in N_y\Sigma=T_y\Omega/T_y\Sigma$ we define $h_\Sigma(y,x)$ by

$$h_\Sigma(y,x)=\lim_{\mu\to 0}\mu^{-r}h(y+\mu x) \tag{2.7}$$

which is well defined on the normal bundle $N\Sigma$ of Σ. Take the local coordinates $x=(x_a,x_b)$ for which Σ is defined by $x_a=0$. In these coordinates, h and h_Σ are given by

$$h(x_a,x_b)=\sum_{|\alpha|=r} C_\alpha(x_b,x_a)x_a^\alpha,\ h_\Sigma(x_a,x_b)=\sum_{|\alpha|=r} C_\alpha(x_b,0)x_a^\alpha$$

where $C_\alpha(x_b,x_a)$ are C^∞. Let $x^0\in\Sigma$, $x^1\in N_{x^0}\Sigma\backslash 0$ be characteristics of h, h_{x^0} of order r and s respectively. In our coordinates, $x^0=(x_b^0,0)$, $x^1=(0,x_a^1)$ and $(x_b^0,x_a^1)\in N\Sigma\backslash\Sigma$. Remark that (x_b^0,x_a^1) is a characteristic of order s of h_Σ because

$$h_{x^0}(x_a^1+x_a)=h_\Sigma(x_b^0,x_a^1+x_a)$$

and hence $h_\Sigma(x_b^0,x_a^1+x_a)$ is hyperbolic with respect to the variable x_1 and has the zero $x_1=0$ of order s when $x_{a'}=0$ with $x_a=(x_1,x_{a'})$.

Lemma 2.5. *Let x be the local coordinates as above. Then $h_{\{x^0,x^1\}}(x_b,x_a)$ is a polynomial of degree s with principal part $h_{\Sigma(x_b^0,x_a^1)}(x_b,x_a)$, the localization of h_Σ at (x_b^0,x_a^1).*

Proof. Set

$$h^*(x_b,x_a,\rho)=\sum_{|\alpha|=r} C_\alpha(x_b,\rho x_a)x_a^\alpha.$$

It is clear that $h^*(x_b^0+x_b, x_a^1+x_a, \rho)$ has the zero $x_1=0$ of order s when $(x_b, x_{a'}, \rho)=0$ and hyperbolic with respect to the variable x_1. Then it follows that

$$h^*(x_b^0+x_b, x_a^1+x_a, \rho)=h_0^*(x_b, x_a, \rho)+O(|x_b|+|x_a|+|\rho|)^{s+1}$$

where $h_0^*(x_b, x_a, \rho)$ is a homogeneous polynomial of degree s which is hyperbolic with respect to x_1. Since

$$h(x^0+\mu(x^1+x_b)+\mu^2 x_a)=\mu^r h^*(x_b^0+\mu x_b, x_a^1+\mu x_a, \mu)$$

it follows that $h_{\{x^0,x^1\}}(x_b, x_a)=h_0^*(x_b, x_a, 1)$. On the other hand, by definition, we have

$$h_\Sigma(x_b^0+\mu x_b, x_a^1+\mu x_a)=\mu^s(h_{\Sigma(x_b^0,x_a^1)}(x_b, x_a)+O(\mu))$$

and hence $h_{\Sigma(x_b^0,x_a^1)}(x_b, x_a)=h_0^*(x_b, x_a, 0)$ because $h_\Sigma(x_b, x_a)=h^*(x_b, x_a, 0)$. This proves the assertion. □

3. Localization of system

Let Ω be an open set in $\boldsymbol{R}^{n+1}$ with local coordinates $x=(x_0, x')$ where $x'=(x_1, \cdots, x_n)$ and let $T^*\Omega$ be the cotangent bundle over Ω with corresponding coordinates (x, ξ). Let L be a first order differential operator on $C^\infty(\Omega, \boldsymbol{C}^m)$ with symbol $L(x, \xi)\in C^\infty(T^*\Omega, \text{Hom}(\boldsymbol{C}^m))$. We denote by $h(x, \xi)$ the determinant of $L(x, \xi)$. Following [8] (see also [1]) we define the localization of $L(x, \xi)$ at a characteristic $z^0=(x^0, \xi^0)\in T^*\Omega\backslash 0$ of order r of h with

$$\dim \operatorname{Ker} L(z^0)=r.$$

Let π be the natural projection π: $\boldsymbol{C}^m\to\boldsymbol{C}^m/\operatorname{Im}L(z^0)$ and ι be the inclusion $\iota:\operatorname{Ker}L(z^0)\to\boldsymbol{C}^m$.

DEFINITION 3.1. We define $L_{z^0}(z)$ by

$$L_{z^0}(z)=\lim_{\mu\to 0}\mu^{-1}\pi L(z^0+\mu z)\iota,\ z\in T_{z^0}(T^*\Omega).$$

Taking bases for $\boldsymbol{C}^m$ and then for $\operatorname{Ker}L(z^0)$, $\operatorname{Ker}{}^tL(z^0)$, where ${}^tL(z^0)$ denotes the transposed of $L(z^0)$, we study $L_{z^0}(z)$. We choose $u_j, v_j\in\boldsymbol{C}^m$ so that

$$\operatorname{Ker}L(z^0)=\text{span}-\{u_1, \cdots, u_r\},\ \operatorname{Ker}{}^tL(z^0)=\text{span}-\{v_1, \cdots, v_r\}.$$

With $U=(u_1, \cdots, u_n)$, $V=(v_1, \cdots, v_n)$, which are $m\times r$ matrices, we set $L_{(U,V)}(z)={}^tVL(z)U$. Then in these bases, $L_{z^0}(z)$ is expressed by $L_{(U,V)z^0}(z)$:

$$L_{(U,V)z^0}(z)=\lim_{\mu\to 0}\mu^{-1}L_{(U,V)}(z^0+\mu z).$$

For another pair of bases $\tilde{U}$, $\tilde{V}$ for $\mathrm{Ker}L(z^0)$, $\mathrm{Ker}^tL(z^0)$ respectively it is clear that $L_{(\tilde{U},\tilde{V})}(z)=M_1L_{(U,V)}(z)M_2$ with some non singular $M_i\in M(r,\boldsymbol{C})$ and hence

$$L_{(\tilde{U},\tilde{V})z^0}(z)=M_1L_{(U,V)z^0}(z)M_2. \tag{3.1}$$

We next examine the effects of a change of basis for $\boldsymbol{C}^m$. Let $L^T(z)=T^{-1}L(z)T$ with a non singular $T\in M(m,\boldsymbol{C})$ and let U_1, V_1 be a pair of bases for $\mathrm{Ker}L^T(z^0)$, $\mathrm{Ker}^tL^T(z)$. Then it is also clear that

$$L^T_{(U_1,V_1)z^0}(z)=N_1L_{(U,V)z^0}(z)N_2 \tag{3.2}$$

with non singular $N_i\in M(r,\boldsymbol{C})$. From (3.1) the determinant of $L_{z^0}(z)$ is well defined up to non-zero multiple constant.

Lemma 3.1. *We have*

$$(\mathrm{det}L)_{z^0}(z)=\mathrm{det}L_{z^0}(z)$$

up to non-zero multiple constant.

Proof. As noted above, it is enough to show the assertion with suitably chosen bases U, V for $\mathrm{Ker}L(z^0)$, $\mathrm{Ker}^tL(z^0)$ and a basis for $\boldsymbol{C}^m$. After a change of basis for $\boldsymbol{C}^m$ we may assume that $L(z^0)=G\oplus O$ where $G\in M(m-r,\boldsymbol{C})$ is non singular and O denotes the zero matrix of order r. Write

$$L=\begin{pmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix}$$

where $L_{ij}(z^0)=O$ unless $(i,j)=(1,1)$ and $L_{11}(z^0)=G$. Thus choosing U, V suitably we have

$$L_{(U,V)}(z)=L_{22}(z).$$

Since $L_{11}(z^0+\mu z)=G+O(\mu)$, $L_{ij}(z^0+\mu z)=\mu L'_{ij}(z)+O(\mu^2)$ as $\mu\to 0$, we see that

$$\mathrm{det}L(z^0+\mu z)=\mu^r\{(\mathrm{det}G)\mathrm{det}L'_{22}(z)+O(\mu)\}$$

and hence $(\mathrm{det}L)_{z^0}(z)=(\mathrm{det}G)\mathrm{det}L'_{22}(z)$. On the other hand, by definition, we have $L_{(U,V)z^0}(z)=L'_{22}(z)$ and hence the assertion. □

From (3.1) it is clear that every s-th minor of $L_{(\tilde{U},\tilde{V})_{z^0}}(z)$ is a linear combination of s-th minors of $L_{(U,V)_{z^0}}(z)$ and *vice versa.*

Lemma 3.2. *Every $(r-1)$-th minor of $L_{z^0}(z)$ is a linear combination of $m_{z^0}(z)$'s where $m(z)$ are $(m-1)$-th minors of $L(z)$.*

Proof. It is enough to show the assertion for $L_{(U,V)_{z^0}}(z)$ with suitably chosen U, V and a basis for $\boldsymbol{C}^m$. As observed in the proof of Lemma 3.1 we may assume that

$$L(z^0+\mu z)=\begin{pmatrix} G+O(\mu) & O(\mu) \\ O(\mu) & \mu L'_{22}(z)+O(\mu^2) \end{pmatrix}.$$

Let $m(z)$ be the $(m-1)$-th minor of $L(z)$ obtained removing i-th row and j-th column of $L(z)$. Similarly we denote by $l(z)$ the so obtained $(r-1)$-th minor of $L'_{22}(z)$. Then it is clear that

$$m_{z^0}(z)=\mu^{r-1}\{(\det G)l(z)+O(\mu)\}$$

as $\mu\to 0$ and hence $l(z)=(\det G)^{-1}m_{z^0}(z)$, which proves the assertion. $\square$

Recall that $L_{z^0}(z)$ is $\mathrm{Hom}(\mathrm{Ker}L(z^0),\ \boldsymbol{C}^m/\mathrm{Im}L(z^0))$ valued linear function in z.

Definition 3.2. Let $L_{z^0}(z)=(\phi^i_j(z))$. We call

$$d(L_{z^0})=\dim\ \mathrm{span}-\{\phi^i_j\}$$

the reduced dimension of L_{z^0}.

