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STRUCTURAL OPERATORS AND SEMIGROUPS ASSOCIATED WITH FUNCTIONAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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1. Introduction

Let $\Omega\subset \boldsymbol{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and $H=L^2(\Omega)$ be the usual L^2-space. A model equation under consideration is described by the following parabolic partial differential equation with time delays

$$(1.1)\qquad \frac{\partial u(t,x)}{\partial t}=\mathcal{A}_0u(t,x)+\mathcal{A}_1u(t-h,x)+\int_{-h}^{0}a(s)\mathcal{A}_2u(t+s,x)ds+f(t,x)\,,\quad t\geq 0\,,\quad x\in\Omega\,,$$

where $\mathcal{A}_i(i=0,1,2)$ are elliptic differential operators of the second order induced by the sesquilinear forms, $f\in L^2_{\mathrm{loc}}(\boldsymbol{R}^+;L^2(\Omega))$ is a forcing function, $h>0$ is a delay time and $a(s)$ is a real scalar function on $[-h,0]$. The boundary condition attached to (1.1) is, for simplicity, given by the Dirichlet boundary condition

$$(1.2)\qquad u|_{\partial\Omega}=0\,,\quad t\geq 0$$

and the initial data is given by

$$(1.3)\qquad u(0,x)=g^0(x)\,,\quad u(s,x)=g^1(s,x)\quad \text{a.e. } s\in[-h,0)\,,\quad x\in\Omega\,,$$

where $g^0\in L^2(\Omega)$, $g^1\in L^2(-h,0;H_0^1(\Omega))$.

Let $H=L^2(\Omega)$, $V=H_0^1(\Omega)$ and $g=(g^0,g^1)\in M_2\equiv H\times L^2(-h,0;V)$ and $A_i(i=0,1,2)$ be the realization of $\mathcal{A}_i$ with the boundary condition (1.2) in $H=L^2(\Omega)$. Then the system (1.1)-(1.3) can be included in the following general class of abstract functional differential equation (**E**) in a Hilbert space H:

$$(1.4)\qquad \frac{du(t)}{dt}=A_0u(t)+A_1u(t-h)+\int_{-h}^{0}a(s)A_2u(t+s)ds+f(t)\,,\quad t\geq 0$$

(1.5) $$u(0)=g^0, \quad u(s)=g^1(s) \quad \text{a.e. } s\in[-h, 0).$$

This type of equations having the highest order spatial derivatives in time delayed terms has been studied by Ardito and Ricciardi [1], Ardito and Velnole [2], Di Blasio [6], Di Blasio, Kunisch and Sinestrari [7, 8] for linear equations, and by Yong and Pan [26] for quasi-linear equations. However, the structural study as well as the adjoint theory for (**E**) as developed in Nakagiri [17], in which general retarded equations with bounded operators in delayed terms are treated, has not been studied except for Tanabe [23], Jeong [10] and Kunisch and Mastinšek [13]. The unbounded operators acting on delayed terms of equations studied in [13] are given by fractional powers of operators which generate analytic semigroups, and the equations do not cover the case studied in [23]. In Tanabe [23] the structural operators are first introduced for (**E**) and the basic properties of structural operators and solution semigroups associated with (**E**) are announced, but the detail of results is not given there. Our approach to the basic questions such as the existence, uniqueness and wellposedness of solutions is close to that in Di Blasio, Kunisch and Sinestrari [7], but our treatment using the sesquilinear forms gives a more general and convenient way for the semigroup theoretical study of (**E**). In fact, our solution semigroup is defined on a larger product space than that in [7].

The purposes of this paper are the followings; the first is to give complete proofs of theorems in [23], the second is to develop a further and comprehensive study of structural properties of solution semigroups in the product Hilbert space M_2 including adjoint semigroup theory.

We will carry out this work in the following manner. First of all the existence and uniqueness result of solutions is proved in the M_2-space structure of initial conditions. Nest, by introducing the fundamental solution and the structural operator F which is unbounded and represents the effect of time delays, a simple form of the variation of constants formula for the solutions is established. According to the formula the structural operator $G(t)$ is introduced to characterize the solution semigroup. It is important to develop the adjoint theory of functional differential equations for practical applications to control problems (cf. Nakagiri and Yamamoto [18, 19], Jeong [11], Salamon [20]). For this the transposed equation is introduced and the adjointness of solutions, fundamental solutions, and transposed solution semigroups is investigated. The surjectivity of the operator F and the image of the specific operator $G=G(h)$ are also studied. The structural operators F and G are used to give the most useful relations between the transpsoed solution semigroup and the adjoint solution semigroup. Finally, an application to a practical partial functional differential equation of parabolic type as well as its transposed equation is given.

2. Functional Differential Equations

First we shall give an exact description of the following functional differential equation (**E**) on a Hilbert space H:

$$(2.1)\qquad \frac{du(t)}{dt}=A_0u(t)+A_1u(t-h)+\int_{-h}^{0}a(s)A_2u(t+s)ds+f(t)\,,\quad \text{a.e. } t\geq 0$$

$$(2.2)\qquad u(0)=g^0\,,\quad u(s)=g^1(s)\quad \text{a.e. } s\in[-h,0)\,.$$

Let H be a 'pivot' complex Hilbert space and V be a complex Hilbert space such that V is dense in H and the inclusion map $i\colon V\to H$ is continuous. The norms of H, V and the inner product of H are denoted by $|\cdot|$, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. By identifying the antidual of H with H we may consider $V\subset H\subset V^*$. The norm of the dual space V^* is denoted by $\|\cdot\|_*$. Let $a(u, v)$ be a bounded sesquilinear form defined in $V\times V$ satisfying Gårding's inequality

$$(2.3)\qquad \operatorname{Re} a(u,u)\geq c_0\|u\|^2-c_1|u|^2\,,$$

where $c_0>0$ and $c_1\geq 0$ are real constants. Let A_0 be the operator associated with this sesquilinear form

$$(2.4)\qquad \langle v, A_0u\rangle=-a(u,v)\,,\quad u,v\in V\,,$$

where $\langle\cdot,\cdot\rangle$ denotes also the duality pairing between V and V^*. The operator A_0 is bounded linear from V into V^*. The realization of A_0 in H, which is the restriction of A_0 to the domain $\mathcal{D}(A_0)=\{u\in V; A_0u\in H\}$ is also denoted by A_0. It is proved in Tanabe [21; Chap. 3] that A_0 generates an analytic semigroup $e^{tA_0}=T(t)$ both in H and V^* and that $T(t)\colon V^*\to V$ for each $t>0$. Throughout this paper it is assumed that each $A_i (i=1,2)$ is bounded and linear from V to V^* (i.e., $A_i\in\mathcal{L}(V, V^*)$) such that A_i maps $\mathcal{D}(A_0)$ endowed with the graph norm of A_0 to H continuously. The real valued scalar function $a(s)$ is assumed to be L^2-integrable on $[-h, 0]$, that is, $a(\cdot)\in L^2(-h, 0)$.

For brevity of notations, we introduce a Stieltjes measure η given by

$$(2.5)\qquad \eta(s)=-\chi_{(-\infty,-h]}(s)A_1-\int_s^0 a(\xi)d\xi A_2\colon V\to V^*\,,\quad s\in[-h,0]\,,$$

where $\chi_{(-\infty,-h]}$ denotes the characteristic function of $(-\infty, -h]$. Then the delayed terms in (2.1) are written simply by $\int_{-h}^{0} d\eta(s)u(t+s)$.

In this section we give the existence and uniqueness result of solutions and establish a variation of constants formula of solutions in terms of the fundamental solution.

Definition A function $u\in L^2_{\mathrm{loc}}(\boldsymbol{R}^+; V)\cap W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)$ is said to be a solution of (**E**) if $u(t)$ satisfies (2.1) and (2.2).

In order to solve the existence and uniqueness problem for (**E**) we start with the following proposition due to Lions [14; Chapter 3] and Tanabe [21 (see also Lions and Magenes [15; Chapter 1]).

Propostion 1 *For any $u_0 \in H$ and $f \in L^2_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)$ the Cauchy problem*

$$\frac{du(t)}{dt} = A_0 u(t) + f(t), \quad \text{a.e. } t > 0, \quad u(0) = u_0 \tag{2.6}$$

admits a unique solution

$$u \in L^2_{\mathrm{loc}}(\boldsymbol{R}^+; V) \cap W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}^+; V^*) \subset C(\boldsymbol{R}^+; H) \tag{2.7}$$

which is represented by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad t \geq 0. \tag{2.8}$$

Further, for each $T > 0$ there exists a constant C_T depending only on T, c_0, c_1 such that

$$\left(\int_0^T \|u(t)\|^2 dt + \int_0^T \left\|\frac{du(t)}{dt}\right\|_*^2 dt\right) \leq C_T\left(|u_0|^2 + \int_0^T \|f(t)\|_*^2 dt\right). \tag{2.9}$$

We shall solve the equation (**E**) by considering the following functional integral equation

$$u(t) = \begin{cases} T(t)g^0 + \int_0^t T(t-s)f(s)ds \\ \quad + \int_0^t T(t-s)\int_{-h}^0 d\eta(\xi)u(\xi+s)ds, & t \geq 0 \\ g^1(t) \quad \text{a.e. } t \in [-h, 0). \end{cases} \tag{2.10}$$

In view of Proposition 1, the solution $u(t)$ of the integral equation (2.10) satisfies the equation (2.1) in the sense of vectorial distribution (cf. Barbu [3; Chap.1]).

Theorem 1 *Let $f \in L^2_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)$ and $g = (g^0, g^1) \in H \times L^2(-h, 0; V)$. Then there exists a unique solution $u(t) = u(t; f, g)$ of* (**E**) *satisfying*

$$u \in L^2_{\mathrm{loc}}(-h, \infty; V) \cap W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}^+; V^*) \subset C([0, \infty); H). \tag{2.11}$$

Further, for each $T > 0$ there is a constant K_T such that

$$\begin{aligned} &\left(\int_0^T \|u(t)\|^2 dt + \int_0^T \left\|\frac{du(t)}{dt}\right\|_*^2 dt\right) \\ &\leq K_T\left(|g^0|^2 + \int_{-h}^0 \|g^1(s)\|^2 ds + \int_0^T \|f(t)\|_*^2 dt\right). \end{aligned} \tag{2.12}$$

Proof. Leg $b>0$ be fixed and define the operator $\mathcal{E}: L^2(-h, b; V)\to L^2(0, b; V^*)$ by

$$(\mathcal{E}w)(s)=\int_{-h}^{0} d\eta(\xi)w(s+\xi) \quad \text{a.e. } s\in[0, b] \tag{2.13}$$

for $w\in L^2(-h, b; V)$. Using Schwartz inequality, from $A_1, A_2\in\mathcal{L}(V, V^*)$ and $a\in L^2(-h, 0)$ we have the following estimate:

$$\|\mathcal{E}w\|_{L^2(0,b;V^*)} \le \begin{cases} \|a\|_{L^2(-h,0)}\|A_2\|_{\mathcal{L}(V,V^*)}b^{1/2}\|w\|_{L^2(-h,b;V)} & \text{if } b<h \\ (\|A_1\|_{\mathcal{L}(V,V^*)}+\|a\|_{L^2(-h,0)}\|A_2\|_{\mathcal{L}(V,V^*)}b^{1/2})\,\|w\|_{L^2(-h,b;V)} & \text{if } b\ge h\,, \end{cases} \tag{2.14}$$

where $\|a\|_{L^2(-h,0)}=(\int_{-h}^{0}|a(s)|^2ds)^{1/2}$.

For any $v\in L^2(0, b; V)$ we define the operator $\mathcal{S}: L^2(0, b; V)\to L^2(0, b; V)$ by

$$(\mathcal{S}v)(t)=T(t)g^0+\int_0^t T(t-s)(f(s)+\mathcal{E}\bar{v}(s))ds\,, \quad t\in[0, b]\,, \tag{2.15}$$

where the extension $\bar{v}\in L^2(-h, b; V)$ of v is given by

$$\bar{v}(t)=\begin{cases} v(t) & \text{a.e. } t\in[0, b] \\ g^1(t) & \text{a.e. } t\in[-h, 0]\,. \end{cases}$$

It is easily verified, by Proposition 1 and the estimate (2.14), that $\mathcal{S}$ is into and the image $\operatorname{Im}\mathcal{S}$ is contained in $W^{1,2}(0, b; V^*)$. Let $v_1, v_2\in L^2(0, b; V)$. If $b<h$, then by (2.9) and (2.14)

$$\begin{aligned}\|\mathcal{S}v_1-\mathcal{S}v_2\|_{L^2(0,b;V)} &\le \left(\int_0^b \left\|\int_0^t T(t-s)\mathcal{E}(\bar{v}_1-\bar{v}_2)(s)ds\right\|^2 dt\right)^{1/2} \\ &\le C_b\|\mathcal{E}(\bar{v}_1-\bar{v}_2)\|_{L^2(0,b;V^*)} \\ &\le C_b\|a\|_{L^2(-h,0)}\|A_2\|_{\mathcal{L}(V,V^*)}b^{1/2}\|v_1-v_2\|_{L^2(0,b;V)}\,.\end{aligned}$$

This implies that the operator $\mathcal{S}$ is a contraction and hence has a unique fixed point $u\in L^2(0, b; V)\cap W^{1,2}(0, b; V^*)$ for sufficiently small $b>0$. This function u gives a unique solution to the equation (2.10) on the interval $[0, b]$. Combining the estimates (2.9) and (2.14) via the equation (2.10) and using that b is small, we can verify that the estimate (2.12) for the solution holds true. By the inclusion $L^2(0, b; V)\cap W^{1,2}(0, b; V^*)\subset C([0, b]; H)$, we see easily that $u(b)\in H$. Hence by the step by step method using Proposition 1 and the estimate (2.14) we have the existence and uniqueness of a global solution $u(t)$ of (2.10) on $\boldsymbol{R}^+$ satisfying the estimate (2.12) for each $T>0$ as shown in Di Blasio, Kunisch and Sinestrari [7;Theorem 3.3]. □

Using Theorem 1 and repeating the similar argument in [7; Theorem 3.4], we can obtain the following theorem on the 'strong' solution of **(E)**.

Theorem 2. *If* $f\in W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)$ *and* $g=(g^0, g^1)$ *satisfy*

$$g^1\in W^{1,2}(-h, 0; V), \quad g^1(0)=g^0, \quad A_0g^0+\int_{-h}^{0} d\eta(s)g^1(s)ds\in H, \tag{2.16}$$

then the solution u *of* (**E**) *satisfies*

$$u\in W^{1,2}_{\mathrm{loc}}(-h, \infty; V)\cap W^{2,2}_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)\subset C^1([0, \infty); H). \tag{2.17}$$

According to Nakagiri [17], we define the fundamental solution $W(t)$ of (**E**) by

$$W(t)g^0=\begin{cases} u(t; 0, (g^0, 0)), & t\geq 0 \\ O & t<0 \end{cases} \quad \text{for} \quad g^0\in H. \tag{2.18}$$

In other words, $W(t)$ is defined as a unique solution of

$$W(t)=\begin{cases} T(t)+\int_0^t T(t-s)\int_{-h}^0 d\eta(\xi)W(\xi+s)ds, & t\geq 0 \\ O & t<0 \end{cases} \tag{2.19}$$

in $\mathcal{L}(H)$. By Theorem 1 we see easily that $W(t)$ is strongly continuous in H and $W(\cdot)g^0\in L^2_{\mathrm{loc}}(\boldsymbol{R}^+; V)\cap W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)$ for each $g^0\in H$.

If we want to get a stronger regularity of $W(t)$ like as that for $T(t)$, we have to solve the equation (2.19) for $W(t)$ in the space $\mathcal{L}(V^*)$. For this sake we assume the following additional condition on $a(s)$, which is used essentially in Tanabe [22, 24].

$$a(s) \text{ is Hölder continuous on } [-h, 0]. \tag{2.20}$$

Modifying Theorem 1 in Tanabe [23] on our Hilbert space setting, we have the following lemma.

Lemma 1. *Under the condition* (3.20), *the fundamental solution* $W(t)$ *is strongly continuous in* V^*, $W(t)\colon V^*\to V$ *for each* $t>0$ *and satisfies*

$$\frac{d}{dt}W(t)=A_0W(t)+\int_{-h}^0 d\eta(s)W(t+s) \quad \text{a.e. } t>0. \tag{2.21}$$

The functions $\frac{d}{dt}W(t)$ *and* $(A_0+c_1)\,W(t)$ *are strongly continuous in* V^* *on each* $(nh, (n+1)h]$, $n=0, 1, 2, \cdots$, *and the following estimates hold for some constants* C_n:

$$\begin{cases} \|(A_0+c_1)W(t)\|_{\mathcal{L}(V^*)}\leq \dfrac{C_n}{t-nh} \\ \|\dfrac{d}{dt}W(t)\|_{\mathcal{L}(V^*)}\leq \dfrac{C_n}{t-nh} \\ \|\int_{nh}^t (A_0+c_1)W(s)ds\|_{\mathcal{L}(V^*)}\leq C_n \end{cases} \quad \text{for} \quad t\in(nh, (n+1)h]. \tag{2.22}$$

By virtue of Lemma 1 we can derive the next proposition which is crucial for the variation of constants formula of solutions.

Proposition 2. *Let $f \in L^2_{\text{loc}}(\boldsymbol{R}; V^*)$. Then the function $v(t)=\int_0^t W(t-s)f(s)ds$ solves the integral equation* (2.10) *with $g^0=0$, $g^1=0$ and satisfies*

$$v \in L^2_{\text{loc}}(\boldsymbol{R}^+; V) \cap W^{1,2}_{\text{loc}}(\boldsymbol{R}^+; V^*) \subset C([0, \infty); H). \tag{2.23}$$

Hence $v(t)$ gives a unique solution of (**E**) *with zero initial data.*

Proof. Since $W(t)$ is strongly continuous in V^*, $v(t)$ makes sense as a Bochner integral in V^* for each $t>0$. Noting that $W(t)\colon V^* \to V$ and $W(t)$ satisfies (2.19) in $\mathcal{L}(V^*)$, we have for $t>0$

$$\begin{aligned} v(t) &= \int_0^t W(t-s)f(s)ds \\ &= \int_0^t T(t-s)f(s)ds \\ &\quad + \int_0^t \Big(\int_0^{t-s} T(t-s-\xi)A_1 W(\xi-h)d\xi\Big) f(s)ds \\ &\quad + \int_0^t \Big(\int_0^{t-s} T(t-s-\xi)\int_{-h}^0 a(\tau)A_2 W(\xi+\tau)d\tau d\xi\Big) f(s)ds \\ &= I_1+I_2+I_3. \end{aligned} \tag{2.24}$$

Using Fubini's theorem seveial times and noting that $W(t)=O$ for $t<0$, we transform the integrals I_2, I_3 in (2.24) as

$$\begin{aligned} I_2 &= \int_0^t \int_s^t T(t-\beta)A_1 W(\beta-s-h)d\beta f(s)ds \quad (\text{by } s+\xi \to \beta) \\ &= \int_0^t T(t-\beta)A_1\Big(\int_0^\beta W(\beta-s-h)f(s)ds\Big)d\beta \\ &= \int_0^t T(t-s)A_1\Big(\int_0^{s-h} W(s-h-\tau)f(\tau)d\tau\Big)ds\,; \\ &\qquad\qquad (\text{by } \beta \to s,\ s \to \tau) \end{aligned} \tag{2.25}$$

$$\begin{aligned} I_3 &= \int_0^t \int_s^t T(t-\beta)\int_{-h}^0 a(\tau)A_2 W(\beta-s+\tau)d\tau d\beta f(s)ds \\ &= \int_0^t T(t-\beta)\Big(\int_0^\beta \int_{-h}^0 a(\tau)A_2 W(\beta-s+\tau)d\tau f(s)ds\Big)d\beta \\ &= \int_0^t T(t-\beta)\Big(\int_{-h}^0 a(\tau)A_2 \int_0^\beta W(\beta-s+\tau)f(s)d\tau ds\Big)d\beta \\ &= \int_0^t T(t-s)\Big(\int_{-h}^0 a(\tau)A_2\Big[\int_0^{s+\tau} W(s+\tau-\xi)f(\xi)d\xi\Big]d\tau\Big)ds. \\ &\qquad\qquad (\text{by } \beta \to s,\ s \to \xi) \end{aligned} \tag{2.26}$$

Hence by (2.24)-(2.26), we can verify that $v(t)$ satisfies the integral equation

$$(2.27)\qquad v(t)=\begin{cases}\int_0^t T(t-s)f(s)ds+\int_0^t T(t-s)\int_{-h}^0 d\eta(\xi)v(\xi+s)ds\,, & t\geq 0\\ 0 \quad \text{a.e. } t\in[-h,0)\end{cases}$$

in the space V. Next we shall prove (2.23). To this end we use the density argument and Lemma 1. Let $T>0$ and assume further that $f\in C^\rho([0,T];V^*)$, $0<\rho<1$, the space of Hölder continuous functions in $[0, T]$ of exponent ρ with values in V^*. Let $||f||_\rho=\sup_{0\leq s<t\leq T}||f(t)-f(s)||_*|t-s|^{-\rho}$. Then by the estimates (2.22), we have for $t\in(nh,(n+1)h],\leq T$