Definition 3.3. Assume that $L(z)$ is real. Let z^0 be a characteristic of order r of h with $\dim\ \mathrm{Ker}L(z^0)=r$. We say that z^0 is non degenerate if

$$d(L_{z^0})\geq r(r+1)/2.$$

Let z^0 be a characteristic of h of order r again. Note that one can take $m\times r$ matrices $U(z)$ and $V(z)$, depending smoothly on z near z^0, verifying

$$\dim\ \mathrm{Ker}L(z)=r\Rightarrow U(z),\ V(z) \text{ are bases for } \mathrm{Ker}L(z),\ \mathrm{Ker}\,{}^tL(z) \text{ respectively.}$$

Then it is also clear that

$$L_z(w)=\lim_{\mu\to 0}\mu^{-1}M(z+\mu w) \tag{3.3}$$

with $M(z) = {}^tV(z)L(z)U(z)$ if $\dim \operatorname{Ker} L(z) = r$.

Lemma 3.3. *Assume that*

$$\Sigma = \{z \in T^*\Omega \mid d^j h(z) = 0, j < r, d^r h(z) \neq 0\}$$

is a C^∞ manifold through z^0 and that $\dim \operatorname{Ker} L(z) = r$, $z \in \Sigma$. *Then we have*

$$L_z(w) = O, \quad w \in T_z\Sigma, \quad z \in \Sigma. \tag{3.4}$$

In particular, if $z \in \Sigma$ then $d(L_z) = d(h_z)$.

Proof. Since $M(z) = O$ on Σ, the first assertion is clear by (3.3). To prove the second assertion, it is enough to note that if $z \in \Sigma$ then $d(L_z) \geq d(h_z)$ by Lemma 3.1. The opposite inequality follows from (3.4). □

4. Necessary condition (1)

Let

$$L(x, D) = \sum_{j=0}^{n} A_j(x) D_j$$

be a differential operator of order 1 on $C^\infty(\Omega, C^m)$. We assume that $h(x, \xi)$ is hyperbolic with respect to $t(x) \in C^\infty(\Omega)$, $dt(x) \neq 0$, that is

$$h(x, \xi + \lambda dt(x)) = 0$$

has only real roots for every $x \in \Omega$, $\xi \in T_x^*\Omega$. Let $\sigma = \Sigma_{j=0}^n d\xi_j \wedge dx_j$ be the canonical symplectic two form on $T^*\Omega$ and for $S \subset T_w(T^*\Omega)$ we denote by S^σ the annihilator of S with respect to σ:

$$S^\sigma = \{z \in T_w(T^*\Omega) \mid \sigma(z, u) = 0, \forall u \in S\}.$$

In what follows we assume that $t(x) = x_0$ and $A_0 = I_m$, the identity matrix of order m without restrictions. Recall that we say that L is strongly hyperbolic near the origin if the Cauchy problem for $L(x, D) + B(x)$ is correctly posed for every $B(x) \in C^\infty(\Omega, M(m, C))$ in both Ω^t, Ω_t with small t, where $\Omega^t = \{x \in \Omega | x_0 < t\}$ and $\Omega = \{x \in \Omega | x_0 > t\}$.

In this section we show the following result.

Theorem 4.1. *Assume that $A_j(x)$ are real analytic in Ω containing the origin. Let $z^0 \in T_0^*\Omega \backslash 0$, $z^1 \in T_{z^0}(T^*\Omega)$ be characteristics of order r and s of h* and $h_{z^0} = \det L_{z^0}$ *respectively with $\Lambda_{z^0}(h)^\sigma \subset \Lambda_{z^0}(h)$. If L is strongly hyperbolic near the origin then* *every $(r-1)$-th minor of L_{z^0} vanishes of order $s-2$ at z^1.*

This result is optimal in a sense. We give an example.

EXAMPLE 4.1: Let

$$L(z)=\begin{pmatrix} \xi_0 & \xi_1 & 0 \\ x_0^2\xi_1 & \xi_0 & 0 \\ 0 & 0 & \xi_0-2x_0\xi_1 \end{pmatrix}, \quad z^0=(0,e_n), \quad n\geq 2.$$

For this $L(z)$, it is not difficult to examine the following.

1) L is strongly hyperbolic near the origin (see Example 1.2. in [6]) and z^0 is a characteristic of order 3 of h with $\Lambda_{z^0}(h)^\sigma \subset \Lambda_{z^0}(h)$.

2)

$$L_{z^0}(z)=\begin{pmatrix} \xi_0 & \xi_1 & 0 \\ 0 & \xi_0 & 0 \\ 0 & 0 & \xi_0 \end{pmatrix}$$

and $z^1=(0,e_1)$ is a characteristic of $\det L_{z^0}(z)$ of order 3.

3) the 2-minor

$$\begin{vmatrix} \xi_1 & 0 \\ 0 & \xi_0 \end{vmatrix}$$

vanishes of order $1=3-2$ at z^1.

We first derive an a priori estimate for the well posed Cauchy problem which will be needed in the following sections also. Let $\sigma=(\sigma_0,\cdots,\sigma_n)\in \boldsymbol{Q}_+^{n+1}$ and set

$$y(\lambda)=y^0+\sum_{j=1}^{s} y^j\lambda^{-\varepsilon_j},\ \eta(\lambda)=\eta^0+\sum_{j=1}^{s}\eta^j\lambda^{-\varepsilon_j} \tag{4.1}$$

where $y^j,\ \eta^j\in \boldsymbol{R}^{n+1}$ and $\varepsilon_j\in \boldsymbol{Q}_+$, $0<\varepsilon_1<\varepsilon_2<\cdots<\varepsilon_s$. For a differential operator P on $C^\infty(\Omega, \boldsymbol{C}^m)$ with $C^\infty(\Omega)$ coefficients we set

$$P_\lambda(y(\lambda),\ \eta(\lambda);\ x,\xi)=P(y(\lambda)+\lambda^{-\sigma}x,\ \lambda^\kappa\eta(\lambda)+\lambda^\sigma\xi) \tag{4.2}$$

with $\kappa \in \boldsymbol{Q}_+$ where $\lambda^{-\sigma}=(\lambda^{-\sigma_0}x_0, \cdots, \lambda^{-\sigma_n}x_n)$ etc. Assuming that the Cauchy problem for $P(x,D)$ is correctly posed in both Ω^t and Ω_t for every small t, we derive an a priori estimate for $P_\lambda(x,D)$.

Proposition 4.2. *Let $\sigma \in \boldsymbol{Q}_+^{n+1}$ and κ, $\varepsilon_j \in \boldsymbol{Q}_+$. Assume that $0 \in \Omega$, $y^0=0$ and the Cauchy problem for $P(x,D)$ is correctly posed in both Ω^t and Ω_t for every small t. Then for every compact set $\tilde{Y}$, $\tilde{H} \subset \boldsymbol{R}^{(n+1)s}$, $W \subset \boldsymbol{R}^{n+1}$ and for every positive $T>0$ we can find $C>0$, $\bar{\lambda}>0$ and $p \in \boldsymbol{N}$ such that*

$$|u|_{C^0(W^t)} \le C\lambda^{(\bar{\sigma}+\kappa)p}|P_\lambda u|_{C^p(W^t)}, \quad |u|_{C^0(W_t)} \le C\lambda^{(\bar{\sigma}+\kappa)p}|P_\lambda u|_{C^p(W_t)}$$

for $\lambda \ge \bar{\lambda}$, $u \in C_0^\infty(W, \boldsymbol{C}^m)$, $|t|<T$, $Y=(y^1, \cdots, y^s) \in \tilde{Y}$, $H=(\eta^1, \cdots, \eta^s) \in \tilde{H}$ where $\bar{\sigma}=\max_j \sigma_j$.

Proof. Set

$$\tilde{P}(x,D)=e^{-i\lambda^\kappa<\eta(\lambda),x>}P(x,D)e^{i\lambda^\kappa<\eta(\lambda),x>}$$

so that $\tilde{P}(x,\xi)=P(x,\lambda^\kappa\mu(\lambda)+\xi)$. Let $K \subset \Omega$ be a compact set and recall Proposition 2.1 in [7]:

$$|v|D_{C^0(K^t)} \le C|Pv|D_{C^p(K^t)}, \quad |t|<\tau, \ v \in C_0^\infty(K, \boldsymbol{C}^m)$$

with an integer $p \in \boldsymbol{N}$ and a $\tau>0$. Since $|u|_{C^p(K^t)} \le C_1\lambda^{\kappa p}|e^{-i\lambda^\kappa<\eta(\lambda),x>}u|_{C^p(K^t)}$ it follows that $|u|_{C^0(K^t)} \le C_2\lambda^{\kappa p}|\tilde{P}(x,D)u|_{C^p(K^t)}$. Repeating the proof of Proposition 2.2 in [7] we get the desired assertion. □

Before going into details we recall what we have proved in [7]: Let $m(z)$ be a $(m-1)$-th minor of $L(z)$ and assume that

$$h_\lambda(y(\lambda), \eta(\lambda); x, \xi)=\lambda^\gamma\{\sigma_0(h)(Y,H;x,\xi)+o(1)\},$$
$$m_\lambda(y(\lambda), \eta(\lambda); x, \xi)=\lambda^{\gamma+\tau}\{\sigma_0(m)(Y,H;x,\xi)+o(1)\}$$

as $\lambda \to \infty$ where $Y=(y^1, \cdots, y^s)$, $H=(\eta^1, \cdots, \eta^s)$. Then if L is strongly hyperbolic near the origin and $\tau>0$, it follows that $\sigma_0(m)(Y,H;x,\xi)$ is divisible by $\sigma_0(h)(Y,H;x,\xi)$ as polynomials in ξ_0.