$$(2.28)\qquad \begin{aligned}||(A_0+c_1)v(t)||_* &= ||\int_0^t (A_0+c_1)W(s)f(t-s)ds||_*\\
&\leq \sum_{j=0}^{n-1}||\int_{jh}^{(j+1)h}(A_0+c_1)W(s)(f(t-s)-f(t-jh))ds||_*\\
&\quad+||\int_{nh}^t(A_0+c_1)W(s)(f(t-s)-f(t-nh))ds||_*\\
&\quad+\sum_{j=0}^{n-1}||\int_{jh}^{(j+1)h}(A_0+c_1)W(s)ds\,f(t-jh)||_*\\
&\quad+||\int_{nh}^t(A_0+c_1)W(s)ds\,f(t-nh)||_*\\
&\leq \sum_{j=0}^{n-1}\int_{jh}^{(j+1)h}||(A_0+c_1)W(s)||_{\mathcal{L}(V^*)}||f(t-s)-f(t-jh)||_*ds\\
&\quad+\int_{nh}^t||(A_0+c_1)W(s)||_{\mathcal{L}(V^*)}||f(t-s)-f(t-nh)||_*ds\\
&\quad+\sum_{j=0}^{n-1}||\int_{jh}^{(j+1)h}(A_0+c_1)W(s)ds||_{\mathcal{L}(V^*)}||f(t-jh)||_*\\
&\quad+||\int_{nh}^t(A_0+c_1)W(s)ds||_{\mathcal{L}(V^*)}||f(t-nh)||_*\\
&\leq \sum_{j=0}^{n-1}C_j||f||_\rho\int_{jh}^{(j+1)h}(s-jh)^{-1}(s-jh)^\rho ds\\
&\quad+C_n||f||_\rho\int_{nh}^{(n+1)h}(s-nh)^{-1}(s-nh)^\rho ds\\
&\quad+\sum_{j=0}^{n-1}C_j||f(t-jh)||_*+C_n||f(t-nh)||_*\\
&\leq \frac{K_1h^\rho}{\rho}+K_2\Big(\sup_{t\in[0,T]}||f(t)||_*\Big)\,;\end{aligned}$$

$$(2.29)\qquad \begin{aligned}&||\int_0^t\Big(\frac{d}{dt}W(t-s)\Big)f(s)ds||_*\\
&\leq \sum_{j=0}^{n-1}||\int_{jh}^{(j+1)h}\frac{d}{ds}W(s)(f(t-s)-f(t-jh))ds||_*\\
&\quad+||\int_{nh}^t\frac{d}{ds}W(s)(f(t-s)-f(t-nh))ds||_*\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{n-1}\|\int_{jh}^{(j+1)h}\left(\frac{d}{ds}W(s)ds\right)f(t-jh)\|_*\\
&+\|\int_{nh}^{t}\left(\frac{d}{ds}W(s)ds\right)f(t-nh)\|_*\\
\le&\sum_{j=0}^{n-1}\int_{jh}^{(j+1)h}\|\left(\frac{d}{ds}W(s)\right)\|_{\mathcal{L}(V^*)}\|f(t-s)-f(t-jh)\|_*ds\\
&+\int_{nh}^{t}\|\left(\frac{d}{ds}W(s)\right)\|_{\mathcal{L}(V^*)}\|f(t-s)-f(t-nh)\|_*ds\\
&+\sum_{j=0}^{n-1}\|W((j+1)h)-W(jh)\|_{\mathcal{L}(V^*)}\|f(t-jh)\|_*\\
&+\|W(t)-W(nh)\|_{\mathcal{L}(V^*)}\|f(t-nh)\|_*\\
\le&\frac{K_1h^\rho}{\rho}+K_2\left(\sup_{t\in[0,T]}\|f(t)\|_*\right)
\end{aligned}
$$

for some constants K_1, K_2. Here in (2.29) we use the uniform boundedness

$$\left(\sup_{t\in[0,T]}\|W(t)\|_{\mathcal{L}(V^*)}\right)<\infty$$

because of the strong continuity of $W(t)$ in V^*. Since A_0+c_1 is an isomorphism from V to V^*, the estimate (2.28) implies $v\in L^2(0, T; V)$. In view of Theorem 2 in Tanabe [22] and the estimate (2.29), $v(t)$ is strongly differentiable in V^* and its derivative is given by

$$\frac{dv(t)}{dt}=f(t)+\int_0^t\frac{d}{dt}W(t-s)f(s)ds\,,\quad t>0\,, \tag{2.30}$$

and hence

$$\|\frac{dv(t)}{dt}\|_*\le\|f(t)\|_*+\frac{K_1h^\rho}{\rho}+K_2\left(\sup_{t\in[0,T]}\|f(t)\|_*\right). \tag{2.31}$$

This estimate implies $v(\cdot)\in W^{1,2}(0, T; V^*)$. Therefore, by differentiating the integral equation (2.27) we can verify that $v(t)$ gives a solution of (**E**) with the data $f\in C^\rho([0, T]; V^*)$ and $g=0$.

Now let $f\in L^2_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)$. Since the Hölder space $C^\rho([0, T]; V^*)$ is dense in $L^2(0, T; V^*)$, there exists a sequence $\{f_n\}\subset C^\rho([0, T]; V^*)$ such that $f_n\to f$ in $L^2(0, T; V^*)$. Set

$$v_n(t)=\int_0^t W(t-s)f_n(s)ds\,. \tag{2.32}$$

It is clear by the strong continuity of $W(t)$ in V^* that

$$v_n(t)\to v(t)\quad\text{in}\quad V^*\quad\text{for each}\quad t>0\,. \tag{2.33}$$

Since $v_n(t)-v_m(t)$ give a solution of (E) with $f=f_n-f_m$ and $g=0$, we have by the boundedness (2.12) in Theorem 1

$$\left(\int_0^T \|v_n(t)-v_m(t)\|^2 dt+\int_0^T \left\|\frac{dv_n(t)}{dt}-\frac{dv_m(t)}{dt}\right\|_*^2 dt\right) \tag{2.34}$$
$$\leq K_T\left(\int_0^T \|f_n(t)-f_m(t)\|_*^2 dt\right),$$

i.e., $\{v_n\}$ and $\{dv_n/dt\}$ are Cauchy sequences in $L^2(0, T; V)$ and in $L^2(0, T; V^*)$, respectively. Then by the closedness of differential operator d/dt in $L^2(0, T; V^*)$, there exists a function $v_0 \in L^2(0, T; V)\cap W^{1,2}(0, T; V^*)$ such that

$$v_n \to v_0 \text{ in } L^2(0, T; V), \quad \frac{dv_n}{dt} \to \frac{dv_0}{dt} \text{ in } L^2(0, T; V^*). \tag{2.35}$$

Combining (2.33) and the easy consequence $v_n(t)\to v_0(t)$ in H from (2.35), we have $v_0(t)=v(t)$ for each $t\in[0, T]$. This proves that $v\in L^2(0, T; V)\cap W^{1,2}(0, T; V^*)$, so that v gives a solution of (E) on $[0, T]$ with the data $f\in L^2_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)$ and $g=0$. This completes the proof of Proposition 2. □

Here we note an important consequence from Proposition 2. Let $f\in L^2_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)$. Then by Lemma 1, we have

$$\int_0^t \frac{d}{dt}W(t-s)f(s)ds \tag{2.36}$$
$$= A_0\int_0^t W(t-s)f(s)ds+A_1\int_0^t W(t-s-h)f(s)ds$$
$$+A_2\int_0^t\int_{-h}^0 a(\xi)W(t-s+\xi)d\xi\, f(s)ds.$$

It is easily verified that all integrals in the right hand side of (2.36) make sense as integrals in V. Hence for each $t\geq 0$, the integral $\int_0^t \frac{d}{dt}W(t-s)f(s)ds$ makes sense as an integral in V^*. Thus from Proposition 2 it is calculated by standard manipulations using the Lebesgue density theorem that

$$\frac{d}{dt}\int_0^t W(t-s)f(s)ds = f(t)+\int_0^t \frac{d}{dt}W(t-s)f(s)ds \quad \text{a.e. } t\geq 0. \tag{2.37}$$

For each $t>0$, we introduce the operator valued function $U_t(\cdot)$ defined by

$$U_t(s) = W(t-s-h)A_1+\int_{-h}^s W(t-s+\xi)a(\xi)A_2 d\xi \tag{2.38}$$
$$= \int_{-h}^s W(t-s+\xi)d\eta(\xi)\colon V\to V, \quad \text{a.e. } s\in[-h, 0].$$

Let $T>0$ be arbitrary fixed. Associated with $U_t(\cdot)$, we consider the operator $\mathcal{U}: L^2(-h, 0; V)\to L^2(0, T; V)$ defined by

$$(\mathcal{U}g^1)(t) = \int_{-h}^{0} U_t(s)g^1(s)ds\,, \quad t\in[0, T] \tag{2.39}$$

for $g^1\in L^2(-h, 0; V)$. We want to show that $\mathcal{U}$ is into and bounded for each $T>0$. For this sake it is convenient to introduce the structural operator $F_1: L^2(-h, 0; V)\to L^2(-h, 0; V^*)$ defined by

$$\begin{aligned}[F_1g^1](s) &= \int_{-h}^{s} d\eta(\xi)g^1(\xi-s) \\ &= A_1g^1(-h-s)+\int_{-h}^{s} a(\xi)A_2g^1(\xi-s)d\xi \quad \text{a.e. } s\in[-h, 0\,].\end{aligned} \tag{2.40}$$

By simple calculations using Schwartz inequality we see that F_1 is into and bounded. Then by using Fubini's theorem as in Nakagiri [17; Lemma 4.1], we have

$$\begin{aligned}(\mathcal{U}g^1)(t) &= \int_{-h}^{0} W(t+s)[F_1g^1](s)ds \\ &= \int_{0}^{t} W(t-s)\phi(s)ds \quad \text{a.e. } s\in[-h, 0]\,,\end{aligned} \tag{2.41}$$

where $\phi(\cdot)=\chi_{[0,h]}[F_1g^1](-\cdot)\in L^2(0, T; V^*)$. Hence by Proposition 2, $\mathcal{U}$ is into and bounded for any $T>0$. At the same time we know $\mathcal{U}g^1\in W^{1,2}(0, T; V^*)$, $T>0$.

The following variation of constants formula of solutions are fundamental in our structural study for (**E**) (cf. Nakagiri [17]). Recently Yong and Pan [26] have derived a similar formula in a slightly different form under some abstract assumptions on the delay terms.

Theorem 3. *For $f\in L^2_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)$ and $g=(g^0, g^1)\in H\times L^2(-h, 0; V)$, the solution $u(t)=u(t; f, g)$, $t\geq 0$ of* (**E**) *is represented by*

$$u(t; f, g) = W(t)g^0+\int_{-h}^{0} U_t(s)g^1(s)ds+\int_{0}^{t} W(t-s)f(s)ds\,. \tag{2.42}$$

Proof. Now it is obvious from the definition of fundamental solution and Proposition 2 that, by the uniqueness of solutions in the class $L^2_{\mathrm{loc}}(\boldsymbol{R}^+; V)\cap W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}^+; V^*)$

$$u(t; f, (g^0, 0)) = W(t)g^0+\int_{0}^{t} W(t-s)f(s)ds\,. \tag{2.43}$$

Thus, it is left to prove that for $g^1\in L^2(-h, 0; V)$

$$u(t; 0, (0, g^1)) = \int_{-h}^{0} U_t(s)g^1(s)ds = (\mathcal{U}g^1)(t)\,. \tag{2.44}$$

Since $\mathcal{U}g^1\in L^2(0, T; V)\cap W^{1,2}(0, T; V^*)$ for each $T>0$, it is sufficient to verify that

$$v(t)=\begin{cases}(\mathcal{U}g^1)(t), & t\geq 0\\ g^1(t) & \text{a.e. } t\in[-h, 0)\end{cases}$$

satisfies the integral equation (2.10) with $g^0=0, f=0$. Let us set $\phi(s)=\chi_{[0,h]}[F_1g^1](-s)$. Then we can write $v(t)$ as $\int_0^t W(t-s)\phi(s)ds$ for $t\geq 0$. If we set $\tilde{v}(t)=\begin{cases}v(t), & t\geq 0\\ 0 & \text{a.e. } t\in[-h, 0)\end{cases}$, then as proved in Proposition 2, $\tilde{v}(t)$ satisfies

$$\tilde{v}(t)=\begin{cases}\int_0^t T(t-s)\phi(s)ds+\int_0^t T(t-s)A_1\tilde{v}(s-h)ds\\ \quad +\int_0^t T(t-s)(\int_{-h}^0 a(\tau)A_2\tilde{v}(s+\tau)d\tau)ds, & t\geq 0\\ 0 & \text{a.e. } t\in[-h, 0).\end{cases}$$

Hence for $t\geq 0$, by defining $T(t)=O$ for $t<0$ for simplicity, we have

$$\begin{aligned}v(t) &= \int_0^h T(t-s)[F_1g^1](-s)ds\\ &\quad +\int_0^h T(t-s)(\int_{-s}^0 a(\tau)A_2v(s+\tau)d\tau)ds\\ &\quad +\int_h^t T(t-s)(A_1v(s-h)+\int_{-h}^0 a(\tau)A_2v(s+\tau)d\tau)ds\\ &= \int_0^h T(t-s)(A_1g^1(s-h)+\int_{-h}^{-s} a(\tau)A_2g^1(s+\tau)d\tau)ds\\ &\quad +\int_0^h T(t-s)(\int_{-s}^0 a(\tau)A_2v(s+\tau)d\tau)ds\\ &\quad +\int_h^t T(t-s)(\int_{-h}^0 d\eta(\tau)v(s+\tau)d\tau)ds\\ &= \int_0^t T(t-s)(\int_{-h}^0 d\eta(\tau)v(s+\tau)d\tau)ds, \quad t\geq 0;\\ v(t) &= g^1(t) \quad \text{a.e. } t\in[-h, 0).\end{aligned}$$

This proves (2.44). Thus, we show the formula (2.42). □

3. Solution Semigroups and Structural Operators

As in previous researches [4–8, 16, 17–19, 20, 22, 23, 25] we take a product state space approach to study the properties of solution smeigroups and the adjoint semigroups associated with the functional differential equations. Our main concern is to establish the relations between the semigroups and structural operators F, $G(t)$ introduced in Nakagiri [17] and Tanabe [23] (see also Bernir and Manitius [4], Delfour and Manitius [5], Manitius [16] for those in Euclidean spaces and Kunisch and Mastinšek [13] in Hilbert spaces). In what follows we assume the condition (2.20) on $a(s)$.

Let $M_2=H\times L^2(-h, 0; V)$ be the state space of the equation (**E**). M_2 is a product Hilbert space with the inner product

$$\langle g, k\rangle_{M_2} = \langle g^0, k^0\rangle + \int_{-h}^{0} \langle g^1(s), k^1(s)\rangle_V \, ds ,$$
$$g = (g^0, g^1), \quad k = (k^0, k^1) \in M_2 ,$$

where $\langle \cdot, \cdot\rangle_V$ denotes the inner product of V.

Let $g\in M_2$ and $u(t; g)$ be the solution of (**E**) with $f=0$. The segment u_t is given by $u_t(s; g)=u(t+s; g)$, $s\in[-h, 0]$. The solution semigroup $S(t)$ associated with (**E**) is defined by

$$S(t)g = (u(t; g), u_t(\cdot\,; g)) , \quad t\geq 0 , \quad g\in M_2 . \tag{3.1}$$

Then we have the following theorem which follows from Thoerem 1 and and Theorem 2 as shown in Di Blasko, Kunisch and Sinestrari [7].

Theorem 4. (i) *The family of operators* $\{S(t); t\geq 0\}$ *is a* C_0*-semigroup on* M_2.
(ii) *The infinitesimal generator* A *of* $S(t)$ *is characterized by*

$$\mathcal{D}(A) = \{g = (g^0, g^1); g^1\in W^{1,2}(-h, 0; V) , \tag{3.2}$$
$$g^1(0) = g^0 , \quad A_0 g^0 + \int_{-h}^{0} d\eta(s) g^1(s) ds \in H\}$$

$$Ag = (A_0 g^0 + \int_{-h}^{0} d\eta(s) g^1(s) ds, \dot{g}^1) \qquad \text{for} \quad g = (g^0, g^1)\in\mathcal{D}(A) , \tag{3.3}$$

where $\dot{g}^1$ *denotes the distribution derivative* dg^1/ds.

In order to decompose the semigroup $S(t)$ on the basis of the variation of constants formula of solutions, we introduce the structural operators F and $G(t)$. The structural operator $F\colon M_2\to M_2^*$ is defined by $F=\begin{pmatrix} I & O \\ O & F_1 \end{pmatrix}$ i.e.,

$$[Fg]^0 = g^0 , \quad [Fg]^1 = F_1 g^1 , \qquad \text{for} \quad g = (g^0, g^1)\in M_2 , \tag{3.4}$$

where F_1 is given in (2.40). As is seen before F is into and bounded.

For $t\geq 0$, the structural operator $G(t)\colon M_2^*\to M_2$ is defined by

$$\begin{aligned} &[G(t)f]^1(s) = W(t+s)f^0 + \int_{-h}^{0} W(t+s+\xi) f^1(\xi) d\xi \quad s\in[-h, 0] , \\ &[G(t)f]^0 = [G(t)f]^1(0) , \qquad\qquad f = (f^0, f^1)\in M_2^* . \end{aligned} \tag{3.5}$$

In view of Proposition 2 we see that $G(t)$ is also into and bounded. We define the structural operator $G: M_2^* \to M_2$ by

$$G = G(h). \tag{3.6}$$

Proposition 3. *The semigroup $S(t)$ is represented by*

$$S(t) = G(t)F + \kappa(t), \qquad t \geq 0, \tag{3.7}$$

where $\kappa(t); M_2 \to M_2$ is given by

$$[\kappa(t)g]^0 = 0, \quad [\kappa(t)g]^1(s) = \chi_{[-h,-t]}(s) g^1(t+s) \quad \text{a.e. } s \in [-h, 0]. \tag{3.8}$$

In particular, $S(h)$ has the following decomposition

$$S(h) = GF. \tag{3.9}$$

Proof. This proposition follows immediately from the equality

$$u(t+s; g) = \begin{cases} W(t+s)g^0 + \int_{-h}^0 W(t+s+\xi)[Fg]^1(\xi)d\xi, & t+s \geq 0 \\ g^1(t+s) & t+s < 0, \end{cases} \tag{3.10}$$

which is proved by Theorem 3 and (2.41), via the definition (3.5).

Using the trivial relation $AS(t) = S(t)A$ on $\mathcal{D}(A)$ and Proposition 3, we can derive the following useful commutative relation between $W(t)$ and A_0, η. A direct proof of the relation is given in Tanabe [23].

Corollary 1. *For any $g^0 \in V$, the following relation holds:*

$$\begin{aligned} & A_0 W(t) g^0 + \int_{-h}^0 d\eta(s) W(t+s) g^0 \\ & = W(t) A_0 g^0 + \int_{-h}^0 W(t+s) d\eta(s) g^0 \qquad \text{for all} \quad t \in \boldsymbol{R}^+. \end{aligned} \tag{3.11}$$

Proof. By improving the proof of Bernir and Manitius [4; Lemma 4.1], we shall show this corollary. For each $g \in W^{1,2}(-h, 0; V)$ it is calculated that

$$\frac{d}{ds}[Fg]^1(s) = a(s)A_2 g^1(0) - [F\dot{g}]^1(s) \quad \text{a.e. } s \in [-h, 0] \tag{3.12}$$

and $\frac{d}{ds}[Fg]^1$, $a(\cdot)A_2 g^1(0)$, $[F\dot{g}]^1 \in L^2(-h, 0; V^*)$. Let $g = (g^0, g^1) \in \mathcal{D}(A)$. Then by (3.3), (3.5), (3.7) and noting that $W(s) = O$ if $s < 0$,

$$\begin{aligned} & [AS(t)g]^0 \\ & = A_0[S(t)g]^0 + \int_{-h}^0 d\eta(s)[S(t)g]^1(s) \\ & = A_0 W(t) g^0 + A_0 \int_{-h}^0 W(t+\xi)[Fg]^1(\xi) d\xi \end{aligned} \tag{3.13}$$

$$+\int_{-h}^{0}\begin{cases} W(t+s)g^0+\int_{-h}^{0} W(t+s+\xi)[Fg]^1(\xi)d\xi\,, & t+s\geq 0 \\ g^1(t+s) & t+s<0 \end{cases}$$

$$= A_0W(t)g^0+\int_{-h}^{0} d\eta(s)W(t+s)g^0$$

$$+\begin{cases} \int_{-t}^{0} A_0W(t+s)[Fg]^1(s)ds+\int_{-h}^{-t} d\eta(s)g^1(t+s) \\ \int_{-h}^{0} A_0W(t+s)[Fg]^1(s)ds \end{cases}$$

$$\left.\begin{matrix} +\int_{-t}^{0} d\eta(s)\left(\int_{-h}^{0} W(t+s+\xi)[Fg]^1(\xi)d\xi\right) & \text{if} \quad 0\leq t<h \\ +\int_{-h}^{0} d\eta(s)\left(\int_{-h}^{0} W(t+s+\xi)[Fg]^1(\xi)d\xi\right) & \text{if} \quad t\geq h \end{matrix}\right\}$$

and, by using Fubini's theorem and changing variables

$$\begin{aligned} (3.14)\qquad & [AS(t)g]^0 \\ &= A_0W(t)g^0+\int_{-h}^{0} d\eta(s)W(t+s)g^0 \\ &+\begin{cases} \int_{-t}^{0}\left(A_0W(t+s)+\int_{-h}^{0} d\eta(\xi)W(t+s+\xi)\right)[Fg]^1(s)ds \\ \int_{-h}^{0}\left(A_0W(t+s)+\int_{-h}^{0} d\eta(\xi)W(t+s+\xi)\right)[Fg]^1(s)ds \end{cases} \\ &\qquad \left.\begin{matrix} +\int_{-h}^{-t} d\eta(s)g^1(t+s) & \text{if} \quad 0\leq t<h \\ +0 & \text{if} \quad t\geq h \end{matrix}\right\}. \end{aligned}$$

On the other hand, by (3.3), (3.7) and (3.12),

$$\begin{aligned} (3.15)\qquad & [S(t)Ag]^0 \\ &= W(t)\left(A_0g^0+\int_{-h}^{0} d\eta(s)g^1(s)\right)+\int_{-h}^{0} W(t+s)[F\dot{g}]^1(s)ds \\ &= W(t)A_0g^0+W(t)\int_{-h}^{0} d\eta(s)g^1(s)+\int_{-h}^{0} W(t+s)a(s)A_2g^1(0)ds \\ &\quad -\int_{-h}^{0} W(t+s)\left(\frac{d}{ds}[Fg]^1(s)\right)ds\,. \end{aligned}$$