To show Theorem 4.1 we prepare several lemmas. Since $\Lambda_{z^0}(h)^\sigma \subset \Lambda_{z^0}(h)$, choosing suitable local coordinates x, preserving the plane $x_0=0$, we may assume that $z^0=(0,e_n)$ and

$$h_{z^0}(z)=Q(x_b, \xi_a) \tag{4.3}$$

with a homogeneous polynomial Q of degree s where $x=(x_a, x_b)$, $x_a=(x_0, \cdots, x_k)$, $x_b=(x_{k+1}, \cdots, x_n)$ is a partition of the variable x and $\xi=(\xi_a, \xi_b)$ is that of ξ (cf. Proposition

2.6 in [7]). Let $z^1=(y,\eta)$ and set

$$\sigma_j=1/3,\ 0\le j\le k,\ \sigma_j=2/3,\ k+1\le j\le n,$$
$$y(\lambda)=\lambda^{-1/3}y,\ \eta(\lambda)=e_n+\lambda^{-1/3}\eta.$$

We now study

$$h_\lambda(y(\lambda),\eta(\lambda);x,\xi)=\lambda^m h(z^0+\lambda^{-1/3}z^1+\lambda^{-1/3}(x_a,\xi_b)+\lambda^{-2/3}(x_b,\xi_a))$$
$$=\lambda^{m^*}(h_{\{z^0,z^1\}}(z_a,z_b)+O(\lambda^{-1/3}))$$

where $m^*=m-(r+s)/3$ and $z_a=(x_a,\xi_b)$, $z_b=(x_b,\xi_a)$. From (2.5) it follows that

$$h_{\{z^0,z^1\}}(z_a,z_b)=h_{(z^0,z^1)}(z_b)+\text{ polynominal in } \xi_0 \text{ of degree } \le s-1.$$

We summarize the above observations as:

Lemma 4.3. *Let $z^0\in T_0^*\Omega\backslash 0$, $z^1\in T_{z^0}(T^*\Omega)$ be characteristics of h and h_{z^0} of order r and s respectively. Then we have*

$$h_\lambda(y(\lambda),\eta(\lambda);x,\xi)=\lambda^{m^*}(\sigma_0(h)(y,\eta;x,\xi)+O(\lambda^{-1/3}))$$

where $\sigma_0(h)(y,\eta;x,\xi)$ is a polynomial in ξ_0 of degree s.

Let $m(x,\xi)$ be a $(m-1)$-th minor of $L(x,\xi)$ and hence of homogeneous of degree $m-1$ in ξ. Study $m_\lambda(y(\lambda),\eta(\lambda);x,\xi)$.

(4.4) $$m_\lambda(y(\lambda),\eta(\lambda);x,\xi)=\lambda^{m-1}m(z^0+\lambda^{-1/3}z^1+\lambda^{-1/3}(x_a,\xi_b)$$
$$+\lambda^{-2/3}(x_b,\xi_a))=\sum_{l,\alpha,\beta}\lambda^{G(l,\alpha,\beta)}M_{l(\beta)}^{(\alpha)}(z^0;z^1)x^\beta\xi^\alpha/\alpha!\beta!$$

where

$$M_{l(\beta)}^{(\alpha)}(z^0;z)=\sum_{|\alpha+\beta|=l}m_{(\beta)}^{(\alpha)}x^\beta\xi^\alpha/\alpha!\beta!,\ z=(x,\xi)$$

and $G(l,\alpha,\beta)=m-1-(l-|\alpha+\beta|)/3-|\beta_a+\alpha_b|/3-2|\alpha_a+\beta_b|/3$ with $\alpha=(\alpha_a,\alpha_b)$, $\beta=(\beta_a,\beta_b)$.

Lemma 4.4. *Assume that $m_{(\beta)}^{(\alpha)}(z^0)=0$ for $|\alpha+\beta|<r-1$ and*

$$\lambda^{-m^*}m_\lambda(y(\lambda),\eta(\lambda);x,\xi)=O(1),\ \lambda\to\infty.$$

Then we have $M_{l(\beta)}^{(\alpha)}(z^0;z^1)=0$ for $l+|\alpha+\beta|<r+s-3$.

Proof. Since $G(l,\alpha,\beta)\geq m-1-(l+|\alpha+\beta|)/3$, it follows that

$$G(l,\alpha,\beta)-m^{*}\geq(r+s-3-(l+|\alpha+\beta|))/3.$$

Now the proof is immediate. □

Lemma 4.5. *Assume that* $m^{(\alpha)}_{(\beta)}(z^0)=0$ *for* $|\alpha+\beta|<r-1$. *If* $G(l,\alpha,\beta)\geq m^{*}$ *then* $\alpha_0\leq s-2$.

Proof. Recall that $G(l,\alpha,\beta)=m-1-(l+|\alpha_a+\beta_b|)/3$. Suppose that $\alpha_0\geq s-1$ and $G(l,\alpha,\beta)\geq m^{*}$ with some l,α,β. Then since $l\geq r-1$ by hypothesis we get $G(l,\alpha,\beta)\leq m^{*}-1/3$ which is an obvious contradiction. □

Proof (of Theorem 4.1). Let $m(z)$ be a $(m-1)$-th minor of $L(z)$. We first remark that the dimension of $\mathrm{Ker}L(z^0)$ is equal to r by Corollary 1.4 in [7] and hence L_{z^0} is well defined. On the other hand the assertion

$$m^{(\alpha)}_{(\beta)}(z^0)=0,\ |\alpha+\beta|<r-1$$

follows from Theorem 1.3 in [7]. From Lemma 3.2, to prove the theorem, it is enough to show that $m^{(\alpha)}_{z^0(\beta)}(z^1)=0$, $|\alpha+\beta|<s-2$ for all $(m-1)$-th minors $m(z)$ of $L(z)$. Suppose that

$$m^{(\alpha)}_{z^0(\beta)}(z^1)\neq 0 \text{ for some } |\alpha+\beta|<s-2. \tag{4.5}$$

Write $\lambda^{-m^{*}}m_\lambda(y(\lambda),\eta(\lambda);x,\xi)=\lambda^{\tau}(\sigma_0(m)(y,\eta;x,\xi)+o(1))$ and note that $m_{z^0}(z)=M_{r-1}(z^0;z)$. From Lemma 4.4 with $l=r-1$ we would have $\tau>0$ if (4.5) were true. On the other hand, by Lemma 4.5 we see that $\sigma_0(m)(y,\eta;x,\xi)$ is a polynomial in ξ_0 of degree at most $s-2$. Since $\sigma_0(m)(y,\eta;x,\xi)$ is divisible by $\sigma_0(h)(y,\eta;x,\xi)$ this shows that $\sigma_0(m)=0$ and hence a contradiction. □

5. Necessary condition (2)

Let

$$\Sigma=\{z\in T^{*}\Omega\,|\,d^{j}h(z)=0, j<r, d^{r}h(z)\neq 0\}$$

be the set of characteristics of order r of h. We assume that Σ is an involutive C^∞ manifold through z^0 and let h_Σ be the localization of h along Σ defined by (2.7). Denote by α the canonical projection from $T^{*}\Omega$ onto Ω: $T^{*}\Omega\to\Omega$ and assume that

$$d\alpha_{z^0}\colon T_{z^0}(T^{*}\Omega)\to T_{\alpha(z^0)}\Omega \text{ is surjective on } T_{z^0}\Sigma. \tag{5.1}$$

Recall that on $N\Sigma$ we have an invariant symplectic two form, denoted by $\tilde{\sigma}$ and called the relative symplectic two form (see [4]) which is given by

$$\tilde{\sigma}=\sum_{j=0}^{k} dx_j^*\wedge dx_j=dx_a^*\wedge dx_a$$

where we have assumed that Σ is defined by $\xi_a=0$ and $N\Sigma$ is parametrized by $(x_a, x_b, \xi_b; x_a^*)$. Let $X\in N\Sigma\backslash 0$ and we introduce the condition;

$$\Lambda_X(h_\Sigma)^{\tilde{\sigma}}\subset\Lambda_X(h_\Sigma) \tag{5.2}$$

where $\Lambda_X(h_\Sigma)^{\tilde{\sigma}}$ denotes the annihilator of $\Lambda_X(h_\Sigma)$ with respect to the relative symplectic two form $\tilde{\sigma}$. Then we have:

Theorem 5.1. *Assume that $A_j(x)$ are real analytic in Ω which contains the origin and L is strongly hyperbolic near the origin. Let Σ be the characteristic set of order r which is assumed to be an involutive C^∞ manifold verifying* (5.1). *Let $z^1\in N_{z^0}\Sigma\backslash 0$ be a characteristic of order s of $h_{z^0}=\det L_{z^0}$ and we assume* (5.2) *with $X=(z^0, z^1)$. Then every $(r-1)$-th minor of the localization L_{z^0} vanishes of order $(s-1)$ at z^1. In particular, we have*

$$\dim \operatorname{Ker} L_{z^0}(z^1)=s.$$

Theorem 5.2. *Assume that $A_j(x)$ are real analytic in Ω which contains the origin and L is strongly hyperbolic near the origin. Let Σ be the characteristic set of order r which is assumed to be an involutive C^∞ manifold verifying* (5.1). *Suppose that z^0 is non degenerate and* (5.2) *holds with $X=(z^0, z^1)$ for every $z^1\in N_{z^0}\Sigma\backslash 0$. Then $L_{z^0}(z)$ is symmetrizable by a non singular constant matrix T;*

$$T^{-1}L_{z^0}(z)T$$

is symmetric for every z. In particular, $L_{z^0}(z)$ is strongly hyperbolic on $T_{z^0}(T^*\Omega)$.

Proof. From Lemma 3.3 and Theorem 5.1 we see that

$$\dim \operatorname{Ker} L_{z^0}(z)=\text{the order of } z$$

for every multiple characteristic z of $\det L_{z^0}(z)$. Then it will suffice to apply Corollary 3.5 in [5]. $\square$

Here we give several examples.

EXAMPLE 5.1: Let

$$L^{(1)}(z)=\begin{pmatrix}\xi_0 & \xi_1 & 0\\ x_0^2\xi_1 & \xi_0 & 0\\ 0 & 0 & \xi_0-\xi_1\end{pmatrix},\quad L^{(2)}(z)=\begin{pmatrix}\xi_0 & \xi_1 & 0\\ x_1^2\xi_1 & \xi_0 & 0\\ 0 & 0 & \xi_0-\xi_1\end{pmatrix},$$

$$L^{(3)}(z)=\begin{pmatrix}\xi_0 & x_0\xi_1 & 0\\ x_0^3\xi_1 & \xi_0 & 0\\ 0 & 0 & \xi_0-\xi_1\end{pmatrix},\quad L^{(4)}(z)=\begin{pmatrix}\xi_0 & \xi_1 & 0\\ x_0^4\xi_1 & \xi_0 & 0\\ 0 & 0 & \xi_0-\xi_1\end{pmatrix}.$$

1) Let $z^0=(0,e_n)$ and $n\geq 2$. It is clear that the characteristic set of order 3 of $L^{(i)}$ near z^0 is an involutive C^∞ manifold given by $\Sigma=\{\xi_0=\xi_1=0\}$. Denoting $h^{(i)}(z)=\det L^{(i)}(z)$, it is obvious that $h^{(i)}_{z^0}(z)=\xi_0^2(\xi_0-\xi_1)$ and hence $z^1=(0,e_1)$ is a double characteristic of $h^{(i)}_{z^0}(z)$.