Since $W(t)$ is differentiable and $dW(t)/dt$ is piecewise strongly continuous on any bounded interval in $\boldsymbol{R}^+$ by Lemma 1, we use the integration by parts and (2.40) to obtain

$$(3.16)\qquad \int_{-h}^{0} W(t+s)\left(\frac{d}{ds}[Fg]^1(s)\right)ds$$

$$=\begin{cases}\int_{-t}^{0} W(t+s)\left(\frac{d}{ds}[Fg]^1(s)\right)ds & \text{if } 0\le t<h \\ \int_{-h}^{0} W(t+s)\left(\frac{d}{ds}[Fg]^1(s)\right)ds & \text{if } t\ge h\end{cases}$$

$$=\begin{cases}[W(t+s)[Fg]^1(s)]_{-t}^{0}-\int_{-t}^{0} \frac{d}{dt}W(t+s)[Fg]^1(s)ds & \text{if } 0\le t<h \\ [W(t+s)[Fg]^1(s)]_{-h}^{0}-\int_{-h}^{0} \frac{d}{dt}W(t+s)[Fg]^1(s)ds & \text{if } t\ge h\end{cases}$$

$$=\begin{cases}W(t)\int_{-h}^{0} d\eta(s)g^1(s)-\int_{-h}^{-t} d\eta(s)g^1(t+s) & \text{if } 0\le t<h \\ W(t)\int_{-h}^{0} d\eta(s)g^1(s)-W(t-h)A_1g^1(0) & \text{if } t\ge h\end{cases}$$

$$-\begin{cases}\int_{-t}^{0} \frac{d}{dt}W(t+s)[Fg]^1(s)ds & \text{if } 0\le t<h \\ \int_{-h}^{0} \frac{d}{dt}W(t+s)[Fg]^1(s)ds & \text{if } t\ge h\end{cases}.$$

From (3.15) and (3.16), we have

$$(3.17)\qquad [S(t)Ag]^0$$

$$=W(t)A_0g^0+\int_{-h}^{0} W(t+s)a(s)A_2g^0ds$$

$$=\begin{cases}\int_{-h}^{-t} d\eta(s)g^1(t+s)+\int_{-t}^{0} \frac{d}{dt}W(t+s)[Fg]^1(s)ds & \text{if } 0\le t<h \\ W(t-h)A_1g^1(0)+\int_{-h}^{0} \frac{d}{dt}W(t+s)[Fg]^1(s)ds & \text{if } t\ge h\end{cases}$$

$$=W(t)A_0g^0+\int_{-h}^{0} W(t+s)d\eta(s)g^0ds$$

$$=\begin{cases}\int_{-t}^{0} \frac{d}{dt}W(t+s)[Fg]^1(s)ds & \text{if } 0\le t<h \\ \int_{-h}^{0} \frac{d}{dt}W(t+s)[Fg]^1(s)ds & \text{if } t\ge h\end{cases}$$

$$+\begin{cases}\int_{-h}^{-t} d\eta(s)g^1(t+s) & \text{if } 0\le t<h \\ 0 & \text{if } t\ge h\end{cases}.$$

Here we note that all integrals containing $[Fg]^1$ in (3.16) and (3.17) make sense by $[Fg]^1\in L^2(-h, 0; V^*)$ and Lemma 1. Since $W(t)$ satisfies the equation (2.21) the sum of the last two terms in (3.17) equals the sum of the last two terms of (3.15). Consequently, the trivial equality $[AS(t)g]^0=[S(t)Ag]^0$ for $g\in\mathcal{D}(A)\subset W^{1,2}(-h,0;V)$ implies the commutative relation (3.11) for any $g^0\in V$. □

The following useful 'quasi-semigroup' property for $W(t)$ is verified by the semigroup property $S(t_1+t_2)=S(t_1)S(t_2)$ (cf. Nakagiri [17]).

Corollary 2. *The fundamental solution $W(t)$ satisfies*

$$\begin{aligned}(3.18)\qquad & W(t_1+t_2)\\ &= W(t_1)W(t_2)+\int_{-h}^{0} W(t_1+s)[F_1W(t_2+\cdot)](s)ds \quad t_1, t_2\geq 0\,,\end{aligned}$$

where

$$(3.19)\qquad [F_1W(t_2+\cdot)](s)g^0=\int_{-h}^{s} d\eta(\xi)W(t_2+\xi-s)g^0d\xi\,,\quad g^0\in H\,.$$

Proof. Let $g^0\in H$. Then by (3.7) the relation $[S(t_1+t_2)(g^0,0)]^0=[S(t_1)S(t_2)(g^0,0)]^0$ implies the equality (3.18). □

Next we shall study the adjoint semigroup $S^*(t)$ on the adjoint space of M_2, which plays important role in the control theory involving functional differential equations. The ajoint space M_2^* of M_2 can be identified with the product space $H\times L^2(-h,0;V^*)$ via the duality pairing (we use the same bracket as that of inner product)

$$\begin{aligned}(3.20)\qquad \langle g,f\rangle_{M_2} &= \langle g^0,f^0\rangle+\int_{-h}^{0}\langle g^1(s),f^1(s)\rangle ds\,,\\ & g=(g^0,g^1)\in M_2\,,\quad f=(f^0,f^1)\in M_2^*\,.\end{aligned}$$

By identifying the second dual V^{**} of V with itself, the second adjoint space M_2^{**} is identified with M_2. That is, M_2 is reflexive in this identification. Hence, by the reflexiveness, the adjoint semigroup $S^*(t)$ is strongly continuous in M_2^* (cf. Hille and Phillips [9]). Further the infinitesimal generator of $S^*(t)$ is given by the adjoint A^* and is characterized precisely by the following theorem. For relevant results concerning abstract functional differential equations we refer to Webb [25], Nakagiri [17] and Kunisch and Mastinšek [13].

Theorem 5. *The infinitesimal generator A^* of $S^*(t)$ is given by*

$$\begin{aligned}(3.21)\qquad \mathcal{D}(A^*) = \{f=(f^0,f^1);\, & f^0\in V, f^1\in W^{1,2}(-h,0;V^*)\,,\\ & A_0^*f^0+f^1(0)\in H\,,\quad f^1(-h)=A_1^*f^0\}\end{aligned}$$

$$(3.22)\quad A^*f=(A_0^*f^0+f^1(0),\, a(\cdot)A_2^*f^0-f^1(\cdot)) \quad \textit{for}\ f=(f^0,f^1)\in\mathcal{D}(A^*)\,.$$

Proof. We follow the argument in [17; Prop. 3.3]. Let $(f^0,f^1)\in\mathcal{D}(A^*)$

and $A^*(f^0,f^1)=(k^0,k^1)\in M_2^*$. Then for each $(g^0,g^1)\in\mathcal{D}(A)$,

$$\langle A(g^0,g^1),(f^0,f^1)\rangle_{M_2}=\langle(g^0,g^1),(k^0,k^1)\rangle_{M_2}, \tag{3.23}$$

that is, by (3.3)

$$\begin{aligned}\langle A_0g^0+A_1g^1(-h)+\int_{-h}^0 a(s)A_2g^1(s)ds,\ f^0\rangle+\int_{-h}^0\langle\dot g^1(s),f^1(s)\rangle\,ds\\ =\langle g^0,k^0\rangle+\int_{-h}^0\langle g^1(s),k^1(s)\rangle\,ds\,.\end{aligned} \tag{3.24}$$

First we assume $f^0\in V$. Let φ be a scalar function such that

$$\varphi\in W^{1,2}(-h,0),\ \ \varphi(-h)=0,\ \ \varphi(0)=1,\ \ \int_{-h}^0 a(s)\varphi(s)ds=0\,. \tag{3.25}$$

For $g^0\in\mathcal{D}(A_0)$ it is clear that $\tilde g=(g^0,\varphi(\cdot)g^0)\in\mathcal{D}(A)$. We substitute this $\tilde g$ to (3.24) and obtain

$$\begin{aligned}&\langle A_0g^0,f^0\rangle\\ &=\langle g^0,k^0\rangle+\langle g^0,\int_{-h}^0\varphi(s)k^1(s)ds\rangle-\langle g^0,\int_{-h}^0\dot\varphi(s)f^1(s)ds\rangle\,.\end{aligned} \tag{3.26}$$

The linear form $l(g^0)=\langle A_0g^0,f^0\rangle$ on $\mathcal{D}(A_0)$ defined by the right hand side of (3.26) can be extended to a bounded form $\tilde l$ on V by the density of $\mathcal{D}(A_0)$ in V (note that $A_0+c_1\colon V\to V^*$ is an isomorphism), and then $\tilde l=A_0^*f^0\in V^*$. This means

$$f^0\in V \text{ and } \langle A_0g^0,f^0\rangle=\langle g^0,A_0^*f^0\rangle\qquad\text{for any}\quad g^0\in V\,. \tag{3.27}$$

Next we set

$$M(s)=\int_{-h}^s k^1(\xi)d\xi\,,\quad N(s)=A_1^*f^0+\int_{-h}^s a(\xi)A_2^*f^0d\xi\,,\quad s\in[-h,0]\,.$$

It is easy to see $M\in W^{1,2}(-h,0;V^*)$, $N\in C^1([-h,0];V^*)$. For any $y\in W^{1,2}(-h,0;\mathcal{D}(A_0))$, we have by using integration by parts

$$\begin{aligned}&\int_{-h}^0\langle\dot y(s),M(s)\rangle\,ds\\ &=\langle y(0),M(0)\rangle-\int_{-h}^0\langle y(s),k^1(s)\rangle\,ds\,;\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\int_{-h}^0\langle\dot y(s),N(s)\rangle\,ds\\ &=\langle y(0),N(0)\rangle-\langle y(-h),N(-h)\rangle-\int_{-h}^0\langle y(s),\dot N(s)\rangle\,ds\\ &=\langle y(0),N(0)\rangle-\langle y(-h),A_1^*f^0\rangle-\int_{-h}^0\langle y(s),a(s)A_2^*f^0\rangle\,ds\,.\end{aligned} \tag{3.29}$$

Then by (3.28), (3.29) and taking adjoints, we find

$$\begin{aligned}
&\int_{-h}^{0} \langle \dot{y}(s), f^1(s)-N(s)+M(s)\rangle ds + \langle A_0 y(0), f^0\rangle \\
&= \int_{-h}^{0} \langle \dot{y}(s), f^1(s)\rangle ds - \langle y(0), N(0)\rangle + \langle y(-h), A_1^* f^0\rangle \\
&\quad + \int_{-h}^{0} \langle y(s), a(s)A_2^* f^0\rangle\, ds + \langle y(0), M(0)\rangle \\
&\quad - \int_{-h}^{0} \langle y(s), k^1(s)\rangle ds + \langle A_0 y(0), f^0\rangle \\
&= \langle A_0 y(0) + A_1 y(-h) + \int_{-h}^{0} a(s)A_2 y(s) ds, f^0\rangle + \int_{-h}^{0} \langle \dot{y}(s), f^1(s)\rangle ds \\
&\quad - \langle y(0), k^0\rangle - \int_{-h}^{0} \langle y(s), k^1(s)\rangle ds \\
&\quad + \langle y(0), k^0 - N(0) + M(0)\rangle .
\end{aligned}$$

Hence if $y \in \mathcal{D}(A)$, then by (3.24)

$$\begin{aligned}
(3.30)\qquad &\int_{-h}^{0} \langle \dot{y}(s), f^1(s)-N(s)+M(s)\rangle ds + \langle A_0 y(0), f^0\rangle \\
&= \langle y(0), k^0 - N(0) + M(0)\rangle .
\end{aligned}$$