2) It is not difficult to show that $L^{(1)}$ and $L^{(3)}$ are strongly hyperbolic near the origin. For $L^{(1)}$ we have $h^{(1)}_{\Sigma X}=-(\xi_0^2-x_0^2)\neq h^{(1)}_{(z^0,z^1)}$, $\Lambda_X(h^{(1)}_\Sigma)^{\tilde{\sigma}}\not\subset\Lambda_X(h^{(1)}_\Sigma)$ where $X=(z^0,z^1)$. On the other hand, it is clear that $h^{(3)}_{\Sigma X}=-\xi_0^2=h^{(3)}_{(z^0,z^1)}$, $\Lambda_X(h^{(3)}_\Sigma)^{\tilde{\sigma}}\subset\Lambda_X(h^{(3)}_\Sigma)$, and every 2-minor of $L^{(3)}_{z^0}$ vanishes at z^1.

3) We have $h^{(2)}_{\Sigma X}=-(\xi_0^2-x_1^2)\neq h^{(2)}_{(z^0,z^1)}$, $\Lambda_X(h^{(2)}_\Sigma)^{\tilde{\sigma}}\subset\Lambda_X(h^{(2)}_\Sigma)$ and $h^{(4)}_{\Sigma X}=-\xi_0^2=h^{(4)}_{(z^0,z^1)}$, $\Lambda_X(h^{(4)}_\Sigma)^{\tilde{\sigma}}\subset\Lambda_X(h^{(4)}_\Sigma)$. The localizations of $L^{(2)}$ and $L^{(4)}$ are given by

$$L^{(2)}_{z^0}=L^{(4)}_{z^0}=\begin{pmatrix}\xi_0 & \xi_1 & 0\\ 0 & \xi_0 & 0\\ 0 & 0 & \xi_0-\xi_1\end{pmatrix}.$$

Since the 2-minor

$$\begin{vmatrix}\xi_1 & 0\\ 0 & \xi_0-\xi_1\end{vmatrix}$$

does not vanish at z^1 then $L^{(2)}$ and $L^{(4)}$ are not strongly hyperbolic near the origin.

To prove Theorem 5.1 we first examine what condition (5.2) implies. Let $x=(x_a,x_b)$, $x_a=(x_0,\cdots,x_k)$, $x_b=(x_{k+1},\cdots,x_n)$ be a partition of the variable x then we sometimes abbreviate $j\in\{0,\cdots,k\}$ to $j\in a$, etc.

Lemma 5.3. *Assume that* (5.1) *holds. Then we can find local coordinates* x^α

and the corresponding coordinates $\phi_\alpha=(x^\alpha,\xi^\alpha)$ *in* $T^*\Omega$, *preserving the plane* $x_0=0$, *such that* $\phi_\alpha(z^0)=(0,e_n)=p$ *and* $T_{z^0}\Sigma$ *is given by*

$$f_j(x^\alpha,\xi^\alpha)=\xi_j^\alpha+l_j(x^\alpha,\xi^\alpha)=0,\quad 0\le j\le k \tag{5.3}$$

with some $k<n$ *where* $l_j(p)=0$, $dl_j(p)=0$ *and the quadratic term of* l_j *at* p *depends only on* $\xi_b^\alpha=(x_{k+1}^\alpha,\cdots,\xi_n^\alpha)$.

Proof. From hypothesis (5.1) and that $T_{z^0}\Sigma$ is involutive, one can assume that $T_{z^0}\Sigma$ is given in the original coordinates by $f_j=\xi_j+l_j(x,\xi)=0$, $j\in a$ with $dl_j(p)=0$. By subtractions we may assume that $f_j=\xi_j+l_j(x,\xi_b)+O^3$ where O^3 denotes the term which is O^3 as $(x,\xi)\to(0,e_n)$. Write

$$l_j(x,\xi_b)=q_j(x)\xi_n+l_j'(x\xi_n,\xi_{b'})\xi_n^{-1}$$

where $q_j(x)\xi_n$ is the quadratic part in x of l_j and $\xi_{b'}=(\xi_{k+1},\cdots,\xi_{n-1})$. Since the involutivity of f_j assures the existence of $T(x)$ satisfying $\partial T(x)/\partial x_j=q_j(x)$, $j\in a$, taking new local coordinates $y\,;\tilde{y}=\tilde{x}$, $y_n=x_n-T(x)$, one can assume that $q_j=0$. Write

$$l_j'(x\xi_n,\xi_{b'})\xi_n^{-1}=\sum_{i=k+1}^{n-1}l_{ji}(x)\xi_i+l_j''(\xi_{b'})\xi_n^{-1}$$

where $l_{ji}(x)$ are linear in x. The same argument as above shows that the system

$$\partial S_i(x)/\partial x_j=l_{ji}(x),\ j\in a,\ i\in b'$$

admits solutions $S_i(x)$. Then taking new coordinates y

$$y_j=x_j,\ j\in a,\ y_n=x_n,\ y_i=x_i-S_i(x),\ i\in b'$$

we get $f_j=\xi_j+\tilde{l}_j(\xi_b)+O^3$ which is the desired assertion. $\square$

Lemma 5.4. *Let* x^α *be the local coordinates for which* $T_{z^0}\Sigma$ *is given by* (5.3). *Then one can find homogeneous symplectic coordinates* $\phi_\beta=(x^\beta,\xi^\beta)$ *such that* $\phi_\beta(z^0)=(0,e_n)=p$ *and* $T_{z^0}\Sigma$ *is given by* $\xi_j^\beta=0$, $j\in a$ *and moreover*

$$\xi_j^\beta(x^\alpha,\xi^\alpha)-f_j(x^\alpha,\xi^\alpha)=O^3,\ j\in a,\ (x,\xi)\to p.$$

In particular, $\phi_{\beta\alpha}'(p)=I$ *with* $\phi_{\beta\alpha}=\phi_\beta\circ\phi_\alpha^{-1}$ *and* $(\xi_j^\beta)''(p)z=(0,0,0,c_j(\xi_b))$, $j\in a$ *with* $z=(x_a,x_b,\xi_a,\xi_b)$.

Proof. To simplify notations, we denote (y,η), (x,ξ) for (x^β,ξ^β), (x^α,ξ^α) respectively. Take $\eta_0=f_0$ and determine η_j successively by solving

$$H_{\eta_i}\eta_j=0,\ 0\leq i\leq j,\ \eta_j|_C=f_j|_C$$

where $C=\{x_i=0, 0\leq i\leq j-1\}$. Since f_j are in involution, we see that $\eta_j=0$ on Σ. Noting that $\eta_i-f_i=O^3$, $0\leq i\leq j-1$ we conclude that $\eta_j-f_j=O^3$. We now extend η_j, $j\in a$ to homogeneous symplectic coordinates (y,η). It is clear that one can take (y,η) so that $(y,\eta)'(p)=I$. □

Let $\phi_\alpha=(x^\alpha,\xi^\alpha)$, $\phi_\beta=(x^\beta,\xi^\beta)$ be the coordinates in Lemmas 5.3 and 5.4 respectively and denote by h^α, p, q the representatives of h, z^0, z^1 in the coordinates ϕ_α. Note that

$$h^\alpha_{\{p,q\}}(w,\xi_a)=h^\beta_{\{p,q\}}(w,\xi_a+\phi''_{\beta\alpha}(p)(w+q))$$

with $w=(x_a,x_b,\xi_b)$. From Lemma 5.4 it follows that $\phi''_{\beta\alpha}(p)(w+q)\equiv F(\xi_b)$ mod $T_p\Sigma$ where the components of $F(\xi_b)$ are polynomials in ξ_b of degree at most two. Then it follows that

$$h^\alpha_{\{p,q\}}(w,\xi_a)=h^\beta_{\{p,q\}}(w,\xi_a+F(\xi_b)). \tag{5.4}$$

Lemma 5.5. *Let $\phi_\alpha=(x^\alpha,\xi^\alpha)$ be local coordinates for which $T_{z^0}\Sigma$ is given by $\xi^\alpha_a=0$ Then for any given $k\times k$ real non singular A with ${}^tAe_0=e_0$ and $k\times k$ real symmetric T, we can find local coordinates $\phi_\gamma=(x^\gamma,\xi^\gamma)$ with $x^\gamma_0=x^\alpha_0$ for which $T_{z^0}\Sigma$ is given by $\xi^\gamma_a=0$ and*

$$h^\gamma_{\{p,q^\gamma\}}(x_a,x_b,\xi_b;\xi_a)=h^\alpha_{\{p,q\}}(A^{-1}x_a,x_b,\xi_b;{}^tA\xi_a+TA^{-1}x_a) \tag{5.5}$$

where ${}^tAq^\gamma=q$ and h^γ, q^γ are representatives of h, q in the coordinates ϕ_γ.

Proof. To simplify notation we write (x,ξ), (y,η) for (x^α,ξ^α), (x^γ,ξ^γ) respectively and set $\phi=\phi_{\alpha\gamma}$. Note that $k\geq 1$, because otherwise we would have $q=0$ which contradicts $q\in N_p\Sigma\backslash 0$. Since the map

$$(x,\xi)\mapsto(A^{-1}x_a,x_b,\xi_b;{}^tA\xi_a+TA^{-1}x_a)$$

with a non singular A and a real symmetric T generates a group, we may suppose that $q=e_k$. Since the assertion when $T=O$ is almost trivial, it suffices to show the assertion with $A=I$. Let

$$y_j=x_j,\ j\neq k,\ y_k=x_k+\langle Tx_a,x_a\rangle/2.$$

It is obvious that $\phi(p)=p$ and $\phi'(p)=I$. It is also easy to see that

$$\xi''_j(p)(q,w)=(Tx_a)_j,\ \xi''_j(p)(w,w)=0,\ j\in a,\ w\in T_p\Sigma$$

with $w=(x_a, x_b, \xi_a, \xi_b)$. This proves that $\phi''(q,w)\equiv Tx_a$, $\phi''(w,w)\equiv 0 \mod T_p\Sigma$ and hence the assertion. □

Lemma 5.6. *Assume* (5.2) *with* $X=(z^0,z^1)$ *and let* ϕ_β *be the local coordinates in Lemma* 5.4. *Then choosing a non singular* A *with* ${}^tAe_0=e_0$ *and a symmetric* T *suitably, we have*

$$h^\beta_{\Sigma(p,q)}(A^{-1}x_a, 0, 0; {}^tA\xi_a+TA^{-1}x_a)=Q(\xi_0,\cdots,\xi_\mu, x_{\mu+1},\cdots,x_\nu)$$

with some $\mu\leq\nu\leq k$ *where* Q *is a polynomial of degree* s.