Let $g^0 \in \mathcal{D}(A_0)$. Applying $y(s) \equiv g^0 \in W^{1,2}(-h, 0; \mathcal{D}(A_0)) \cap \mathcal{D}(A)$ to (3.30), we get

$$\langle A_0 g^0, f^0\rangle = \langle g^0, k^0 - A_1^* f^0 - \int_{-h}^{0} a(\xi) A_2^* f^0 d\xi + \int_{-h}^{0} k^1(\xi) d\xi\rangle ,$$

so that by the extension and (3.27),

$$(3.31)\qquad k^0 = A_0^* f^0 + A_1^* f^0 + \int_{-h}^{0} a(\xi) A_2^* f^0 d\xi - \int_{-h}^{0} k^1(\xi) d\xi \in H .$$

Since $W^{1,2}(-h, 0; \mathcal{D}(A_0)) \cap \mathcal{D}(A)$ is dense in $W^{1,2}(-h, 0; V)$, it follows from (3.30) and (3.31) that

$$\begin{aligned}
(3.32)\qquad f^1(s) &= N(s) - M(s) \\
&= A_1^* f^0 + \int_{-h}^{s} a(\xi) A_2^* f^0 d\xi - \int_{-h}^{s} k^1(\xi) d\xi , \quad s \in [-h, 0] ,
\end{aligned}$$

and

$$(3.33)\qquad f^1 \in W^{1,2}(-h, 0; V^*) , \quad f^1(-h) = A_1^* f^0 .$$

Therefore by (3.31)-(3.33), $\mathcal{D}(A^*)$ is contained in the right hand side of (3.21). The reverse inclusion can be verified analogously. So the proof is complete. □

Finally in this section we note that the condition (2.20) on $a(s)$ is not necesary in Theorem 4 and Theorem 5.

4. Adjoint Theory for Transposed Equations

The purpose of this section is to give an elementary adjoint theory for transposed equations and study the relations between transposed semigroups, adjoint semigroups and structural operators.

The 'transposed' equation $(\mathbf{E_T})$ in H with an initial data $(\varphi^0, \varphi^1)\in M_2$ and a forcing function $h\in L^2_{loc}(\boldsymbol{R}^+; V^*)$ is defined by

$$(4.1)\quad \frac{dv(t)}{dt} = A_0^* v(t) + A_1^* v(t-h) + \int_{-h}^{0} a(s) A_2^* v(t+s)ds + h(t)\,, \quad \text{a.e. } t\geq 0$$

$$(4.2)\qquad v(0)=\varphi^0\,,\quad v(s)=\varphi^1(s)\quad \text{a.e. } s\in[-h, 0)\,,$$

where A_ι^* denote the adjoint operator of A_ι $(\iota=0, 1, 2)$.

It is well known (cf. Hille and Phillips [9] and Tanabe [21]) that the adjoint operator A_0^* generates an analytic semigroup $T^*(t)$, which is the adjoint of $T(t)$, both in H and V^* and that $T^*(t)$: $V^*\rightarrow V$ for each $t>0$. Then we can construct the fundamental solution $W_T(t)$ of $(\mathbf{E_T})$, which is strongly continuous both in H and V^* and $W_T(t)$: $V^*\rightarrow V$, $t>0$ satisfying the same estimates in Lemma 1. For simplicity we use the adjoint Stieltjes measure $\eta^*(s)$ defined by

$$(4.3)\qquad \eta^*(s) = -\chi_{(-\infty,-h]}(s)A_1^* - \int_s^0 a(\xi)d\xi A_2^*\colon V\rightarrow V^*\,,\quad s\in[-h, 0]\,.$$

Hence, as shown in Theorem 1 and Theorem 3 the solution $v(t)$ of $(\mathbf{E_T})$ exists uniquely and is represented by

$$(4.4)\qquad v(t) = W_T(t)\varphi^0 + \int_{-h}^{0} V_t(s)\varphi^1(s)ds + \int_0^t W_T(t-s)h(s)ds \quad t\geq 0\,,$$

where

$$(4.5)\qquad V_t(s) = \int_{-h}^{0} W_T(t-s+\xi)d\eta^*(\xi)\colon V\rightarrow V\,,\quad \text{a.e. } s\in[-h, 0]\,.$$

Now we denote the adjoint of $W(t)$ by $W^*(t)$. First we shall show the following important relationship in the adjoint theory.

Lemma 2. *For $t\in\boldsymbol{R}$,*

$$(4.6)\qquad W_T(t) = W^*(t)\,.$$

Proof. By the identification of V^{**} with V, it is obvious that $W^*(t)$: $V^*\rightarrow V$ for $t>0$. The commutative relation (3.11) in Corollary 1 implies, by taking adjoint, the adjoint relation

$$(4.7)\quad A_0^* W^*(t) + \int_{-h}^{0} d\eta^*(s) W^*(t+s) = W(t)^* A_0^* + \int_{-h}^{0} W(t+s)^* d\eta^*(s)\,.$$

On the other hand, taking adjoint of (2.21), we have that $W^*(t)$ satisfies

$$\frac{d}{dt}W^*(t) = W^*(t)A_0^* + \int_{-h}^0 W^*(t+s)d\eta^*(s)\,, \quad \text{a.e. } t>0\,, \tag{4.8}$$

so by (3.11),

$$\frac{d}{dt}W^*(t) = A_0^* W^*(t) + \int_{-h}^0 d\eta^*(s)W^*(t+s)\,, \quad \text{a.e. } t>0\,. \tag{4.9}$$

This shows that $W^*(t)$ is the fundamental solution of $(\mathbf{E_T})$. Hnece by uniqueness of solutions the assertion (4.6) follows. □

Calculating the adjoints of F and $G(t)$ straightforwardly by means of the pairing (3.20) and using Lemma 2, we can deduce the following representation formulas.

Proposition 4. (i) *The adjoint $F^*: M_2 \to M_2^*$ of F is given by*

$$[F^*g]^0 = g^0\,, \quad [F^*g]^1 = F_1^* g, \qquad \text{for} \quad g = (g^0, g^1) \in M_2\,, \tag{4.10}$$

where $F_1^*: L_2(-h, 0; V) \to L_2(-h, 0; V^*)$ denotes the adjoint of F_1 and is represented by

$$[F_1^* g^1](s) = \int_{-h}^s d\eta^*(\xi) g^1(\xi - s) \quad \text{a.e. } s \in [-h, 0]\,. \tag{4.11}$$

(ii) *The adjoint $G^*(t): M_2^* \to M_2$ of $G(t)$ is given by*

$$\begin{aligned} &[G^*(t)f]^1(s) = W_T(t+s)f^0 + \int_{-h}^0 W_T(t+s+\xi)f^1(\xi)d\xi \;\; s \in [-h, 0]\,, \\ &[G^*(t)f]^0 = [G^*(t)f]^1(0)\,, \qquad\qquad f = (f^0, f^1) \in M_2^*\,. \end{aligned} \tag{4.12}$$

By Proposition 4, we see that the solution $v(t) = v(t; \varphi)$ of $(\mathbf{E_T})$ with $h=0$ is written as

$$v(t; \varphi) = W_T(t)\varphi^0 + \int_{-h}^0 W_T(t+s)[F^*\varphi]^1(s)ds\,. \tag{4.13}$$

Let us denote by $\{S_T(t)\}_{t\geq 0}$ the transposed C_0-semigroup on M_2 associated with the transposed equation $(\mathbf{E_T})$ and by A_T its infinitesimal generator. Now the following proposition is immediate from (4.13) and Proposition 4.

Proposition 5. *The semigroup $S_T(t)$ is expressed by*

$$S_T(t) = G^*(t)F^* + \kappa(t)\,, \qquad t \geq 0\,, \tag{4.14}$$

where $\kappa(t): M_2 \to M_2$ is given by (3.8). *In particular, $S_T(h)$ has the following decomposition*

$$S_T(h) = G^*F^* . \tag{4.15}$$

The adjoint semigroup of $S_T(t)$ and its infinitesimal generator are denoted by $S_T^*(t)$ and A_T^*, respectively. Taking adjoints of $S(t)$ and $S_T(t)$ and using Proposition 3 and Proposition 5, we have the following result.

Proposition 6. *For $t \geq 0$, the semigroups $S^*(t)$ and $S_T^*(t)$ are represented by*

$$S^*(t) = F^*G^*(t) + \kappa^*(t) , \quad S_T^*(t) = FG(t) + \kappa^*(t) , \tag{4.16}$$

where the adjoint $\kappa^*(t)\colon M_2^* \to M_2^*$ of $\kappa(t)$ is given by

$$[\kappa^*(t)f]^0 = 0 , \quad [\kappa^*(t)f]^1(s) = f^1(s-t)\chi_{[0,s+h)}(s) \quad \text{a.e. } s \in [-h, 0] \tag{4.17}$$

for $f = (f^0, f^1) \in M_2^*$. In particular,

$$S^*(h) = F^*G^* , \quad S_T^*(h) = FG . \tag{4.18}$$

On the ranges and the invertibility of the operators G, G^*, we have the following characterizations.

Proposition 7. (i) $\operatorname{Im} G$ *equals* $\operatorname{Im} G^*$ *and is given by*

$$\mathcal{W}(-h, 0) = \{\varphi \in M_2;\ \varphi^1 \in W^{1,2}(-h, 0; V^*) \cap L^2(-h, 0; V) ,\ \varphi^1(0) = \varphi^0 \in H\} . \tag{4.19}$$

(ii) *$G\colon M_2^* \to \mathcal{W}(-h, 0)$ is bijective and the inverse $G^{-1}\colon \mathcal{W}(-h, 0) \to M_2^*$ is given explicitly by*

$$\begin{aligned} &[G^{-1}\varphi]^1(s) = \dot{\varphi}^1(-s-h) - A_0\varphi^1(-s-h) - \int_s^0 d\eta(\xi)\varphi^1(\xi-s-h) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } s \in [-h, 0] \\ &[G^{-1}\varphi]^0 = \varphi^1(-h) , \qquad\qquad \varphi = (\varphi^0, \varphi^1) \in \mathcal{W}(-h, 0) . \end{aligned} \tag{4.20}$$

$G^*\colon M_2^* \to \mathcal{W}(-h, 0)$ is also bijective and the inverse $(G^*)^{-1}\colon \mathcal{W}(-h, 0) \to M_2^*$ is given by the formula (4.20) *in which A_0, $\eta(\xi)$ are replaced by A_0^*, $\eta^*(\xi)$, respectively. In particular,*

$$\operatorname{Ker} G = \operatorname{Ker} G^* = \{0\} ; \quad \operatorname{Cl}(\operatorname{Im} G) = \operatorname{Cl}(\operatorname{Im} G^*) = M_2 . \tag{4.21}$$

Proof. Since the inclusion $\operatorname{Im} G \subset \mathcal{W}(-h, 0)$ is clear form Theorem 1 and Proposition 2, it sufficies to show the reverse inclusion $\operatorname{Im} G \supset \mathcal{W}(-h, 0)$. Let $\varphi = (\varphi^1(0), \varphi^1) \in \mathcal{W}(-h, 0)$ and define $k = (k^0, k^1)$ by the right hand side of (4.20). It is easy to see $k \in M_2^*$. If we set $u(t) = \varphi^1(t-h)$, $t \in [0, h]$ and $u(0) = \varphi^1(-h)$, $u(s) = 0$ a.e. $s \in [-h, 0]$, then from (4.20) $u(\cdot) \in W^{1,2}(0, h; V^*) \cap L^2(0, h; V)$ gives a solution of (E) on $[0, h]$ with $g^0 = \varphi^1(-h)$, $g^1 \equiv 0$, $f(t) = k^1(-t)$ a.e. $t \in [0, h]$