Proof. Note that the lineality of $h^\beta_{\Sigma(p,q)}(x_a,0,0;\xi_a)$ is given by $l_i(x_a,\xi_a)=0$, $0\leq i\leq\nu$ where l_i are linear functions in (x_a,ξ_a). Then we get $h^\beta_{\Sigma(p,q)}(x_a,0,0;\xi_a)=Q(l_i(x_a,\xi_a))$ with a homogeneous polynomial Q of degree s. From hypothesis $\Lambda_{(p,q)}(h_\Sigma)^{\tilde\sigma}\subset\Lambda_{(p,q)}(h_\Sigma)$, it follows that $l_i(x_a,\xi_a)$ are in involution with respect to $d\xi_a\wedge dx_a$. Then to prove the assertion, it is enough to follow the arguments in [3], page 127. □

Let A, T be given in Lemma 5.6. For these A, T we choose the local coordinates x^γ in Lemma 5.5 so that (5.5) holds. From (5.4) it follows that

$$h^\gamma_{\{p,q^\gamma\}}(x_a,x_b,\xi_b;\xi_a)=h^\beta_{\{p,q\}}(A^{-1}x_a,x_b,\xi_b;{}^tA\xi_a+TA^{-1}x_a+F(\xi_b)).$$

It is clear that the principal part of $h^\gamma_{\{p,q^\gamma\}}(x_a,x_b,\xi_b;\xi_a)$ with respect to (x_a,ξ_a) coincides with that of $h^\beta_{\{p,q\}}(A^{-1}x_a,x_b,\xi_b;{}^tA\xi_a+TA^{-1}x_a)$. Then from Lemma 2.5 it follows that the principal part of $h^\gamma_{\{p,q^\gamma\}}(x_a,x_b,\xi_b;\xi_a)$ with respect to (x_a,ξ_a) is $h^\beta_{\Sigma(p,q)}(A^{-1}x_a,0,0;{}^tA\xi_a+TA^{-1}x_a)$ and hence equal to

$$Q(\xi_0,\cdots,\xi_\mu,x_{\mu+1},\cdots,x_\nu)$$

by Lemma 5.6. Thus the principal part with respect to ξ_a is $Q(\xi_0,\cdots,\xi_\mu,0,\cdots,0)$ and then we have $h^\gamma_{\{p,q^\gamma\}}(\xi_a)=Q(\xi_0,\cdots,\xi_\mu,0,\cdots,0)$. Since $h^\gamma_{\{p,q^\gamma\}}(x_a,x_b,\xi_b;\xi_a)$ is hyperbolic with respect to the ξ_0 variable, Corollary 12.4.8 in [2] shows that $h^\gamma_{\{p,q^\gamma\}}(x_a,x_b,\xi_b;\xi_a)$ is a polynomial in $(\xi_0,\cdots,\xi_\mu,x_{\mu+1},\cdots,x_\nu,x_b,\xi_b)$. Hence we see that $h^\gamma_{\{p,q^\gamma\}}(x_a,x_b,\xi_b;\xi_a)$ is constant along $\{(x_0,\cdots,x_\mu,0,\cdots,0)\}=\Lambda_{(p,q^\gamma)}(h^\gamma)^\sigma$.

Here we summarize our observations.

Proposition 5.7. *Let* Σ *be the characteristic set of* h *of order* r *which is assumed to be an involutive* C^∞ *manifold verifying* (5.1). *Let* $z^1\in N_{z^0}\Sigma\backslash 0$ *be a characteristic of order* s *of* h_{z^0} *and suppose* (5.2) *with* $X=(z^0,z^1)$. *Then one can find local coordinates* x, *preserving the plane* $x_0=0$, *for which* $T_{z^0}\Sigma$ *is given by* $\xi_a=0$ *and that*

$$h_{\{z^0,z^1\}}(v,\xi_a)=h_{\{z^0,z^1\}}(0,\xi_a),\ \ \forall v\in\Lambda_{(z^0,z^1)}(h)^\sigma.$$

6. Proof of Theorem

In this section we shall prove Theorem 5.1. To do so, without assuming that Σ is a manifold, we give more general results which are not coordinate free though. We denote by ρ the radial vector field on $T^*\Omega$. Recall that $\Lambda_{z^0}(h) \subset \Lambda_{(z^0,z^1)}(h)$ and hence

$$\Lambda_{(z^0,z^1)}(h)^\sigma \subset \Lambda_{z^0}(h)^\sigma.$$

Proposition 6.1. *Assume that $A_j(x)$ are real analytic in Ω which contains the origin and L is strongly hyperbolic near the origin. Let $z^0 \in T_0^*\Omega\backslash 0$, $z^1 \in T_{z^0}(T^*\Omega)$ be characteristics of order r and s of h and $h_{z^0} = \det L_{z^0}$ respectively with*

$$(6.1) \qquad \rho(z^0) \notin \Lambda_{z^0}(h)^\sigma \subset \Lambda_{z^0}(h).$$

Assume that we can find local coordinates x near the origin with $t(x) = x_0$ such that

$$(6.2) \qquad h_{\{z^0,z^1\}}(v, z) = h_{\{z^0,z^1\}}(0, z), \ \forall v \in \Lambda_{(z^0,z^1)}(h)^\sigma.$$

Then every $(r-1)$-th minor of L_{z^0} vanishes of order $s-1$ at z^1. In particular, we have

$$\dim \operatorname{Ker} L_{z^0}(z^1) = s.$$

The proof of Theorem 5.1 follows immediately from Propositions 5.7 and 6.1 because $\rho(z^0) \notin \Lambda_{z^0}(h)$ by (5.1).

Proposition 6.2. *Assume that $A_j(x)$ are real analytic in Ω which contains the origin and L is strongly hyperbolic near the origin. Let $z^0 \in T_0^*\Omega\backslash 0$ be a non degenerate characteristic of h of order r with (6.1). Assume that for every multiple characteristic $z^1 \in T_{z^0}(T^*\Omega)$ of $h_{z^0} = \det L_{z^0}$ we can find local coordinates x with $t(x) = x_0$ verifying (6.2). Then $L_{z^0}(z)$ is symmetrizable by a non singular constant matrix.*

This result clearly generalizes Theorem 1 in [8].

Corollary 6.3. *Assume that $A_j(x)$ are real analytic in Ω which contains the origin and L is strongly hyperbolic near the origin. Let $z^0 \in T_0^*\Omega\backslash 0$ be a characteristic of order r of h with (6.1). Assume that for every $z^1 \in \Lambda_{z^0}(h)$ one can find local coordinates x with $t(x) = x_0$ verifying (6.2). Then we have $d(L_{z^0}) = d(\det L_{z^0})$.*

Proof. The proof follows immediately from Proposition 1 in [8] and Proposition 6.1 above. □

To show Propositions 6.1 and 6.2, we look for local coordinates x for which

(6.2) remains true and $\Lambda_{\{z^0,z^1\}}(h)$ is in a special position.

Lemma 6.4. *Let $z^0\in T_0^*\Omega\backslash 0$, $z^1\in T_{z^0}(T^*\Omega)$ be multiple characteristics of h and h_{z^0} with* (6.1). *Let $S\subset\Lambda_{z^0}(h)^\sigma$ be a linear subspace and assume that*

$$h_{\{z^0,z^1\}}(w,z)=h_{\{z^0,z^1\}}(0,z),\quad \forall w\in S \tag{6.3}$$

in some local coordinates. Then we can find new local coordinates x preserving the plane $x_0=0$, and the corresponding coordinates (x,ξ) in $T^*\Omega$ such that

(i) *the coordinates of z^0 are $(0,e_n)=p$,*

(ii) *$\Lambda_{z^0}(h)$ is given by $\xi_j=0, 0\le j\le k, x_j=0,\ k+1\le j\le N$,*

(iii) (6.3) *remains true in these coordinates.*

Lemma 6.5. *Let $z^0\in T_0^*\Omega\backslash 0$, $z^1\in T_{z^0}(T^*\Omega)$ be multiple characteristics of h and h_{z^0} with* (6.1). *Assume that* (6.2) *holds in some local coordinates. Then we can find new local coordinates x verifying all conditions stated in* Lemma 6.4 *with $S=\Lambda_{\{z^0,z^1\}}(h)^\sigma$ and furthermore satisfying*

(iv) *$\Lambda_{\{z^0,z^1\}}(h)$ is given by $\xi_j=0$, $0\le j\le k_1$, $x_j=0$, $k+1\le j\le N_1$ with some $k_1\le k$, $N_1\le N$.*

Let $z=(x,\xi)$ be the coordinates for which (6.3) holds. Without restrictions we may assume that $z^0=(0,e_n)$. To simplify notations we denote by $S_z,\Lambda_z\subset\boldsymbol{R}^{2n+2}$ the representatives of S, $\Lambda_{z^0}(h)$ in the coordinates z. We first recall that with $\tilde{h}(w)=h(z(w))$ we have

$$\tilde{h}_{\{w^0,w^1\}}(v,u)=h_{\{z^0,z^1\}}(\tilde{v},\tilde{u}+z''(w^0)(w^1+v))$$

where $\tilde{u}=z'(w^0)u$, $\tilde{v}=z'(w^0)v$ and $w^0=w(z^0)$, $w^1=w'(z^0)z^1$. Thus if

$$z''(w^0)(w^1,v),\ z''(w^0)(v,v)\in\Lambda_z, \tag{6.4}$$

it follows that $\tilde{h}_{\{w^0,w^1\}}(v,u)=h_{\{z^0,z^1\}}(\tilde{v},\tilde{u}+z''(w^0)(w^1))$ and then $\tilde{h}_{\{w^0,w^1\}}$ verifies (6.3) if $h_{\{z^0,z^1\}}(h)$ does. Remark that successive change of coordinates $z=z(\zeta)=z(w(\zeta))$ verifies (6.4) provided if each change of coordinates $z=z(w)$, $w=w(\zeta)$ verifies (6.4).

Since Λ_z contains the $(0,e_n)$-axis we can assume that Λ_z is given by $\phi_j(x,\xi)=0$, $0\le j\le N$ where $\phi_j(x,\xi)$ are linearly independent linear functions in $(x,\tilde{\xi})$, $\tilde{\xi}=(\xi_0,\cdots,\xi_{n-1})$. We first study the case Λ_z contains $(e_n,0)$-axis and hence ϕ_j contains no x_n. Note that some ϕ_j actually depend on ξ_0 because h_{z^0} is hyperbolic with respect to the ξ_0 variable. By a linear change of coordinates $\tilde{x}=(x_0,\cdots,x_{n-1})$ preserving the plane $x_0=0$, the $(0,e_n)$-axis and a renumbering of the ϕ_j's, we can assume that $\phi_0=\xi_0+l_0(\tilde{x})$ in the original coordinates. Choose new coordinates y so that

$$\tilde{x}=\tilde{y},\ y_n=<Q\tilde{x},\tilde{x}>/2$$

where Q is a real symmetric matrix. With suitable choice of Q, Λ_w is given by

$$\eta_j=0,\ 0\leq j\leq k,\ y_j=0,\ k+1\leq j\leq N$$

(cf.[3]) which proves (ii). Since $v\in\Lambda_w^\sigma$ implies that $v_{y_n}=v_{\eta_n}=0$ it is clear that $\xi_j''(w^0)(v,u)=0$ if $v\in\Lambda_w^\sigma$, $u_{\eta_n}=0$. Note that we can assume that $w_{\eta_n}^1=0$ because Λ_w contains the $(0,e_n)$-axis. Then $z''(w^0)(w^1,v)$, $z''(w^0)(v,v)$ are proportional to $(e_n,0)\in\Lambda_z$ and hence (iii).

We turn to the case Γ_z does not contain the $(e_n,0)$-axis. Renumbering ϕ_j if necessary, we can assume that ϕ_N contains x_n. Subtracting constant times ϕ_N from the other ϕ_j's we can assume that $\phi_j(j<N)$ contains no x_n. We first treat the case ϕ_j $(j<N)$ contains no ξ_0. Renumbering ϕ_j again, one can assume that

$$\phi_0=\xi_0+l_0(\tilde{x},\xi'')+ax_n,\ \phi_j=l_j(\tilde{x},\xi''),\ 1\leq j\leq N$$

where $\xi''=(\xi_1,\cdots,\xi_{n-1})$, $a\neq 0$. By a linear change of coordinates $\tilde{x}$, preserving the plane $x_0=0$, one can assume that

$$\phi_0=\xi_0+l_0^2(\tilde{x})+ax_n,\ \phi_j=l_j^1(\xi'')+l_j^2(\tilde{x}),\ 1\leq j\leq N$$

in the original coordinates. Take new coordinates y so that $\tilde{y}=\tilde{x}$, $y_n=x_n-ax_0x_n$ and hence Λ_w is given by

$$\eta_0+l_0^2(\tilde{y})=0,\ l_j^1(\eta'')+l_j^2(\tilde{y})=0,\ 1\leq j\leq N.$$

It is clear that $x_j''(w^0)=0$ unless $j=n$ and $\xi_j''(w^0)=0$ unless $j=0$, n. It is also easy to see that

$$x_n''(w^0)(u,v)=au_{y_n}v_{y_0},\ \xi_0''(w^0)(u,v)=-a^2u_{y_n}v_{y_0}$$

if $v_{y_n}=v_{\eta_n}=0$. Thus we have $z''(w^0)(u,v)\in\Lambda_z$, $\forall v\in\Lambda_w^\sigma$, $\forall u$. Since Λ_w contains the $(e_n,0)$-axis the proof is reduced to the preceding case.

We next treat the case that some $\phi_j(j<N)$ depend on ξ_0. Renumbering ϕ_j's if necessary we may assume that $\phi_0=\xi_0+l(\tilde{x},\xi'')$. As before, after a linear change of coordinates $\tilde{x}$, we may assume $\phi_0=\xi_0+l_0^2(\tilde{x})$. Subtracting constant times ϕ_0 from the other ϕ_j's one can assume that ϕ_j, $1\leq j\leq N$ contains no ξ_0. Repeating this argument we arrive at

$$\phi_j=\xi_j+l_j(\tilde{x}),\ 0\leq j\leq k-1,\ \phi_j=l_j(\tilde{x}),\ k\leq j\leq N-1,$$

$$\phi_N=l_N(\tilde{x})+ax_n \text{ or } \phi_N=\xi_k+l_N(\tilde{x})+ax_n$$

with some k. Suppose $\phi_N=\xi_k+l_N(\tilde{x})+ax_n$. Since ϕ_j are in involution we have

$\phi_j = l_j(x_b, x_c)$, $k \le j \le N-1$ where $x_b = (x_{k+1}, \cdots, x_N)$, $x_c = (x_{N+1}, \cdots, x_{n-1})$. By a linear change of coordinates (x_b, x_c) we may assume that $\phi_j = x_{j+1}$, $k \le j \le N-1$. Subtractions give

$$l_j = l_j(x_a, x_c),\ 0 \le j \le k-1,\ l_N = l_N(x_a, x_c)$$

with $x_a = (x_0, \cdots, x_k)$. Thus one can assume that Λ_z is given by

$$\xi_j + l_j(x_a, x_c) = 0,\ 0 \le j \le k-1,\ x_j = 0,\ k+1 \le j \le N,$$

$$\xi_k + l_N(x_a, x_c) + a x_n = 0$$

in the original coordinates. Take new coordinates y; $\tilde{y} = \tilde{x}$, $y_n = x_n - a x_k x_n$. Then similar arguments as before prove that $z''(w^0)(u, v) \in \Gamma_z$, $\forall v \in \Gamma_w^\sigma$, $\forall u$ and the proof is reduced to the previous case.

Finally let $l_j(\tilde{x}) = l_j(x_{b'}, x_N, x_{c'})$, $k \le j \le N$ where $x_{b'} = (x_k, \cdots, x_{N-1})$, $x_{c'} = (x_{N+1}, \cdots, x_{n-1})$. Since $(0, e_n) \notin \Lambda_z^\sigma$ by hypothesis, it follows that $l_N(\tilde{x}) \neq 0$. Therefore by a linear change of coordinates $\tilde{x}$ and subtractions we may assume that Λ_z is given by

$$\xi_j + l_j(x_{a'}, x_{c'}) + b_j x_N = 0,\ 0 \le j \le k-1,$$

$$x_j = 0,\ k \le j \le N-1,\ x_N + a x_n = 0$$

where $x_{a'} = (x_0, \cdots, x_{k-1})$. Make a linear change $y_N = x_N + a x_n$, $y_j = x_j$, $j \neq N$. After subtractions one can assume that Λ_z is given by

$$\xi_j + l_j(x_{a'}, x_{c'}) + c_j x_n = 0,\ 0 \le j \le k-1,\ x_j = 0,\ k \le j \le N$$

in the original coordinates. We take new coordinates y

$$\tilde{y} = \tilde{x},\ y_n = x_n - \sum_{j=0}^{k-1} c_j x_j x_n$$

so that Λ_w is given by $\eta_j + l_j(y_{a'}, y_{c'}) = 0$, $0 \le j \le k-1$, $y_j = 0$, $k \le j \le N$. It is now easy to check that $z''(w^0)(u, v) \in \Lambda_z$, $\forall v \in \Lambda_w^\sigma$, $\forall u$. The rest of the proof is simply a repetition. $\square$

Proof (of Lemma 6.6). By Lemma 6.5 we can assume that (i), (ii) and (iii) are satisfied in the original coordinates. Since $\Lambda_{z^0}(h) \subset \Lambda_{(z^0, z^1)}(h)$ one may assume that $\Lambda_{(z^0, z^1)}(h)$ is given by $\psi_j(\xi_a, x_b) = 0$, $0 \le j \le N_2$ with $\xi_a = (\xi_0, \cdots, \xi_k)$, $x_b = (x_{k+1}, \cdots, x_N)$. After a linear change of coordinates one can assume that

$$\psi_j = \xi_j + l_j(x_{N_1+1}, \cdots, x_N),\ 0 \le j \le k_1,\ \psi_j = x_{k-k_1+j},\ k_1 + 1 \le j \le N_2$$

with the original coordinates where $k - k_1 + N_2 = N_1$. Take new coordinates y

$$\tilde{y}=\tilde{x},\ y_n=x_n-\sum_{j=0}^{k_1} l_j(x_{N_1+1},\cdots,x_N)x_j$$

so that Λ_w is given by $\eta_j=0$, $0\le j\le k_1$, $y_j=0$, $k+1\le j\le N_1$. It is easy to see that $z''(w^0)(u,v)$, $z''(w^0)(v,v)\in\Lambda_z$, $\forall v\in\Lambda_w^\sigma$, $\forall u$ and hence the proof is complete. □

We now complete the proof of Propositions 6.1 and 6.2. Choose the coordinates found in Lemma 6.6 and let $z^1=(y,\eta)$ be a characteristic of h_{z^0} of order s verifying the hypotheses of Proposition 6.1. Recall that

$$h(z^0+\mu z^1+\mu^2(x,\xi))=\mu^{r+s}\Bigg(\sum_{l+|\alpha+\beta|=r+s} H_{l(\beta)}^{(\alpha)}(z^0;z^1)x^\beta\xi^\alpha/\alpha!\beta!+o(1)\Bigg)$$

where

$$H_l(z^0;x,\xi)=\sum_{|\alpha+\beta|=l} h_{(\beta)}^{(\alpha)}(z^0)x^\beta\xi^\alpha/\alpha!\beta!.$$

From Lemma 6.6 there is a homogeneous polynomial Q of degree r such that

$$h_{z^0}(x,\xi)=H_r(z^0;x,\xi)=Q(x_b,\xi_a)$$

where $x=(x_a,x_b,x_c)=(x_0,\cdots,x_k,x_{k+1},\cdots,x_N,x_{N+1},\cdots,x_n)$ and $\xi=(\xi_a,\xi_b,\xi_c)$ is the corresponding partition. We set

$$x_a=(x_{a_1},x_{a_2})=(x_0,\cdots,x_{k_1},x_{k_1+1},\cdots,x_k),$$

$$x_b=(x_{b_1},x_{b_2})=(x_{k+1},\cdots,x_{N_1},x_{N_1+1},\cdots,x_N)$$

and $\xi_a=(\xi_{a_1},\xi_{a_2})$, $\xi_b=(\xi_{b_1},\xi_{b_2})$. Lemma 6.6 asserts that for $|\alpha+\beta|+l=r+s$

$$H_{l(\beta)}^{(\alpha)}(z^0;z^1+z)=H_{l(\beta)}^{(\alpha)}(z^0;z^1),\ \forall z\in\Lambda_{(z^0,z^1)}(h)^\sigma. \tag{6.5}$$