(note that $\varphi^1(-h)\in H, f\in L^2(0, h; V^*)$). Thus, by the uniqueness of solution and the variation of constants formula,

$$\begin{aligned} u(t) &= W(t)\varphi^1(-h)+\int_0^t W(t-s)f^1(-s)ds \\ &= [Gk]^1(t-h) = \varphi^1(t-h)\,, \quad t\in[0, h]\,, \\ u(h) &= [Gk]^1(0) = [Gk]^0 = \varphi^1(0)\,. \end{aligned}$$

This implies $Gk=\varphi$, and hence $\operatorname{Im} G=\mathcal{W}(-h, 0)$. At the same time we see that G is bijective and the inverse G^{-1} is given by (4.20). The proof for G^* is quite same. Now the first relation in (4.21) is obvious and the second one follows readily from $\mathcal{D}(A)\subset\mathcal{W}(-h, 0)$. $\square$

Related to the operators F, F^* we have the following Proposition

Proposition 8. (i) *If A_1: $V\to V^*$ has a bounded inverse A_1^{-1}: $V^*\to V$, then*

$$\operatorname{Im} F = \operatorname{Im} F^* = M_2^* \tag{4.22}$$

and

$$\operatorname{Ker} F = \operatorname{Ker} F^* = \{0\}\,. \tag{4.23}$$

(ii) *If $A_1=O$, $a\in C^1[-h, 0]$, $a(-h)\neq 0$ and A_2: $V\to V^*$ has a bounded inverse A_2^{-1}: $V^*\to V$, then*

$$\operatorname{Ker} F = \operatorname{Ker} F^* = \{0\}\,. \tag{4.24}$$

Proof. (i) Let $f=(f^0, f^1)\in M_2^*$, $g=(g^0, g^1)\in M_2$. In order to show $\operatorname{Im} F=M_2^*$, we shall solve the equation $Fg=f$. Since A_1^{-1} exists, this equation is equivalent to $g^0=f^0$ and

$$g^1(-h-s)+\int_{-h}^s a(\xi)A_1^{-1}A_2g^1(\xi-s)d\xi = A_1^{-1}f^1(s) \quad \text{a.e. } s\in[-h, 0]\,. \tag{4.25}$$

The equation (4.25) for g^1 is a Volterra integral equation of the second kind, so by the contraction mapping principle, we have a unique solution $g^1\in L^2(-h, 0; V)$. Hence we can construct a unique solution $g\in M_2$ of the equation $Fg=f$. This proves $\operatorname{Im} F=M_2^*$ and $\operatorname{Ker} F=\{0\}$. Since A_1 is boundedly invertible if and only if the adjoint A_1^* is boundedly invertible (cf. Kato [12; p. 169]), as seen as above we have $\operatorname{Im} F^*=M_2^*$ and $\operatorname{Ker} F^*=\{0\}$.

(ii) Assume $A_1=0$ and let $g=(g^0, g^1)\in M_2$ satisfy $Fg=0$. Then $g^0=0$ and g^1 satisfies the Volterra integral equation of the first kind

$$\int_{-h}^s a(\xi)A_2g^1(\xi-s)d\xi = 0 \quad \text{a.e. } s\in[-h, 0]\,. \tag{4.26}$$

Since $a(s)$ is continuously differentiable, we differentiate the equality (4.26) and apply A_2^{-1} to obtain

$$(4.27) \qquad a(-h)g^1(-h-s)+\int_{-h}^{s}\frac{da(\xi)}{d\xi}g^1(\xi-s)d\xi = 0 \quad \text{a.e. } s\in[-h,0].$$

The Volterra integral equation (4.28) is of the second kind by $a(-h)\neq 0$, and so this equation admits a trivial solution $g^1(s)=0$, and hence $g=(g^0,g^1)=0$. This shows $\operatorname{Ker} F=\{0\}$. Noting Kato [12; p. 169] again, we see $\operatorname{Ker} F^*=\{0\}$ readily. □

The next Proposition extends the key relation $G(t+h)=S(t)G$ proved in Nakagiri [17; Prop. 3.7], which plays a central role in the structural study of retarded functional differential equation (**E**).

Proposition 9. (i) *For* $t_1, t_2\geq 0$,

$$(4.28) \qquad S(t_1)G(t_2) = G(t_1+t_2)E(t_2), \quad S_T(t_1)G^*(t_2) = G^*(t_1+t_2)E(t_2),$$

where $E(t_2)\colon M_2^*\to M_2^*$ *is given by*

$$(4.29) \qquad [E(t_2)f]^0 = f^0, \quad [E(t_2)f]^1(s) = \chi_{(-t_2,0]}(s)f^1(s) \quad \text{a.e. } s\in[-h,0].$$

(ii) *For* $t_1, t_2\geq 0$,

$$(4.30) \qquad G^*(t_1)S^*(t_2) = E^*(t_1)G^*(t_1+t_2), \quad G(t_1)S_T^*(t_2) = E^*(t_1)G(t_1+t_2),$$

where $E^*(t_1)\colon M_2\to M_2$ *is given by*

$$(4.31) \qquad [E^*(t_1)g]^0 = g^0,\ [E^*(t_1)g]^1(s) = \chi_{(-t_1,0]}(s)g^1(s) \quad \text{a.e. } s\in[-h,0].$$

(ii) *For* $t_1\geq 0$ *and* $t_2\geq h$,

$$(4.32) \qquad S(t_1)G(t_2) = G(t_2)S_T^*(t_1) \quad S_T(t_1)G^*(t_2) = G^*(t_2)S^*(t_1).$$

Proof. (i) Let $f=(f^0,f^1)\in M_2^*$. By definitions of $S(t)$, $G(t)$ and (3.10), we have for $t_1+s\geq 0$,

$$\begin{aligned}(4.33) \qquad & [S(t_1)G(t_2)f]^1(s)\\ &= W(t_1+s)[G(t_2)f]^0+\int_{-h}^{0}W(t_1+s+\xi)\Big[F[G(t_2)f]^1\Big](\xi)d\xi\\ &= \Big(W(t_1+s)W(t_2)f^0+\int_{-h}^{0}W(t_1+s+\xi)[F_1W(t_2+\cdot)f^0](\xi)d\xi\Big)\\ &\quad +\int_{-h}^{0}W(t_1+s)W(t_2+\xi)f^1(\xi)d\xi\\ &\quad +\int_{-h}^{0}W(t_1+s+\xi)\Big[F_1[\int_{-h}^{0}W(t_2+\cdot+\alpha)f^1(\alpha)d\alpha]\Big](\xi)d\xi\\ &= I_1+I_2+I_3.\end{aligned}$$

By the change of variables $\alpha\to\xi$, the term I_3 is calculated as

$$(4.34)\qquad \begin{aligned} I_3 &= \int_{-h}^{0}\int_{-h}^{\xi} W(t_1+s+\xi)d\eta(\tau)\int_{-h}^{0} W(t_2+\alpha+\tau-\xi)f^1(\alpha)d\alpha d\xi \\ &= \int_{-h}^{0}\int_{-h}^{0} W(t_1+s+\alpha)\int_{-h}^{\alpha} d\eta(\tau)W(t_2+\xi+\tau-\alpha)d\alpha f^1(\xi)d\xi \\ &= \int_{-h}^{0}\left(\int_{-h}^{0} W(t_1+s+\alpha)[F_1W(t_2+\xi+\cdot)](\alpha)d\alpha\right)f^1(\xi)d\xi\,, \end{aligned}$$

so that

$$(4.35)\qquad \begin{aligned} I_2+I_3 = \int_{-h}^{0} &(W(t_1+s)W(t_2+\xi) \\ &+\int_{-h}^{0} W(t_1+s+\alpha)[F_1W(t_2+\xi+\cdot)](\alpha)d\alpha)f^1(\xi)d\xi\,. \end{aligned}$$

Noting that $W(t_2+\xi)=O$ if $t_2+\xi<0$ and applying the quasi-semigroup property (3.18) of $W(t)$ we obtain that

$$(4.36)\qquad \begin{aligned} &[S(t_1)G(t_2)f]^1(s) \\ &= W(t_1+t_2+s)f^0+\int_{-t_2}^{0} W(t_1+t_2+s+\xi)f^1(\xi)d\xi \\ &= W(t_1+t_2+s)f^0+\int_{-h}^{0} W(t_1+t_2+s+\xi)\chi_{(-t_2,0]}f^1(\xi)d\xi \\ &= [G(t_1+t_2)E(t_2)f]^1(s)\,. \end{aligned}$$

On the other hand, we have for $t_1+s<0$,

$$(4.37)\qquad \begin{aligned} &[S(t_1)G(t_2)f]^1(s) \\ &= [G(t_2)f]^1(t_1+s) \\ &= W(t_1+t_2+s)f^0+\int_{-h}^{0} W(t_2+t_1+s+\xi)f^1(\xi)d\xi \\ &= W(t_1+t_2+s)f^0+\int_{-h}^{0} W(t_2+t_1+s+\xi)\chi_{(-t_2,0]}f^1(\xi)d\xi \\ &= [G(t_1+t_2)E(t_2)f]^1(s)\,. \end{aligned}$$

Substituting $s=0$ in (4.36) we have $[S(t_1)G(t_2)f]^0=[G(t_1+t_2)E(t_2)f]^0$. Hence $S(t_1)G(t_2)=G(t_1+t_2)E(t_2)$ is proved. Similarly we have $S_T(t_1)G^*(t_2)=G^*(t_1+t_2)E(t_2)$.

(ii) Taking adjoints of the equalities in (4.28) and changing t_1 and t_2, we have the relations (4.30).

(iii) The assertion (4.32) follows from (i), (ii) and that $E(h)$ is the identity operator on M_2^*.

The following result gives the interconnected property between the structural operator G and the semigroup $S(t)$.

Theorem 6. *The following relations on G hold:*

(4.38) $S(t)G = GS_T^*(t)\,,\quad G^*S^*(t) = S_T(t)G^*\,,\quad t\geq 0\,;$

(4.39) $G\mathcal{D}(A_T^*)\subset\mathcal{D}(A)$ *and* $AG = GA_T^*$ *on* $\mathcal{D}(A_T^*)\,;$

(4.40) $G^*\mathcal{D}(A^*)\subset\mathcal{D}(A_T)$ *and* $G^*A^* = A_TG^*$ *on* $\mathcal{D}(A^*)\,.$

Proof. Substituting $t_1=t$ and $t_2=h$ in (4.32) we get (4.38). It is well known that the first and the second equalities in (4.38) are equivalent to (4.39) and (4.40), respectively (cf. Salamon [20; Lemma I. 3.8]). □

Theorem 7. *The following relations on F hold:*

(4.41) $FS(t) = S_T^*(t)F\,,\quad S^*(t)F^* = F^*S_T(t)\,,\quad t\geq 0\,;$

(4.42) $F\mathcal{D}(A)\subset\mathcal{D}(A_T^*)$ *and* $FA = A_T^*F$ *on* $\mathcal{D}(A)\,;$

(4.43) $F^*\mathcal{D}(A_T)\subset\mathcal{D}(A^*)$ *and* $A^*F^* = F^*A_T$ *on* $\mathcal{D}(A_T)\,.$

Proof. It is sufficient to prove (4.41). By (4.38) and (3.9),

$$G(S_T^*(t)F) = S(t)GF = GFS(t) = G(FS(t))$$

and, by Ker $G=\{0\}$ in (6.21), the equality $S_T^*(t)F=FS(t)$ follows. The second equality in (4.41) can be proved similarly. □

5. Examples

In this section we give an application of the results obtained in the preceding sections to practical partial functional differential equations.

Let $\Omega\subset \boldsymbol{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We set $H=L^2(\Omega)$ and $V=H_0^1(\Omega)$. Let $a(u, v)$ be the sesquilinear form in $H_0^1(\Omega)\times H_0^1(\Omega)$ defined by

$$(5.1)\quad a(u, v) = \int_\Omega \{\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\,\overline{\frac{\partial v}{\partial x_j}}+\sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i}\bar{v}+c(x)u\bar{v}\}\,dx,\quad x\in\Omega.$$

Here in (5.1) we assume that the real valued coefficients a_{ij}, b_i, c satisfy

$$a_{ij}\in C^1(\bar{\Omega})\,,\quad b_i\in C^1(\bar{\Omega})\,,\quad c\in L^\infty(\Omega)\,,$$

$a_{ij}=a_{ji}$, $(1\leq i,j\leq n)$ and the uniform ellipticity

$$\sum_{i,j=1}^n a_{ij}(x)y_iy_j\geq \nu\,|y|^2\quad y = (y_1, \cdots, y_n)\in \boldsymbol{R}^n$$

for some positive ν. As is well known (see e.g. Tanabe [21; Chap.2]) this sesquilinear form is bounded and satisfies the Gårding's inequality (2.3). The operator $A_0: H_0^1(\Omega) \to H^{-1}(\Omega)$ defined through (2.4) in this case has the following realization in $L^2(\Omega)$. Let

$$\mathcal{A} = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial}{\partial x_j}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x), \quad x \in \Omega \tag{5.2}$$

be the associated uniformly elliptic differential operator of second order. Then the realization of $-\mathcal{A}$ in $L^2(\Omega)$ under the Dirichlet boundary condition is exactly A_0, i.e.,

$$\mathcal{D}(A_0) = W^{2,2}(\Omega) \cap H_0^1(\Omega),$$
$$A_0 u = -\mathcal{A}u \quad \text{for} \quad u \in \mathcal{D}(A_0).$$

It is not difficult to verify that $A_0 u = -\mathcal{A}u$ for $u \in H_0^1(\Omega)$ in the sense of distribution and $u|_{\partial\Omega} = 0$ for $u \in H_0^1(\Omega)$ also in the sense of distribution (cf. Lions and Magenes [15]).

Next, let A_ι, $\iota = 1, 2$, be the restriction to $H_0^1(\Omega)$ of the second order differential operator $\mathcal{A}_\iota$, $\iota = 1, 2$, given by

$$\mathcal{A}_\iota = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}^\iota(x)\frac{\partial}{\partial x_j}) + \sum_{i=1}^{n} b_i^\iota(x)\frac{\partial}{\partial x_i} + c^\iota(x) \quad x \in \Omega, \tag{5.3}$$

where

$$a_{ij}^\iota = a_{ji}^\iota \in C^1(\bar{\Omega}), \quad b_i^\iota \in C^1(\bar{\Omega}), \quad c^\iota \in L^\infty(\Omega).$$

It is clear that each $A_\iota: H_0^1(\Omega) \to H^{-1}(\Omega)$ is bounded without the ellipticity condition. The kernel function $a(s)$ is assumed to be an element of $L^2(-h, 0)$. Now we consider the following parabolic partial functional differential equation

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \mathcal{A}_0 u(t,x) + \mathcal{A}_1 u(t-h, x) \\ &\quad + \int_{-h}^{0} a(s)\mathcal{A}_2 u(t+s, x)ds + f(t,x), \quad 0<t<T, \quad x \in \Omega, \end{aligned} \tag{5.4}$$

having the Dirichlet boundary condition

$$u|_{\partial\Omega} = 0, \quad 0<t<T, \tag{5.5}$$

and the initial data given by

$$u(0,x) = g^0(x), \quad u(s,x) = g^1(s,x) \quad \text{a.e. } s \in [-h, 0), \quad x \in \Omega. \tag{5.6}$$

Here in (5.4) and (5.6) we suppose that

$$f \in L^2_{\mathrm{loc}}(\boldsymbol{R}^+; H^{-1}(\Omega)) \tag{5.7}$$

$$g^0 \in L^2(\Omega), \quad g^1 \in L^2(-h, 0; H_0^1(\Omega)). \tag{5.8}$$

Under the above conditions the system (5.4)-(5.6) takes the abstract equation form (**E**). Thus by applying Theorem 1, we have the following result.

Theorem 8. *Under the conditions* (5.7) *and* (5.8) *there exists a unique solution* $u(t,x)$ *of the system* (5.4)-(5.6) *such that* $t \to u(t,\cdot)$ *belongs to* $L^2_{loc}(-h,\infty;H^1_0(\Omega)) \cap W^{1,2}_{loc}(\boldsymbol{R}^+;H^{-1}(\Omega))$ *and satisfies the equation* (5.4) *and the boundary condition* (5.5) *in the sense of distribution and also satisfies the initial data* (5.6). *Further, for each* $T>0$ *there is a constant* K_T *such that*

$$\left(\int_0^T \|u(t,\cdot)\|^2_{W^{1,2}(\Omega)}dt+\int_0^T\left\|\frac{\partial u(t,\cdot)}{\partial t}\right\|^2_{H^{-1}(\Omega)}dt\right) \tag{5.9}$$
$$\leq K_T\left(\|g^0\|^2_{L^2(\Omega)}+\int_{-h}^0\|g^1(s)\|^2_{W^{1,2}(\Omega)}ds+\int_0^T\|f(t)\|^2_{H^{-1}(\Omega)}dt\right).$$

Next, we shall introduce the transposed system of (5.4)-(5.6) in the space $L^2(\Omega)$. Let $a(u,v)$ be the sesquilinear form given in (5.1) and the adjoint operator $A_0^*: H^1_0(\Omega)\to H^{-1}(\Omega)$ be defined through the equation

$$\langle u, A_0^* v\rangle = -a(u,v)\,, \quad u, v\in V\,. \tag{5.10}$$

The realization of A_0^* in $L^2(\Omega)$ is characterized as follows. Let us denote the formal adjoint of $\mathcal{A}$ in (5.2) by $\mathcal{A}^*$, that is,

$$\mathcal{A}^* = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial}{\partial x_j})-\sum_{i=1}^n\frac{\partial}{\partial x_i}(b_i(x)\cdot)+c(x) \quad x\in\Omega\,. \tag{5.11}$$

The realization of $-\mathcal{A}^*$ in $L^2(\Omega)$ under the Dirichlet boundary condition coincides with the realization of A_0^* in $L^2(\Omega)$. That is to say,

$$\mathcal{D}(A_0^*) = W^{2,2}(\Omega)\cap H^1_0(\Omega)\,.$$
$$A_0^* u = -\mathcal{A}^* u \qquad \text{for} \quad u\in\mathcal{D}(A_0)\,.$$

The adjoint operators A_ι^* of A_ι, $\iota=1,2$, are given by the restrictions to $H^1_0(\Omega)$ of the following formal adjoint operators of $\mathcal{A}_\iota$, $\iota=1,2,$, respectively:

$$\mathcal{A}_\iota^* = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}(a^\iota_{ij}(x)\frac{\partial}{\partial x_j})-\sum_{i=1}^n\frac{\partial}{\partial x_i}(b^\iota_i(x)\cdot)+c^\iota(x) \quad x\in\Omega\,. \tag{5.12}$$

Evidently, $A_\iota^*: H^1_0(\Omega)\to H^{-1}(\Omega)$ is bounded.

The transposed system of (5.4)-(5.6) is now given by the following system of equations

$$\frac{\partial v(t,x)}{\partial t} = \mathcal{A}_0^* v(t,x)+\mathcal{A}_1^* v(t-h,x) \tag{5.13}$$
$$+\int_{-h}^0 a(s)\mathcal{A}_2^* v(t+s,x)ds+h(t,x)\,, \quad t\geq 0\,, \quad x\in\Omega\,,$$

$$v|_{\partial\Omega}=0\,, \quad t\geq 0\,, \tag{5.14}$$

(5.15) $\quad v(0,x)=\varphi^0(x)\,,\quad v(s,x)=\varphi^1(s,x)\quad$ a.e. $s\in[-h,0)\,,\quad x\in\Omega\,.$

In (5.13) and (5.15) it is assumed that

$$h\in L^2_{\mathrm{loc}}(\boldsymbol{R}^+;H^{-1}(\Omega))\,,\quad \varphi^0\in L^2(\Omega)\,,\quad \varphi^1\in L^2(-h,0;H^1_0(\Omega))\,.$$

Under the above condition the system (5.13)-(5.15) can be considered as the transposed equation ($\mathbf{E_T}$). Hence as shown in Theorem 8, we have the similar existence, uniqueness and well posedness result for the transposed system (5.13)-(5.15) as for the system (5.4)-(5.6).

Finally we give an explicit form of the fundamental solution for some special case. Let $b_i=0$, $i=1,\cdots,n$ in (5.2) and assume that $\mathcal{A}_1=\mathcal{A}$ and $\mathcal{A}_2=O$. Then it is easy to see that A_0 is selfadjoint with compact resolvent in $L^2(\Omega)$, and generates an analytic semigroup $T(t)$ both in $L^2(\Omega)$ and $H^{-1}(\Omega)$. Further, there exists a set of eigenvalues and eigenfunctions $\{\lambda_n,\phi_n\}_{n\geq 1}$ of A_0 such that

$$\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\to-\infty\quad n\to\infty\,;$$
$$A\phi_n=\lambda_n\phi_n\,,\quad \langle\phi_n,\phi_m\rangle_{L^2}=\delta_{nm}\quad n,m=1,2,\cdots\,;$$

and

$$T(t)u=\sum_{n=1}^{\infty}e^{\lambda_n t}\langle u,\phi_n\rangle_{L^2}\phi_n\,,\quad t\geq 0\,,\quad u\in L^2(\Omega)\,,$$

where $\langle u,\phi\rangle_{L^2}=\int_\Omega u(x)\phi(x)dx$ and δ_{mn} denotes the Kronecker's delta. Hence, in this special case the fundamental solution $W(t)$ is given by

$$W(t)u=\sum_{n=1}^{\infty}\sum_{k=1}^{m}\frac{(t-(k-1)h)^{k-1}}{(k-1)!}\lambda_n^{k-1}e^{\lambda_n(t-(k-1)h)}\langle u,\phi_n\rangle_{L^2}\phi_n\,,$$
$$t\in[(m-1)h,mh]\,,\quad u\in\mathcal{D}(A_0^m)\,,\quad m=1,2,\cdots\,.$$

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