Let us take

$$\sigma_0=1/3-\varepsilon_1-\varepsilon_0,\ \sigma_j=1/3-\varepsilon-\varepsilon_1,\ 1\le j\le k_1,$$

$$\sigma_j=1/3+\varepsilon_1+\nu,\ k_1+1\le j\le k,\ \sigma_j=2/3+\varepsilon+\varepsilon_1,\ k+1\le j\le N_1,$$

$$\sigma_j=2/3-\varepsilon_1-\nu,\ N_1+1\le j\le N,\ \sigma_j=1/2,\ N+1\le j\le n$$

where $\varepsilon_1>\varepsilon>\varepsilon_0>0$, $\nu>0$ which will be determined later. Put

$$y(\lambda)=\lambda^{-1/3-\varepsilon_1}y,\ \eta(\lambda)=e_n+\lambda^{-1/3-\varepsilon_1}\eta$$

and study

$$h_\lambda(y(\lambda), \eta(\lambda); x, \xi) = h(y(\lambda) + \lambda^{-\sigma}x,\ \lambda\eta(\lambda) + \lambda^{\sigma}\xi)$$
$$= \lambda^m h(z^0 + \lambda^{-1/3-\varepsilon_1}z^1 + (\lambda^{-\sigma}x, \lambda^{\sigma-1}\xi)).$$

To simplify notations we set

$$p_1 = 1/3 - \varepsilon - \varepsilon_1,\ p_2 = 2/3 + \varepsilon + \varepsilon_1,\ p = 1/3 + \varepsilon_1,\ t = 2\varepsilon_1 + \varepsilon = p - p_1$$

and note that

$$\lambda^{-p}z^1 + (\lambda^{-\sigma}x, \lambda^{\sigma-1}\xi) = \lambda^{-p}z^1 + \lambda^{-p_1}(\lambda^{\varepsilon_0-\varepsilon}x_0, x_{a_1'}\xi_{b_1}) + \lambda^{-p-v}(x_{a_2}, \xi_{b_2})$$
$$+ \lambda^{-p_2}(x_{b_1}, \lambda^{t+v}x_{b_2}, \lambda^{\varepsilon-\varepsilon_0}\xi_0, \xi_{a_1'}, \lambda^{t+v}\xi_{a_2}) + \lambda^{-1/2}(x_c, \xi_c) = \lambda^{-p_2}A + \lambda^{-p}B$$

where $x_{a_1} = (x_0, x_{a_1'})$ and

$$A = (x_{b_1}, \lambda^{t+v}x_{b_2}, \lambda^{\varepsilon-\varepsilon_0}\xi_0, \xi_{a_1'}, \lambda^{t+v}\xi_{a_2}),$$
$$B = z^1 + \lambda^t(\lambda^{\varepsilon_0-\varepsilon}x_0, x_{a_1'}, \xi_{b_1}) + \lambda^{-v}(x_{a_2}, \xi_{b_2}) + \lambda^{p-1/2}(x_c, \xi_c).$$

Note that B is a characteristic of h_{z^0} of order s because $\Lambda_{z^0}(h) = \{\xi_a = 0, x_b = 0\}$. Then one can write

$$h_\lambda = \sum_{l+|\alpha_a+\beta_b| \geq r+s} \lambda^{F(l,\alpha_a,\beta_b)} H_{l(\beta_b)}^{(\alpha_a)}(z^0; B) x_b^{\beta_b} \xi_a^{\alpha_a} / \alpha_a! \beta_b! \tag{6.6}$$

where

$$F(l, \alpha_a, \beta_b) = m - p(l - |\alpha_a + \beta_b|) - p_2|\alpha_a + \beta_b| + (\varepsilon - \varepsilon_0)\alpha_0 + (t+v)|\alpha_{a_2} + \beta_{b_2}|.$$

We first study the term with $l + |\alpha_a + \beta_b| = r + s$ in (6.6). For $l + |\alpha_a + \beta_b| = r + s$ it follows from Lemma 2.4 that $H_{l(\beta_b)}^{(\alpha_a)}(z^0; B) = 0$ unless $\alpha_{a_2} = 0$, $\beta_{b_2} = 0$ because $\Lambda_{(z^0,z^1)}(h) = \{\xi_{a_1} = 0, x_{b_1} = 0\}$. Therefore we get

$$H_{l(\beta_b)}^{(\alpha_a)}(z^0; B) = H_{l(\beta_{b_1})}^{(\alpha_{a_1})}(z^0; z^1 + \lambda^t(\lambda^{\varepsilon_0-\varepsilon}x_0, x_{a_1'}, \xi_{b_1}))$$

$$+ \sum_{|\gamma_c+\delta_c+\gamma_{b_2}+\delta_{a_2}| \geq 1} \lambda^{F_2(l,\alpha_{a_1},\beta_{b_1},\gamma_c,\delta_c,\gamma_{b_2},\delta_{a_2})} H_{l(\beta_{b_1}+\delta_c+\delta_{a_2})}^{(\alpha_{a_1}+\gamma_c+\gamma_{b_2})}(z^0; \lambda^{-t}z^1$$
$$+ (\lambda^{\varepsilon_0-\varepsilon}x_0, x_{a_1'}, \xi_{b_1})) x_c^{\delta_c} \xi_c^{\gamma_c} x_{a_2}^{\delta_{a_2}} \xi_{b_2}^{\gamma_{b_2}} / \delta_c! \gamma_c! \delta_{a_2}! \gamma_{b_2}!$$

where

$$F_2=t(l-|\alpha_{a_1}+\beta_{b_1}+\gamma_c+\delta_c+\gamma_{b_2}+\delta_{a_2}|)-v|\delta_{a_2}+\gamma_{b_2}|+(p-1/2)|\gamma_c+\delta_c|.$$

Since $l\leq r+s$ because $l+|\alpha_{a_1}+\beta_{b_1}|=r+s$ it follows that

$$F_2\leq t(r+s-1)-v|\delta_{a_2}+\gamma_{b_2}|-(1/2-p)|\delta_c+\gamma_c|$$

$$\leq t(r+s-1)-\min\{v,1/2-p\}=(2\varepsilon_1+\varepsilon)(r+s-1)-\min\{v,1/6-\varepsilon_1\}$$

because $|\gamma_c+\delta_c+\gamma_{b_2}+\delta_{a_2}|\geq 1$. Take $\varepsilon_1>0$, $v>0$ so that

$$3\varepsilon_1(r+s-1)<v<1/6-\varepsilon_1 \tag{6.7}$$

and hence $F_2<0$. Then from (6.5) and $(\lambda^{\varepsilon_0-\varepsilon}x_0,x_{a'},\xi_b)\in\Lambda_{(z^0,z^1)}(h)^\sigma$ it follows that

$$h_\lambda=\sum_{l+|\alpha_{a_1}+\beta_{b_1}|=r+s}\lambda^{F(l,\alpha_{a_1},\beta_{b_1})}\left(H^{(\alpha_{a_1})}_{l(\beta_{b_1})}(z^0;z^1)x_{b_1}^{\beta_{b_1}}\xi_{a_1}^{\alpha_{a_1}}/\alpha_{a_1}!\beta_{b_1}!+O(\lambda^{-v_1})\right)$$

$$+\sum_{l+|\alpha_a+\beta_b|\geq r+s+1}\lambda^{F_1(l,\alpha_a,\beta_b)}H^{(\alpha_a)}_{l(\beta_b)}(z^0;\lambda^{-t}B)x_b^{\beta_b}\xi_a^{\alpha_a}/\alpha_a!\beta_b!$$

with some $v_1>0$ and $F_1=F+t(l-|\alpha_a+\beta_b|)$. We next observe F_1:

$$\begin{aligned}F_1&=m-p_1(l-|\alpha_a+\beta_b|)-(p^2-t-v)|\alpha_{a_2}+\beta_{b_2}|\\&\qquad-p^2|\alpha_{a_1}+\beta_{b_1}|+(\varepsilon-\varepsilon_0)\alpha_0\\&=m-p_1(l+|\alpha_a+\beta_b|)-(\varepsilon_1+2\varepsilon-v)|\alpha_{a_2}+\beta_{b_2}|\\&\qquad-3(\varepsilon+\varepsilon_1)|\alpha_{a_1}+\beta_{b_1}|+(\varepsilon-\varepsilon_0)\alpha_0\end{aligned}$$

because $p^2-t-2p_1=\varepsilon_1+2\varepsilon$, $p_2-2p_1=3(\varepsilon+\varepsilon_1)$. Let $l+|\alpha_a+\beta_b|=r+s+u$ with $u\geq 1$. Then we see that with $m^*=m-(1/3+\varepsilon_1)r-(1/3+\varepsilon_0)s$

$$\begin{aligned}F_1-m^*&=(1/3+\varepsilon_1-p_1)r+(1/3+\varepsilon_0-p_1)s-p_1u-(\varepsilon_1+2\varepsilon-v)|\alpha_{a_2}+\beta_{b_2}|\\&\quad-3(\varepsilon+\varepsilon_1)|\alpha_{a_1}+\beta_{b_1}|+(\varepsilon-\varepsilon_0)\alpha_0\leq(2\varepsilon_1+\varepsilon)r+(\varepsilon+\varepsilon_1+\varepsilon_0)s-(1/3-\varepsilon-\varepsilon_1)u\\&\quad-(\varepsilon_1+2\varepsilon-v)(s+u)=(2\varepsilon_1+\varepsilon)r+(\varepsilon_0-\varepsilon+v)s-(1/3+\varepsilon-v)u.\end{aligned}$$

Here we take $\varepsilon_1>0$, $v>0$ so that

(6.8) $$3\varepsilon_1 r+(\varepsilon_1+\nu)s<1/3-\nu$$

and hence we have $F_1(l,\alpha_a,\beta_b)<m^*$ if $l+|\alpha_a+\beta_b|\geq r+s+1$. We summarize our observations.

Lemma 6.6. *Choose* $\nu>0$, $\varepsilon_1>\varepsilon>\varepsilon_0>0$ *sufficiently small so that* (6.7) *and* (6.8) *hold. Then we have*

$$h_\lambda=\lambda^{m^*}(\partial_{\xi_0}^s H_r(z^0;z^1)\xi_0^s/s!+o(1))=\lambda^{m^*}(\sigma_0(h)(y,\eta;x,\xi)+o(1)).$$

Proof. It is enough to examine the term

$$\lambda^{F(l,\alpha_{a_1},\beta_{b_1})}H_{l(\beta_{b_1})}^{(\alpha_{a_1})}(z^0;z^1)x_{b_1}^{\beta_{b_1}}\xi_{a_1}^{\alpha_{a_1}}/\alpha_{a_1}!\beta_{b_1}!$$

for $l+|\alpha_{a_1}+\beta_{b_1}|=r+s$. Recall that

$$F(l,\alpha_{a_1},\beta_{b_1})=m-p(l+|\alpha_{a_1}+\beta_{b_1}|)+(\varepsilon_1-\varepsilon)|\alpha_{a_1}+\beta_{b_1}|+(\varepsilon-\varepsilon_0)\alpha_0$$
$$\leq m-p(r+s)+(\varepsilon_1-\varepsilon)s+(\varepsilon-\varepsilon_0)s=m-pr-(p+\varepsilon_0-\varepsilon)s=m^*.$$

Here the maximum is attained if and only if $l=r$, $\alpha_0=s$ since $\varepsilon>\varepsilon_0$. This proves the assertion. □

We turn to study $(m-1)$-th minors of L. Let $m(z)$ be a $(m-1)$-th minor of $L(z)$. Assume that m vanishes of order $r-1$ at z^0. Recall that

$$m_\lambda=\lambda^{m-1}m(y(\lambda)+\lambda^{-\sigma}x,\eta(\lambda)+\lambda^{\sigma-1}\xi)$$
$$=\lambda^{m-1}m(z^0+\lambda^{-p_2}A+\lambda^{-p}B)=\sum\lambda^{G(l,\alpha,\beta)}M_{l(\beta)}^{(\alpha)}(z^0;z^1)x^\beta\xi^\alpha/\alpha!\beta!$$

where

$$M_l(z^0;z)\sum_{|\alpha+\beta|=l}m_{(\beta)}^{(\alpha)}(z^0)x^\beta\xi^\alpha/\alpha!\beta!,\ z=(x,\xi)$$

and

$$G=m-1-p(l-|\alpha+\beta|)-p_1|\beta_{a_1}+\alpha_{b_1}|+(\varepsilon_0-\varepsilon)\beta_0-(p+\nu)|\beta_{a_2}+\alpha_{b_2}|$$
$$-p_2|\beta_{b_1}+\alpha_{a_1}|+(\varepsilon-\varepsilon_0)\alpha_0-(p_2-t-\nu)|\beta_{b_2}+\alpha_{a_2}|-|\beta_c+\alpha_c|/2$$

which is equal to

$$m-1-p(l+|\alpha+\beta|)+(1/3+\varepsilon+3\varepsilon_1)|\beta_{a_1}+\alpha_{b_1}|(\varepsilon_0-\varepsilon)\beta_0$$
$$+(1/3+\varepsilon_1-v)|\beta_{a_2}+\alpha_{b_2}|+(\varepsilon_1-\varepsilon)|\beta_{b_1}+\alpha_{a_1}|+(\varepsilon-\varepsilon_0)\alpha_0$$
$$+(3\varepsilon_1+v)|\beta_{b_2}+\alpha_{a_2}|+(1/6+2\varepsilon_1)|\alpha_c+\beta_c|$$

because we have

$$p_1-2p=-(1/3+\varepsilon+3\varepsilon_1),\ p_2-2p=-(\varepsilon_1-\varepsilon)$$
$$p_2-t-2p=-3\varepsilon_1,\ 1/2-2p=-(1/6+2\varepsilon_1).$$

Lemma 6.7. *Assume that $m(z)$ vanishes of order $r-1$ at z^0 and $(\varepsilon_1-\varepsilon_0)s<3\varepsilon_1$. Then for l, α, β with $l+|\alpha+\beta|<r+s-2$, $M^{(\alpha)}_{l(\beta)}(z^0;z^1)\neq 0$ we have $G(l,\alpha,\beta)>m^*$.*

Proof. Set $l+|\alpha+\beta|=r+s-3+u$, $u\leq 0$. Then we see that

$$G-m^*=-1-(\varepsilon_1-\varepsilon_0)s+p(3-u)+(1/3+\varepsilon+3\varepsilon_1)|\beta_{a_1}+\alpha_{b_1}|$$
$$+(\varepsilon_0-\varepsilon)\beta_0+(1/3+\varepsilon_1-v)|\beta_{a_2}+\alpha_{b_2}|+(\varepsilon_1-\varepsilon)|\beta_{b_1}+\alpha_{a_1}|$$
$$+(\varepsilon-\varepsilon_0)\alpha_0+(3\varepsilon_1+v)|\beta_{b_2}+\alpha_{a_2}|+(1/6+2\varepsilon_1)|\alpha_c+\beta_c|$$
$$\geq -1+3p-(\varepsilon_1-\varepsilon_0)s-pu=3\varepsilon_1-(\varepsilon_1-\varepsilon_0)s-pu.$$

This shows the assertion because $u\leq 0$. □

Lemma 6.8. *Assume that $m(z)$ vanishes of order $r-1$ at z^0 and $\varepsilon_1(3r-1)<1/3$. Then if $G(l,\alpha,\beta)\geq m^*$ it follows that $\alpha_0\leq s-2$.*

Proof. We first note that

$$G=m-1-pl-(p_1-p)|\beta_{a_1}+\alpha_{b_1}|+(\varepsilon_0-\varepsilon)\beta_0-v|\beta_{a_2}+\alpha_{b_2}|$$
$$-(p_2-p)|\beta_{b_1}+\alpha_{a_1}|+(\varepsilon-\varepsilon_0)\alpha_0-(p_2-p-t-v)|\beta_{b_2}+\alpha_{a_2}|$$
$$-(1/2-p)|\alpha_c+\beta_c|$$

which is equal to

$$m-1-pl+(\varepsilon+2\varepsilon_1)|\beta_{a_1}+\alpha_{b_1}|+(\varepsilon_0-\varepsilon)\beta_0-v|\beta_{a_2}+\alpha_{b_2}|$$
$$-(1/3+\varepsilon)|\beta_{b_1}+\alpha_{a_1}|+(\varepsilon-\varepsilon_0)\alpha_0-(1/3-2\varepsilon_1-v)|\beta_{b_2}+\alpha_{a_2}|-(1/6-\varepsilon_1)|\alpha_c+\beta_c|$$

since

$$p_1-p=-(\varepsilon+2\varepsilon_1),\ p_2-p=1/3+\varepsilon,\ p_2-p-t=1/3-2\varepsilon_1,\ 1/2-p=1/6-\varepsilon_1.$$

Since $M_l(z^0;z)$ is a polynomial of degree l in z and then $|\alpha+\beta|\le l$. From this it follows that $G(l,\alpha,\beta)\le m-1-pl+(\varepsilon+2\varepsilon_1)l-(1/3+\varepsilon_0)\alpha_0$. Suppose that $\alpha_0\ge s-1$ then, taking $l\ge r-1$ into account we get

$$G\le m-1-(1/3-\varepsilon-\varepsilon_1)l-(1/3+\varepsilon_0)(s-1)$$
$$\le m^*-(1/3+\varepsilon+\varepsilon_1-\varepsilon_0)+(\varepsilon+2\varepsilon_1)r\le m^*+\varepsilon_1(3r-1)-1/3.$$

This clearly gives a contradiction. □

We now summarize what we have proved. We choose $\nu>0$, $\varepsilon_1>\varepsilon>\varepsilon_0>0$ so that Lemmas 6.6, 6.7 and 6.8 hold and then fix them.

Lemma 6.9. *Assume that $m(z)$ vanishes of order $r-1$ at z^0 and there are l, α, β such that $l+|\alpha+\beta|<r+s-2$, $M_{l(\beta)}^{(\alpha)}(z^0;z^1)\neq 0$. Then we have*

$$m_\lambda=\lambda^{m^*+\tau}(\sigma_0(m)(y,\eta;x,\xi)+o(1))$$

with $\tau>0$ where $\sigma_0(m)(y,\eta;x,\xi)$ is a polynomial in ξ_0 of degree at most $s-2$.

Proof (of Proposition 6.1). Since $\sigma_0(m)$ must be divisible by $\sigma_0(h)$ it follows that $\sigma_0(m)(y,\eta;x,\xi)=0$. Since $m_{z^0}(z)=M_{r-1}(z^0;z)$ it follows from Lemma 6.9 with $l=r-1$ that

$$m_{z^0(\beta)}^{(\alpha)}(z^1)=0 \text{ if } |\alpha+\beta|<s-1.$$

Applying Lemma 3.2 we get the desired result. □

Proof (of Proposition 6.2). It suffices to repeat the proof of Theorem 5.2. □

References

[1] M.F. Atiyah, R. Bott, L. Gårding: *Lacunas for hyperbolic differential operators with constant coefficients, I,* Acta Math. **124** (1970), 109–189.

[2] L. Hörmander: The Analysis of Linear Partial Differential Operators, II, Springer, Berlin-Heiderberg-New York-Tokyo, 1985.

[3] L. Hörmander: *The Cauchy problem for differential equations with double characteristics,* J. Analyse Math. **32** (1977), 118–196.

[4] Y. Laurent: Théorie de la deuxième microlocalisation dans le domaine complexe, Progress in Math. Birkhäuser 53, 1985.

[5] T. Nishitani: *Symmetrization of a class of hyperbolic systems with real constant coefficients,* Ann. Scuo. Sup. Pisa **21** (1994), 97–130.

[6] T. Nishitani: *Systèmes effectivement hyperboliques*, In Calcul d'opérateurs et fronts d'ondes, J. Vaillant ed., Hermann Paris 1988, pp108–132.
[7] T. Nishitani: *Necessary conditions for strong hyperbolicity of first order systems*, J. Analyse Math. **61** (1993), 181–229.
[8] J. Vaillant: *Symétrisabilité des matrices localisées d'une matrice fortement hyperbolique*, Ann. Scuo. Sup. Pisa **5** (1978), 405–427.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560, Japan